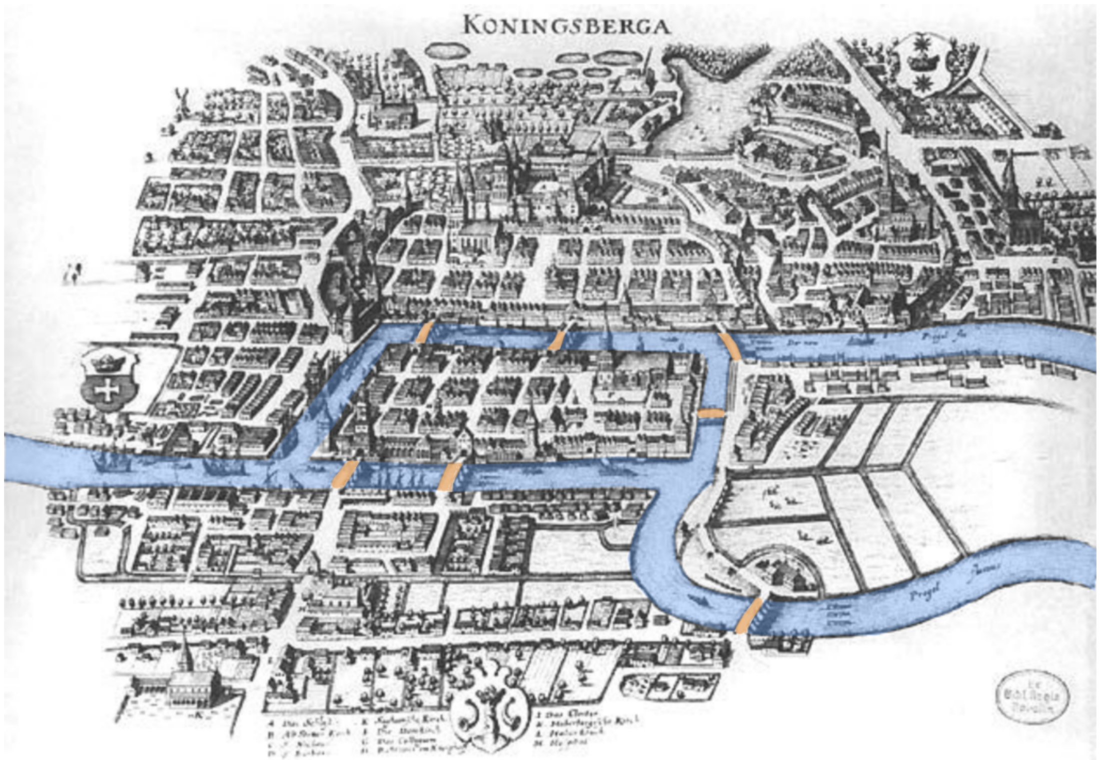
1 The Bridges of Konigsberg

- The city of Konigsberg consisted of two sides of the Pregel River and two large islands, all connected to each other by seven bridges. Is it possible to walk across each of the bridges once and only once?



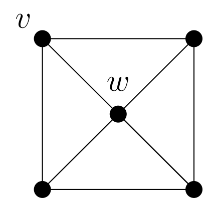
Merian Erban/ Wikimedia Commons/ Public domain

- Euler noticed that the shapes of the islands and the river do not change the answer to this question.

- The only important information from the picture is connectivity. This means that we should be able to represent the bridges and land masses as merely a set of points and lines.

- Draw a set of points and lines that convey the important information from the map of Konigsberg. In other words, we want to represent the information on the map as a set of *objects* and *connections* between them.

- What you have just drawn is called a **graph**, or a finite set of **vertices** (the points/objects) and **edges** (the lines/connections).
- The number of edges connected to a vertex v is called the **degree** of the vertex, denoted $\deg(v)$.
  - Example: What is $\deg(v)$ in the following graph? What is $\deg(w)$?



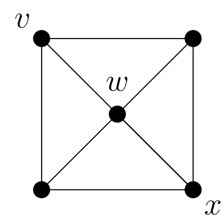
- For which values of n can we draw a graph on n vertices where each vertex has degree 3?
  - Can you draw a graph on four vertices where each vertex has degree 3?
  - Can you draw a graph on five vertices where each vertex has degree 3?
  - **Degree-Sum Theorem** (or The Handshaking Theorem): Let G be a graph with n vertices $\{v_1, v_2, \ldots, v_n\}$ and m edges. Then

$$\sum_{i=1}^{n} \deg(v_i) = 2m.$$

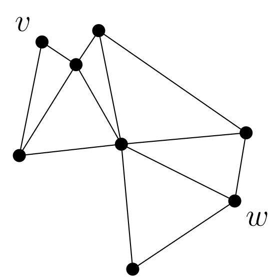
  - Why does this make sense?
  - Can a graph have an odd number of vertices of odd degree?

- So, to answer our question from earlier, can you draw a graph on five vertices where each vertex has degree 3?

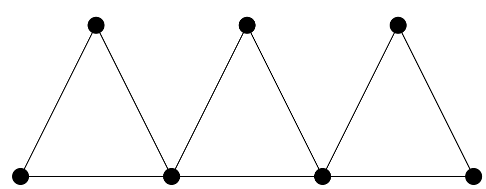
- We say a vertex v is **adjacent** to a vertex w if they are connected by an edge.
  - Example: Is v adjacent to w? Is v adjacent to x?



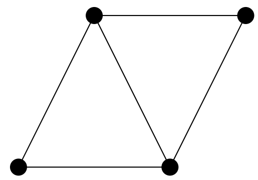
- A set of distinct edges $e_1 e_2 \cdots e_k$ is called a **walk** if we can trace these edges in the graph without picking up our pencil.
  - In other words, the endpoint of edge e_i is the starting point of edge e_{i+1}.
  - Example: Draw a walk from v to w.



- A **closed walk** is a walk $e_1 e_2 \cdots e_k$ that starts and ends at the same vertex.
- If a walk uses all edges of a graph G, it is called an **Eulerian walk**.
  - Example: Is there a closed Eulerian walk in the graph below? If so, draw one.



- If a walk does not go through any vertex twice, we call it a **path**.
  - Example: Is there a closed path in the graph below? If so, draw one.



- Suppose we want to see if there is a way to walk across each of the bridges once and only once, and end up where we started. Interpret this problem (let's call it Problem 1) in terms of our new definitions. **Problem 1**:

- A graph G is **connected** if for any two vertices x and y of G, we can find a path from x to y along edges of G.
  - Draw two graphs: one that is connected and one that is not.

- **Theorem 1**: Let G be a connected graph. Then G has a closed Eulerian walk if and only if all vertices of G have even degree.
  - Theorem 1 is an "if and only if" statement. This means that we can interpret the theorem as two separate statements:
    1. If all vertices of a connected graph G have even degree, then G has a closed Eulerian walk.
    2. If a connected graph G has a closed Eulerian walk, then all vertices of G have even degree.
  - So, if one side of the statement holds, the other must hold.
  - It also means that if one side of the statement does not hold, then the other side cannot hold.

- What does this theorem tell us about the answer to Problem 1?

- Suppose we remove the requirement to start and end at the same place. Interpret this problem (problem 2) in terms of our definitions.
  Problem 2:

- We can use the following theorem to help us determine if our problem has a solution:

- **Theorem 2**: Let G be a connected graph. Then G has an Eulerian walk starting at vertex x and ending at vertex y if and only $deg(x)$ and $deg(y)$ are odd, and all other vertices of G have even degree.

- What does this tell us about the answer to problem 2?

2 Vertex colorings

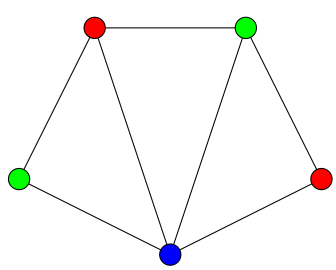
2.1 Coloring maps

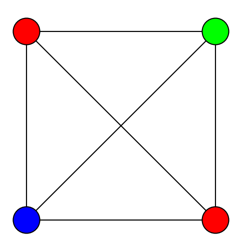
- Suppose we are given a map, and we want to know the minimal number of colors required to color the map so that no two bordering countries are the same color.
- Let's look at the map given below. How can we represent this as a graph?

- Coloring countries on our map is now the same as coloring vertices on our graph. How can we translate the requirement "no two bordering countries are the same color" as a requirement on the coloring of the vertices?

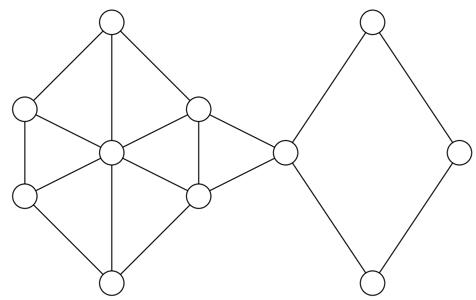
2.2 Colorings and the chromatic number

- A **coloring** of a graph G is an assignment of a color to each vertex so that no two adjacent vertices are the same color.
- A k**-coloring** of G is a coloring that uses exactly k colors.
  - Example: Determine whether the following are colorings of the graphs.

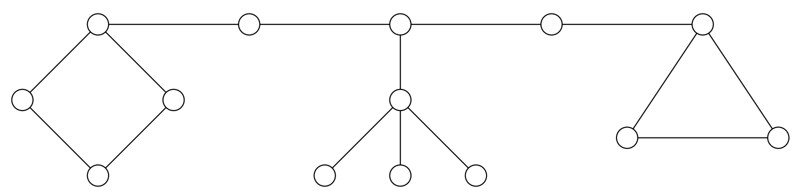




- Example: Give an example of a 3-coloring of the following graph.

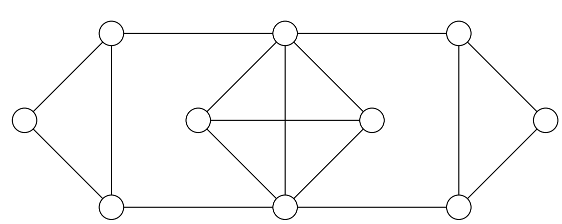


- Example: Is there a 2-coloring of the following graph? If so, give an example. If not, explain why.



- What is the smallest number k such that there exists a k-coloring of the graph above?

* The **chromatic number** of a graph G is the smallest number k such that there exists a k-coloring of G. We write this as $\chi(G)$.
  - Example: Find $\chi(G)$ for the following graph.

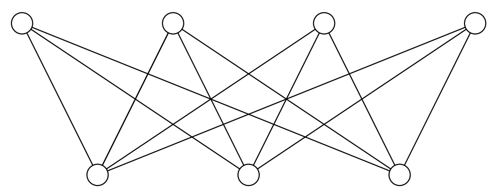


* Interpret our map coloring problem in terms of the definitions we learned today:

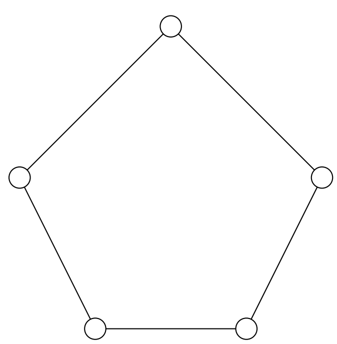
- It turns out that every map can be colored with four or fewer colors:
  - **The Four Color Theorem**: Every map can be colored with four or fewer colors.
  - In other words, $\chi(G) \leq 4$ for any graph G that comes from a map (these are called planar graphs, and some of you will study these for your project).
  - Kenneth Appel and Wolfgang Haken proved this theorem in 1976, but much of their proof was done by checking thousands of special cases on a computer. This proof is very unsatisfying, and mathematicians still want to see a more elegant proof!

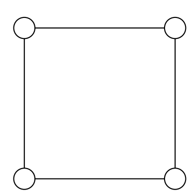
2.3 Bipartite graphs

- A graph is called **bipartite** if it has a 2-coloring.
  - Example: Is the following graph bipartite? Why or why not?



  - Example: Are the following graphs bipartite?

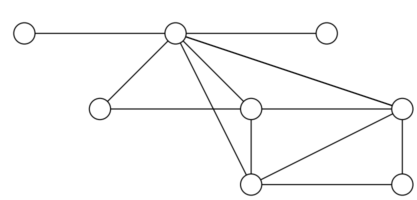




- A **cycle** in a graph is a path that starts and ends at the same vertex.
- Suppose we have a cycle in a graph G of **length** n, meaning there are n edges in the cycle.
  - What is the relationship between the number of vertices and the number of edges in a cycle?
  - A cycle can have even or odd length. What do you think is true about the relationship between the length of a cycle and whether or not the cycle is bipartite? (Think of the examples we did above.)
  - **Theorem**:
- This means that the following theorem is true:
- **Theorem**: A graph G is bipartite if and only if it does not contain a cycle of odd length.
- Example: Suppose Alicia, Bob, Chelsea, Diego, Edward, and Fred were hired for an internship. The company wants to split them into two teams for a team-building activity so that no person knows any other person on their team. Suppose Alice knows Bob, Chelsea, and Diego. Edward knows Bob and Chelsea. Fred knows Alice and Diego. Is this possible? Prove your answer. The number of people on each team does not matter.

2.4 Complete graphs

- A graph G is **complete** if every pair of vertices of G is adjacent (connected by an edge).
- We denote the complete graph on n vertices by K_n.
  - Example: Draw K_1 through K_5. Which ones can you draw so that no edges intersect, except at vertices?
  - Example: Is K_3 bipartite?
  - Example: Is K_4 3-colorable?
  - Is K_n $(n-1)$-colorable?
  - Is K_n n-colorable? Why or why not?
  - Can a graph G be $(n-1)$-colorable if it contains a copy of K_n?
  - Example: is the following graph 3-colorable? Prove your answer.



3 The chromatic polynomial

- Suppose you're hosting a party, and you've decided you want to place your guests at tables so that no person knows any other person at the table. How many ways are there to separate the group into four different tables? Rephrase this problem in terms of graph theory.

- What if we want to know how many ways there are to separate the group into five, six, or seven tables? Instead of computing these values separately, we would like to instead come up with a function that tells us the answer for any number of tables.

- The **chromatic polynomial** $P_k(G)$ for a graph G gives a formula for the number of possible k-colorings of a graph (assume we've already chosen the k colors).

- This is a polynomial with variables in k, i.e., something like

  $$P_k(G) = k^5 + 4k^3 + 2k + 7,$$

  and if you plug in any positive integer value for k, say, 4, it will tell you the number of 4-colorings of the graph G.

- Let G be a path (not closed) of length six. How many different 4-colorings of G are there? (Let's say the colors are red, green, blue, and yellow.)

- How many 7-colorings of G are there?

- How many k-colorings of G are there? (Your answer is $P_k(G)$, the chromatic polynomial of G.)

- What is $P_k(G)$, where G is a path of length n?

- Use your answer above to give the number of 10-colorings of a path of length 8.

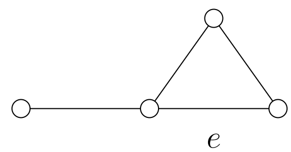
- What is $P_k(K_5)$?

- What is $P_k(K_n)$?

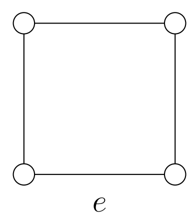
- Find the chromatic polynomial for a cycle of length four (caution: you will have to consider two cases!).

3.1 Deletions and contractions

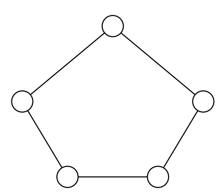
- Pick an edge e in your graph G. Then $G - e$ is the graph G with edge e removed.
- Example: For the following graph G with labeled edge e, draw $G - e$.



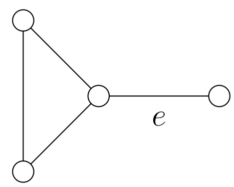
- $G \setminus e$ is the graph G with the edge e contracted; i.e., we shrink edge e down to a single vertex.
- Example: For the following graph G with labeled edge e, draw $G \setminus e$.



- Sometimes the following theorem is helpful for computing the chromatic polynomial:
- **Theorem:** $P_k(G) = P_k(G - e) - P_k(G \setminus e)$.
- Example: Find the chromatic polynomial for the following graph using the theorem above:



- Example: First find the chromatic polynomial directly (without using the theorem above), and then find the chromatic polynomial by using the theorem above, with the edge e selected. Show that your answers are the same.

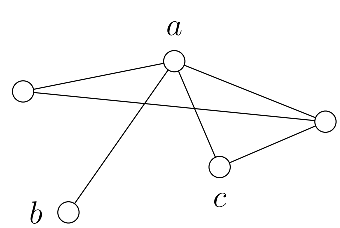


4 Cliques, independent sets, and graph complements

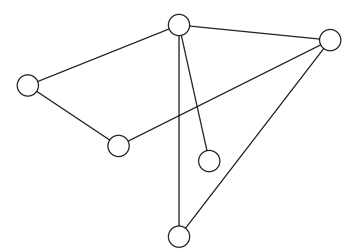
4.1 Cliques

- In today's lecture we will only focus on simple graphs.
- Suppose you browse through your Facebook friends list, and you find the following information:
  - Alex is friends with Diego, Elena, and Fred.
  - Brianna is friends with Charlie, Diego, and Fred.
  - Charlie is friends with Brianna.
  - Diego is friends with Alex, Brianna, and Elena.
  - Elena is friends with Alex and Diego.
  - Fred is friends with Alex and Brianna.
- You want to invite the largest number of people possible to lunch such that each person in the group is friends with everyone else.
- Draw a graph that represents the data given above.

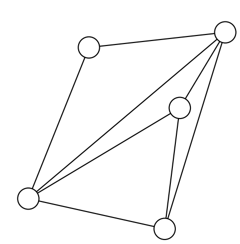
- Let G be a graph, and let $S = \{v_1, \ldots, v_n\}$ be a subset of the vertices of G. Then the **induced subgraph** $G[S]$ is the subgraph of G containing only the vertices in S and the edges of G that have endpoints only in S.
- Let $S = \{a, b, c\}$. Draw the induced subgraph $G[S]$ for the graph G below.



- A **clique** of a graph G is a set of vertices of G such that every two vertices in this set are adjacent.
- Find a clique of size 3 in the graph below.



- In order for a set of vertices S to be a clique, what must be true about the induced subgraph $G[S]$ (what type of graph is it?)?
- What is the largest clique size in the graph below?



- The **clique number**, $\omega(G)$, of a graph G is the size of the largest clique in G.
- Suppose G is a graph with seven vertices $\{v_1, v_2, \ldots, v_7\}$. The degrees of the vertices are: $\deg(v_1) = 4$, $\deg(v_2) = 5$, $\deg(v_3) = 4$, $\deg(v_4) = 3$, $\deg(v_5) = 7$, $\deg(v_6) = 3$, $\deg(v_7) = 3$. Is it possible that $\omega(G) = 5$?
- How can you tell how many 2-cliques there are in a graph?

- What is the clique number of a cycle? (There is not necessarily one answer for all cycles.)

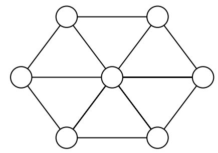
- Solve the problem from the beginning of the lecture.

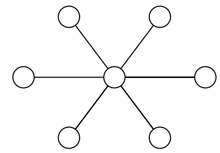
4.2 Independent sets

- Now suppose, given the same group of friends and relations as before, you want to invite as many friends as you can who don't know each other, so that everyone can make new friends.
- An **independent set** of a graph G is a set of vertices of G such that no two are adjacent.
- The **independence number**, denoted $\alpha(G)$, is the size of the largest possible independent set of G.
- What is $\alpha(K_n)$ for any n?

- How can we obtain independent sets of a graph if we are given a coloring of the graph?

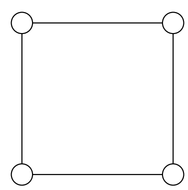
- Find $\alpha(G)$ for the following graphs.





4.3 The complement of a graph

- The **complement** of a graph G, denoted G^c, is the graph with the same set of vertices, and with an edge connecting two vertices if and only if there is no edge connecting those vertices in G.
- In other words, we obtain G^c from G by filling in the remaining edges to get the complete graph and then erasing the edges of G.
- Draw G^c for the graph below.



- What is the complement of the complete graph, K_n?
- What is the complement of a graph with no edges?
- Is it true that $(G^c)^c = G$? Why or why not?

- What happens to a clique of G in G^c?

- What happens to an independent set of G in G^c?

- What does this tells us about the relationship between $\omega(G)$ and $\alpha(G^c)$?

5 Hamiltonian paths and cycles

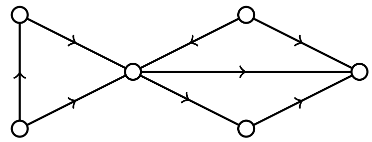
5.1 Motivation

- Suppose you are biking through a city that only has one-way streets, and you want to know if you can plan a route that visits all of the ice cream parlors in the city. We are given the following one-way streets:
  - A street that heads from parlor B to A.
  - A street that heads from D to A.
  - A street that heads from D to B.
  - A street that heads from E to D.
  - A street that heads from E to F.
  - A street that heads from B to C.
  - A street that heads from C to F.
  - A street that heads from C to E.
- How can we model this scenario with a graph?

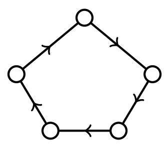
5.2 Directed graphs

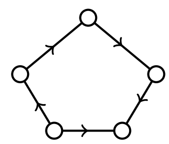
- In the graph we have drawn to represent our problem above, each edge was assigned a direction. This is called a **directed graph**.
- Recall that an Eulerian walk is a walk (a sequence of edges that we can trace in the graph without picking up our pencils) that uses all of the edges of G. In a directed graph, we must move across an edge in the direction the arrow is pointing.
- A **directed walk** is a walk that follows the arrows of G.
- A **directed path** is a path that follows the arrows of G.
- And so on...

- Example: Does the following graph contain a directed Eulerian walk? If so, find one. If not, explain why.

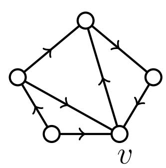


- We call a directed graph G **strongly connected** if for any pair of vertices v and w, there is a directed path from v to w.
  - Example: Are the following graphs strongly connected?



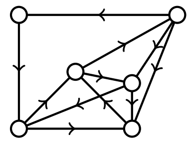


- The **in-degree** of a vertex v, denoted $\text{indeg}(v)$, is the number of edges heading in to v.
- Similarly, the **out-degree** of v, denoted $\text{outdeg}(v)$, is the number of edges heading out of v.
  - Example: What is the $\text{indeg}(v)$? $\text{outdeg}(v)$?



- We call a directed graph G **balanced** if $\text{indeg}(v) = \text{outdeg}(v)$ for all vertices v of G.
- **Theorem**: A directed graph G has a closed Eulerian walk if and only if it is balanced and strongly connected.
  - Why is this true? Let's look at one direction of the proof. Assume a directed graph G has a closed Eulerian walk. Why does that mean that G is balanced?
  - Still assuming that G has a closed Eulerian walk, why is G strongly connected?

– Example: Determine if the graph below has a closed Eulerian walk without at first trying to find such a walk. If there is a closed Eulerian walk, find it.

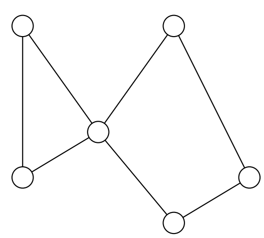


– Example: Draw a directed K_5 that has a closed Eulerian walk.

- A **tournament** is a complete directed graph.

5.3 Hamiltonian paths and cycles

- A **Hamiltonian path** is a path that visits each vertex exactly once.
- Similarly, a **Hamiltonian cycle** is a cycle that visits each vertex exactly once.
- Does the following graph have a Hamiltonian cycle?



- There are several theorems that can help us determine if a graph has a Hamiltonian cycle:
- **Theorem One:** A graph with n vertices, $n > 2$, has a Hamiltonian cycle if the degree of each vertex is at least $n/2$.
- **Theorem Two:** Every tournament has a (directed) Hamiltonian path.

- Suppose a connected graph G has four vertices v_1, v_2, v_3, and v_4. Also assume that $\deg(v_1) = 3$, $\deg(v_2) = 2$, $\deg(v_3) = 2$, and G does not have a Hamiltonian cycle. What is $\deg(v_4)$?

- Example: draw a tournament on four vertices and find a Hamiltonian path in it.

- Suppose we are given a set of results of a round robin tournament (each team plays every other team). How can we model this as a graph? Is it always possible to rank the teams in order from winner to loser?

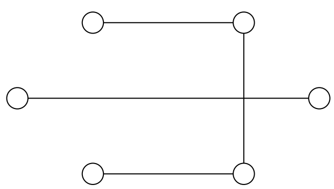
- Write the integers 1 through 15 in a line so that each two adjacent integers add up to a perfect square.

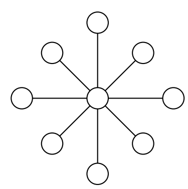
6 The *Good Will Hunting* problem

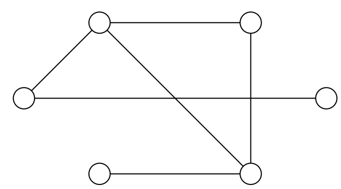
- Problem: Draw all homeomorphically irreducible trees with $n = 10$.

6.1 Trees

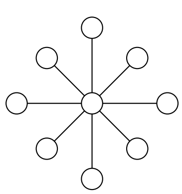
- Remember:
  - a walk is a set of distinct edges $e_1 e_2 \cdots e_k$ that we can trace in the graph without picking up our pencils
  - a path is a walk that does not go through any vertex twice
  - a cycle is a path that starts and ends at the same vertex
  - a graph is connected if there is a walk between each pair of vertices
- A **tree** is a connected graph with no cycles.
  - Example: Are the following graphs trees? Why or why not?







- **Theorem:** Let G be a connected graph. The following statements are equivalent:
  1. G is a tree.
  2. There is a unique path between any pair of vertices v and w in G.
  3. G is minimally connected, i.e., if we remove any edge of G, we will disconnect it.
- We can choose one vertex in our tree to be the **root**, and we can redraw the tree in the following way:

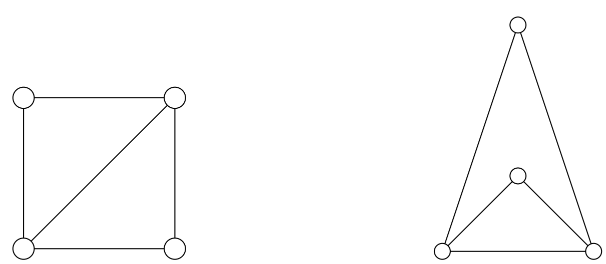


- How many edges does a tree with n vertices have?

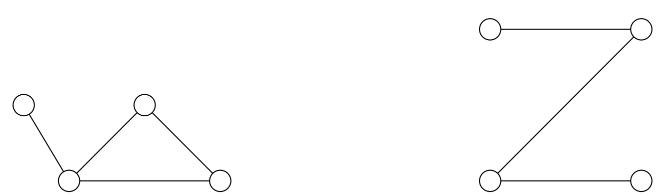
- A tree is homeomorphically irreducible if there is no vertex of degree 2.

6.2 Graph isomorphisms

- Two graphs G_1 and G_2 are *isomorphic* if their vertices can be labeled so that for any two vertices v and w, v and w are adjacent in G_1 if and only if they are adjacent in G_2.
- You can think of this as moving around the vertices of a graph, stretching the edges without breaking them, to get to another graph that is essentially "the same" as the first one.
- Example: Are the following graphs isomorphic?



- Example: Are the following graphs isomorphic?



- The **degree sequence** of a graph G with n vertices is a list of the degrees of the vertices as $d_1, d_2, \ldots, d_n$, where $d_1 \geq d_2 \geq \cdots \geq d_n$.
- Example: Write down the degree sequence for the two graphs in the previous example.
- **Theorem**: If two graphs do not have the same degree sequence, they are not isomorphic.
- *Good Will Hunting* problem, restated: Draw all homeomorphically irreducible trees with 10 vertices, up to isomorphism.
- This means we do not include graphs that are isomorphic to graphs we've already written down.
- Solve the problem (there are ten answers)!

- Writing down the answer is not useful unless we can actually *prove* that it is correct!
- We can prove our answer is correct (i.e., that none of the graphs are isomorphic to each other and that there aren't any more) by looking at possible degree sequences.
- Think back to the Degree-Sum Theorem (the Handshaking Theorem) from Lecture 1. What is the sum of all of the degrees of the vertices?
- Since trees are connected graphs, what is the minimum degree of a vertex in any answer to the problem?
- What is the maximum degree of a vertex in any answer to the problem?
- Write down all possible degree sequences for answers to the problem (remember that none of the vertices can have degree 2).
- Go through each degree sequence and reason why there are no other trees with that degree sequence besides the ones you drew (it might help to draw them as rooted trees).