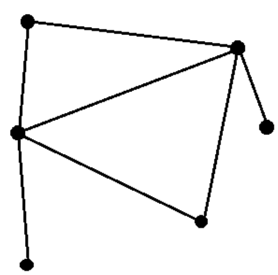
# 1 Vertex covers and river crossings

## 1.1 River crossings

- In the mid-700s, a monk by the name of Alcuin published one of the oldest combinatorial puzzles in the history of mathematics:
  - Suppose a farmer wants to transport his wolf, goat, and cabbage across a river, but he only has one extra seat in his boat. Since the wolf wants to eat the goat and the goat wants to eat the cabbage, he can't leave the wolf and the goat or the goat and the cabbage alone. What is the best way to transport all three across the river?
- Find a solution to this problem (you can do this by moving around pieces of paper; there's no need to use graph theory yet).
- What happens if we add a rabbit (the rabbit wants to eat the cabbage, and the wolf wants to eat the rabbit)? Is it possible to transport all four across the river if the farmer only has one extra seat in his boat?
- How many extra seats does the boat need to have in order to transport all four across the river?
- We also want to generalize this problem: what if we have $n$ objects that we want to transport across the river, with a given set of conflicts? How can we use graph theory to make this problem easier to solve? Explain how we can associate a graph to each initial scenario (what are your vertices and edges?), when all $n$ objects start out on the first bank.

## 1.2 Vertex covers

- A **vertex cover**, $S$, of a graph $G$ is a set of vertices such that every edge in $G$ contains at least one vertex in $S$ as an endpoint.
- What is an obvious vertex cover for any graph?
- A **minimum vertex cover** of a graph $G$ is a vertex cover of the smallest possible size. The **vertex cover number**, denoted $\tau(G)$, is the size of a minimum vertex cover for $G$.
- Find $\tau(G)$ for the following graph.



- Recall that the complete graph on $n$ vertices, denoted $K_n$, is the graph on $n$ vertices such that each vertex is adjacent to every other vertex.
- There is also a graph called the **complete bipartite graph**, denoted $K_{m,n}$, with the property that the vertices can be split into two groups, $U$ (with $m$ vertices) and $V$ (with $n$ vertices), such that there are no edges joining vertices in $U$ to vertices in $U$, no edges joining vertices in $V$ to vertices in $V$, and exactly one edge joining each vertex in $U$ to each vertex in $V$.
- Draw $K_{3,2}$ and $K_{3,3}$.

- What is $\tau(K_{m,n})$, for any $m$ and $n$?

- Find $\tau(K_4)$, $\tau(K_5)$, and $\tau(K_6)$.

### 1.3 Alcuin numbers

- Think back to the river crossing problem. How can we find an acceptable (and smallest possible) choice of objects in the boat in terms of a vertex cover and the graph you described in section one? Explain your answer.

- This choice of vertices gives us a set of vertices in the boat (let's call it $B_1$), and a set of vertices on the (let's say) right river bank (where every object was initially placed) (let's call this $R_1$). There are no objects on the left river bank (let's call this set of vertices $L_1$, and in this case, $L_1$ is the empty set).

- The boat now transports the set $B_1$ to the left river bank. We have to construct a new graph in order to decide which objects we can leave at the left bank. Once we place these new objects in the boat, we have new sets $B_2$, $L_2$, and $R_2$.

- How does $R_2$ relate to $R_1$?

- How many edges are in the graph for the vertices in $R_1$?

- If we continue this process, how will $L_2$ relate to $L_3$?

- How does $R_n$ relate to $R_{n-1}$ if $n$ is even?

- How does $L_n$ relate to $L_{n+1}$, if $n$ is even?

- Let $G$ be the graph representing our initial situation (where all objects are on the right river bank), and let $H$ be the graph representing the situation at any step $k$. How does $\tau(H)$ compare to $\tau(G)$?

- If there is a solution to the river crossing problem, then this process will end after $k$ steps for some $k$, and our set $L_k$ will contain all $n$ objects. The minimum number of extra seats in our boat needed, called the **Alcuin number**, will be the maximum of the vertex cover numbers for every graph we drew.

- We will denote the Alcuin number for a given initial graph $G$ as $A(G)$.

- How many extra seats do we need in our boat to guarantee that we can transport any set of $n$ objects with any set of conflicts?

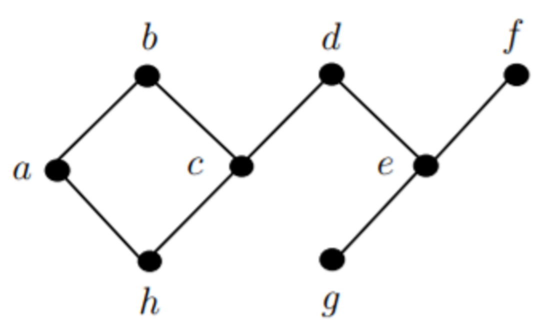
- We will now find an upper and lower bound for $A(G)$.
- How does $A(G)$ compare to $\tau(G)$ (Hint: what does $\tau(G)$ represent in our scenario?)?
- Explain why $A(G) \leq \tau(G) + 1$ by finding a scenario where you can transport all of your objects with $\tau(G) + 1$ seats in the boat.
- This tells us that, given a graph $G$, $A(G)$ only has two possible values. What are they?

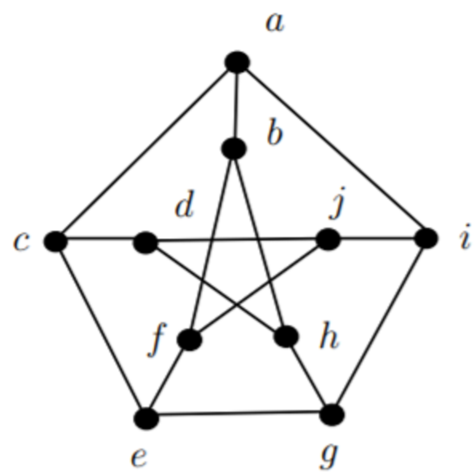
### 1.4 Solving the problem with graph theory

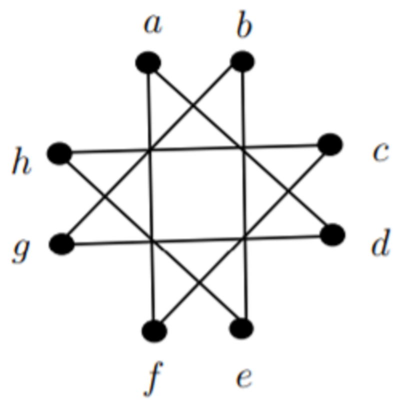
- Solve the problems listed at the beginning of this project using graphs and vertex covers.
- Add in new objects and conflicts, and solve these problems using graphs and vertex covers.

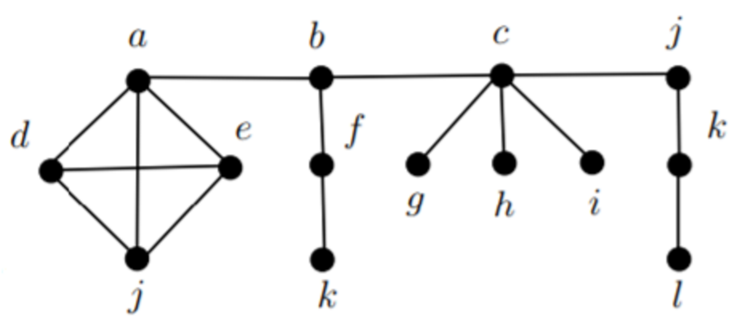
## 1.5 Vertex cover versus dominating set

- One of the other groups is working on a project involving dominating sets, which are closely related to vertex covers.
- A **dominating set** of a graph $G$ is a set of vertices $S$ such that each vertex of $G$ is either in $S$ or adjacent to a vertex in $S$.
- What is the biggest dominating set for any graph $G$?
- The **dominating number** of a graph, denoted $\gamma(G)$, is the smallest number of vertices in any dominating set of a graph $G$.
- Recall that the complete graph on $n$ vertices, $K_n$, is the graph with $n$ vertices such that each vertex is adjacent to every other vertex (they are all connected by an edge). What is $\gamma(K_n)$?
- Exercise: Find $\gamma(G)$ and $\tau(G)$ for the following graphs ([Smi88]). Look for and discuss patterns that might help come up with algorithms for finding dominating sets and vertex covers. For example, does it help to redraw certain graphs? To break them up into pieces? Make sure you are able to justify why there cannot be a smaller dominating set or vertex cover.






- Come up with a graph and a dominating set for that graph which is not a vertex cover.

- Come up with a graph with a dominating set that is also a vertex cover.

- What is the relationship between the dominating number and the vertex cover number for a connected graph? Justify your claim.

# References

[Smi88] Dayna Brown Smithers, *Graph Theory for the Secondary School Classroom*, Electronic Theses and Dissertations, Paper 1015 (2005).