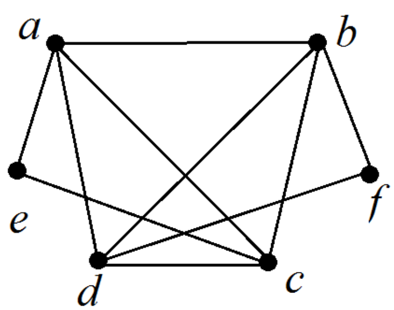
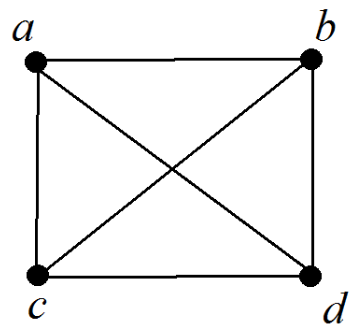
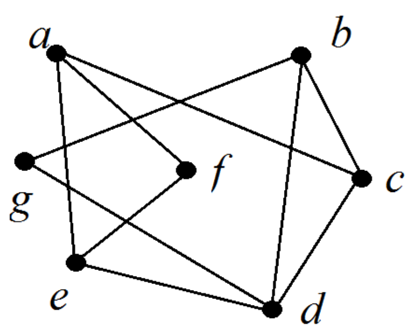
1 Planar Graphs, Euler's Formula, and Brussels Sprouts

1.1 Planarity and the circle-chord method

- A graph is called **planar** if it can be drawn in the plane (on a piece of paper) without the edges crossing.
- We call the graph drawn without edges crossing a **plane graph**.
- Application: When designing electronic circuits, multiple layers are often needed to ensure that the wires don't cross. But, since space is an issue, a major problem in circuit design is breaking down a large circuit into a smaller number of planar circuits.
- Show that the following graphs are planar:

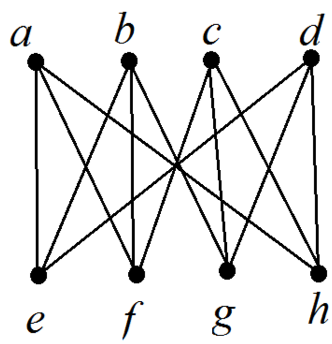






- In order to show that a graph is planar, you simply need to redraw it so that no edges cross. It is harder to show that a graph is **nonplanar**. Just because we can't think of any ways to draw the graph without edges crossing doesn't necessarily mean there isn't a way.

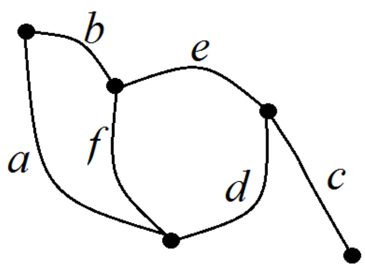
- The **circle-chord method** is a construction that either redraws a graph to show that it is planar or shows that it is not planar.
- The construction is as follows:
  1. First, find a cycle that contains all the vertices of the graph. Draw this cycle as a large circle. (Note: this step is not always possible, and it is sometimes very difficult.)
  2. Choose an edge that is not in the circle. Then draw it outside the circle.
  3. This edge forces other edges to be drawn inside the circle. Draw these.
  4. The edges drawn inside now force other edges to be drawn outside. Draw these.
  5. Eventually you will get a planar graph (showing that your initial graph was planar), or you will come to a point where you cannot draw an edge without crossing another (showing that the graph is nonplanar).
- Use the circle-chord method to determine if the following graph is planar:



1.2 Euler's formula

- Notice that in a planar graph, the edges sometimes enclose areas called **faces**.
- Sometimes the edges do not enclose an area, and the space is part of the unbounded region on the page that surrounds the graph. We will also call this region a face.
- You might be interested in knowing if there is some relationship between the number of vertices, edges, and faces in a graph. First we must see what happens to these numbers when we shrink an edge in a graph.

- Let's start with the planar graph below. Suppose we want to shrink an edge until it completely disappears and its endpoints become one vertex. We assume the edges can curve, so that when an edge shrinks, it doesn't cause any of the other edges to fuse together. Let's shrink the edge d. Draw the resulting graph.



- Notice that this graph has something called **parallel edges**, where there are two distinct edges that have the same endpoints. Parallel edges always enclose a face.

- What happened to the number of vertices? Edges? Faces?

- Now let's shrink edge c. Draw the resulting graph.

- What happened to the number of vertices? Edges? Faces?

- Think more abstractly. What will happen to the number of vertices, edges, and faces of any connected planar graph if we shrink any edge of the graph? (Note: you will have to account for two cases: the case where we shrink an edge that is a loop (an edge where the two endpoints are the same vertex) and the case where we shrink an edge that is not a loop.)

- Let's call the number of vertices V, the number of edges E, and the number of faces F. What happens to the quantity $V - E + F$ when we shrink an edge?

- Suppose we start with a planar graph, and we shrink the edges one by one until we are left with just a single vertex. What is $V - E + F$?

- Has $V - E + F$ changed at all in this process?

- You have just proven **Euler's formula for planar graphs:**

  A connected planar graph with V vertices, E edges, and F faces satisfies

  $$V - E + F = 2.$$

- Since Euler's formula only applies to planar graphs, we can use it to show that a graph is not planar.

- Recall that the complete graph on n vertices, denoted K_n, is the graph on n vertices such that each vertex is adjacent to every other vertex.

- There is also a graph called the **complete bipartite graph**, denoted $K_{m,n}$, with the property that the vertices can be split into two groups, U (with m vertices) and V (with n vertices), such that there are no edges joining vertices in U to vertices in U, no edges joining vertices in V to vertices in V, and exactly one edge joining each vertex in U to each vertex in V.

- Draw $K_{3,2}$ and $K_{3,3}$.

- Is $K_{3,2}$ a planar graph? Do you think $K_{3,3}$ is a planar graph?

- Are K_1, K_2, K_3, and K_4 planar graphs?

- Is $K_{m,1}$ planar for any positive integer m?

- Show that $K_{m,2}$ is planar for any positive integer m by drawing a plane graph.

- Draw K_5. Do you think it is planar?

- Regardless of whether or not you believe K_5 is planar, we can figure out the answer using Euler's formula.
- Let's begin by assuming that K_5 is in fact planar. Then we would be able to draw it in the plane (on a piece of paper) so that no edges cross.
- What are E and V for K_5? Since we are assuming that K_5 is planar, what does Euler's formula tell us the number of faces of this graph will be?
- How many faces does each edge border in a planar graph?
- Write a formula for the average number of sides on all faces in our graph–call it p–using E and F.
- Does K_5 have any loops? Any parallel edges? Are there any faces bounded by one edge? By two edges?
- This tells us that $p \geq k$ for some integer k. What is k?
- Plug your formula for p into this inequality.
- Plug in the actual values for E and F. Do you arrive at a contradiction?

- What does this tell you about the planarity of K_5?

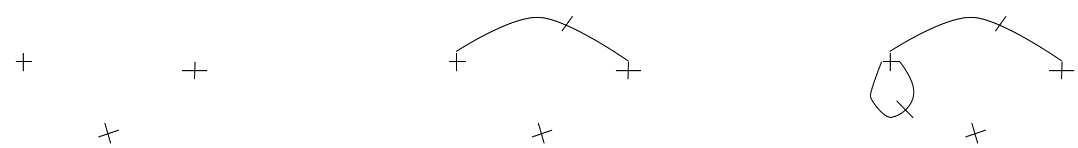
- You can follow the steps of this proof with $K_{3,3}$ to show that this graph is nonplanar. (Try this if you have time at the end.) A key difference in this proof is that since $K_{3,3}$ is bipartite, any circuit must have an even number of edges, so there can't even be any faces bounded by three edges.
- It turns out that K_5 and $K_{3,3}$ are in a sense the only ways that a graph can be nonplanar. We have the following theorem (proof omitted):
- **Kuratowski's Theorem:** A graph is planar if and only if it does not contain K_5 or $K_{3,3}$ as a subgraph.
- Discuss shortcuts for determining if a graph contains K_5 or $K_{3,3}$ as a subgraph (or if it does not).

- Using Kuratowski's Theorem, what can you say about the planarity of K_m for $m \geq 5$ and $K_{m,n}$ for $m, n \geq 3$?

1.3 Brussels Sprouts

- We'll apply our knowledge of Euler's formula to a game called sprouts.
- Here are the rules:
  1. Draw any number of dots on a sheet of paper.
  2. Player one either draws a curve between any two dots or a loop at any dot.
  3. Player one then draws a new dot somewhere along that curve or loop.
  4. Player two then follows steps 2 and 3.
  5. Play alternates until one player can't take a turn.

- The only other rules are that a curve cannot pass through an existing dot, and a dot cannot have more than three lines meeting it.
- The winner is the last player who is able to take a turn.
- Later, a variation of the game, humorously titled "Brussels sprouts," was invented. Here are the rules:
  1. Draw any number of plus signs on a sheet of paper.
  2. Player one connects any of the two free arms with a curve and then makes another cross on that curve.
  3. Player two then takes a turn.
  4. Play continues until one player can't take a turn.
- Here is an example of how the game can start:



- Play a round or two to get the hang of the game.
- Let's investigate this game further.
- If we begin the game with n crosses, how many free arms are there?
- How does the number of free arms change after each turn?
- What is the minimum number of free arms that can point inside a face after any given turn?

- Using the answer to the last question, what is the maximum number of faces that our graph can have after any given turn?

- Suppose we have fewer than the maximum possible number of faces. What must be true about at least one of the faces in your graph, in terms of the number of free arms pointing inside that face?

- If the above scenario occurs after a turn, can the game be over?

- How many faces must there be in our final graph, when the game is over?

- Let's consider the crosses to be vertices, and let's say the game ends after m turns. How many vertices are in our final graph?

- How many edges are in our final graph?

- Now use Euler's formula to solve for m.

- Notice that once you fix the number of crosses at the beginning, you know how many turns the game will take until it ends. Suppose you give your friend the choice of either choosing the number of crosses or choosing who goes first (and you choose the other). Can you always guarantee that you will win?