1 Kirchoff's Matrix-Tree Theorem

1.1 The Laplacian matrix

- An $m \times n$ **matrix** is a set of m times n elements that are indexed in an array such that the element numbered (i, j), denoted $a_{i,j}$, is in the i^{th} row and j^{th} column of the array.

$$\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & a_{m,3} & \cdots & a_{m,n} \end{bmatrix}$$

- In the following matrix, what is $a_{2,3}$? What is $a_{3,2}$? What are the dimensions of this matrix?

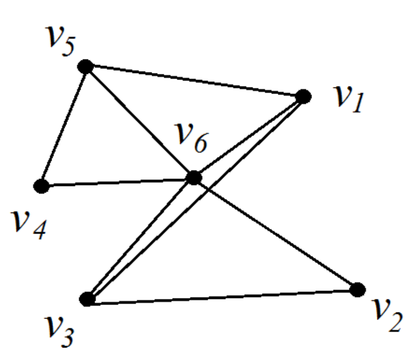
$$\begin{bmatrix} 3 & 4 & 8 & 7 \\ 1 & 2 & 5 & 6 \\ 9 & 10 & -1 & 2 \end{bmatrix}$$

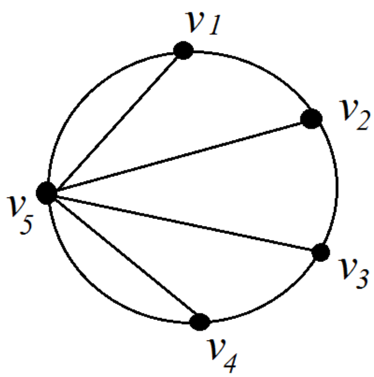
- Let G be a graph with vertices $\{v_1, v_2, \ldots, v_n\}$. Associated to G is a matrix called the **Laplacian**. The entries $L_{i,j}$ of the Laplacian are defined as follows:

$$L_{i,j} = \begin{cases} deg(v_i) & \text{if } i = j \\ -1 & \text{if } i \neq j \text{ and } (v_i, v_j) \text{ is an edge of } G \\ 0 & \text{otherwise} \end{cases}$$

  - Here $deg(v_i)$ is the degree of vertex v_i, and (v_i, v_j) is the edge that connects vertices v_i and v_j.

- Find the Laplacians of the following graphs (it is helpful to label the rows and columns of your matrix with the vertices $v_1, \ldots, v_n$):





- Let D be the diagonal matrix (nonzero entries only on the main diagonal, from top left to bottom right) where $D_{i,i} = deg(v_i)$.
- Let A be the matrix whose entries are defined as follows:

$$A_{i,j} = \begin{cases} 1 & \text{if } (v_i, v_j) \text{ is an edge of } G \\ 0 & \text{otherwise.} \end{cases}$$

  - This is called the **adjacency matrix**.
- There is another way of defining the Laplacian, L, in terms of A and D. What is it? (If you need help, try writing down D and A for the graphs in the example above.)
- We say a matrix is **symmetric** if the entries are symmetric across the main diagonal. Rephrase this by stating when entry $a_{i,j}$ is equal to $a_{k,l}$. Is the Laplacian symmetric? Why or why not?
- By turning all of the rows of a matrix A into columns (and thus the columns become rows), we obtain a matrix called the **transpose** of A, denoted A^T. Row 1 of A is column 1 of A^T, row 2 of A is column 2 of A^T, and so on.
- If A is a symmetric matrix, then what is A^T equal to?

1.2 The determinant of a matrix

- For any $n \times n$ matrix A, there is a numerical value associated to A called the **determinant**, denoted $det(A)$ or $|A|$.
- The determinant of a 2×2 matrix can be computed as follows:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

- Calculate the determinant of

$$A = \begin{bmatrix} 1 & 4 \\ 6 & 2 \end{bmatrix}$$

- The determinant of a 3×3 matrix can be computed as follows:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \times \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \times \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \times \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

- Notice that the formula above is the alternating sum of the entries of the first row times the determinants of the 2×2 submatrices that are formed when we block out the row and column that contain the given element of the first row.
- The formula for the determinant of a 4×4 matrix follows the same pattern:

$$\begin{vmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{vmatrix} = a \times \begin{vmatrix} f & g & h \\ j & k & l \\ n & o & p \end{vmatrix} - b \times \begin{vmatrix} e & g & h \\ i & k & l \\ m & o & p \end{vmatrix} + c \times \begin{vmatrix} e & f & h \\ i & j & l \\ m & n & p \end{vmatrix} - d \times \begin{vmatrix} e & f & g \\ i & j & k \\ m & n & o \end{vmatrix}$$

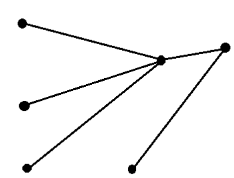
- Compute the determinant of the following matrix:

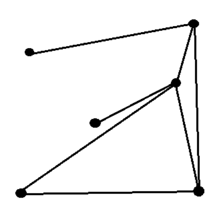
$$\begin{bmatrix} 1 & 2 & 4 \\ 5 & 3 & 8 \\ 1 & 1 & 3 \end{bmatrix}$$

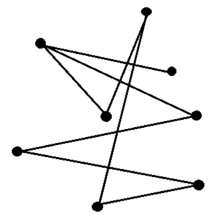
- We say that the rows $r_1, \ldots, r_n$ of a matrix are **linearly dependent** if there exist real numbers $c_1, \ldots, c_n$ such that $c_1 r_1 + \cdots + c_n r_n = 0$, and not all of the c_i are zero. The definition is the same for columns.
- Here are some useful properties of the determinant:
  1. $\det(A^T)$=det(A)
  2. $\det(AB)$=$\det(A)\det(B)$
  3. For an $n \times n$ matrix A, $\det(cA)$=$c^n\det(A)$
  4. If the rows or columns of A are linearly dependent, then $\det(A)$=0.

1.3 Spanning trees

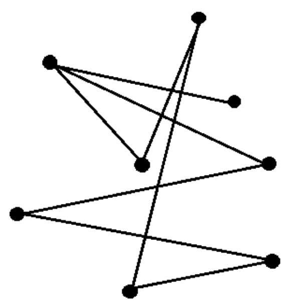
- Remember:
  - a walk is a set of distinct edges $e_1 e_2 \cdots e_k$ that we can trace in the graph without picking up our pencils
  - a path is a walk that does not go through any vertex twice
  - a cycle is a path that starts and ends at the same vertex
- We say that a graph G is **connected** if there is a walk in G between each pair of vertices.
- A **tree** is a connected graph with no cycles.
  - Example: Are the following graphs trees? Why or why not?







- A **spanning tree** of a graph G is a tree that is a subgraph of G (every edge of the subgraph is an edge of G) and includes every vertex of G.
  - Example: Find all 7 spanning trees of the following graph.



1.4 Other types of graphs

- A **directed graph** is a graph where each edge is oriented toward one of its endpoints (we can think of this as a graph with arrows for edges).
- A **simple graph** is a graph with no loops (a loop is a single edge whose endpoints are the same vertex), and for every pair of vertices, there is at most one edge with those vertices as endpoints.

1.5 The statement of the theorem

- Amazingly, we can determine the number of spanning trees of a graph G by looking at the determinant of a graph related to the Laplacian.
- **Theorem** [see Bona 02]: Let G be a directed graph without loops, and let A be the adjacency (or incidency) matrix of G. Remove any row from A, and let A_0 be the remaining matrix. Then the number of spanning trees of G is $\det(A_0 A_0^T)$.
- As a corollary, we have the Matrix-Tree Theorem:
- **The Matrix-Tree Theorem** [see Bona 02]: Let U be a simple undirected graph. Let L be the Laplacian matrix of U. Delete the last row and column of L, and call this L_0. Then the number of spanning trees of U is $\det(L_0)$.
- You should have the tools you need to read through section 10.4 of [Bona 02] and understand the proofs. This may be more of a challenge, so you might want to save it for the end. The examples below do not require a knowledge of the proofs.

1.6 Examples

- Recall that the complete graph on n vertices, denoted K_n, is the graph on n vertices such that each vertex is adjacent to every other vertex.
- There is also a graph called the **complete bipartite graph**, denoted $K_{m,n}$, with the property that the vertices can be split into two groups, U (with m vertices) and V (with n vertices), such that there are no edges joining vertices in U to vertices in U, no edges joining vertices in V to vertices in V, and exactly one edge joining each vertex in U to each vertex in V.
- Draw $K_{3,2}$ and $K_{3,3}$.

- Calculate the number of spanning trees of K_n, the complete graph on n vertices (this is the graph where each vertex is adjacent to every other vertex):
  - What is the Laplacian for K_n?
  - Delete the last row and column to get L_0.
  - Add each of the rows of L_0 to the first row of L_0.
  - Now add the first row to each of the other rows of L_0.
  - What is the determinant of this matrix?

  - A property of the determinant is that when you add a row of A to another row of A, $\det(A)$ is unchanged. So, how many spanning trees does K_n have?

- Compute the number of spanning trees of $K_{m,n}$:
  - What is L_0 for $K_{m,n}$? (You may want to do some examples first, to see the pattern.)
  - Add all of the rows of L_0 to the first row.
  - Add the first row to all other rows.

  - This should allow you to calculate $\det(L_0)$. How many spanning trees does $K_{m,n}$ have?

- Create your own graphs, and calculate the number of spanning trees of each.