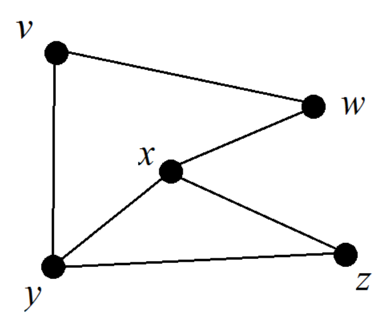
# 1 Domination and The Five Queens Problem

What is the smallest number of queens that we can place on a chessboard so that every space on the board is either occupied by a queen or being attacked by at least one queen? How can you use graph theory to find a suitable arrangement of queens on an $n \times n$ chessboard? To find a solution to this problem, you will learn about the dominating set of a graph.

## 1.1 Domination

- A **dominating set** of a graph $G$ is a set of vertices $S$ such that each vertex of $G$ is either in $S$ or adjacent to a vertex in $S$.
- What is the *biggest* dominating set of the graph below?



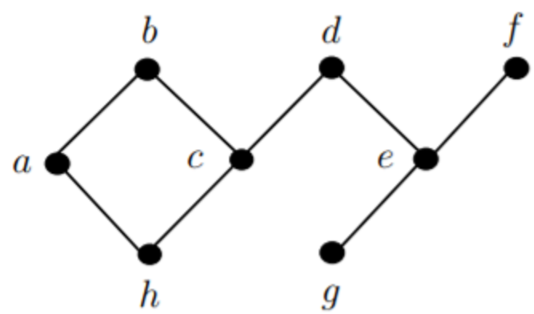

- What is the biggest dominating set for any graph $G$?
- Can you find a dominating set with three vertices? Two vertices? One? Give an example, or explain why not.

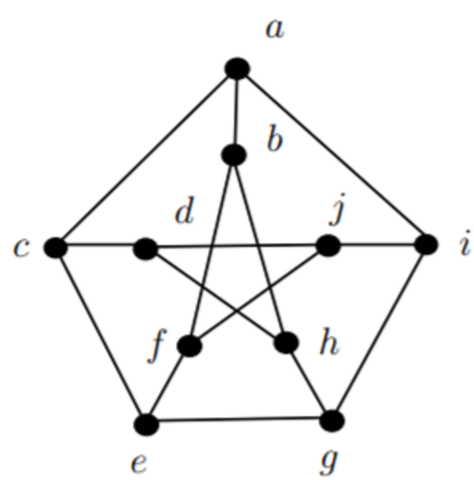
## 1.2 The domination number

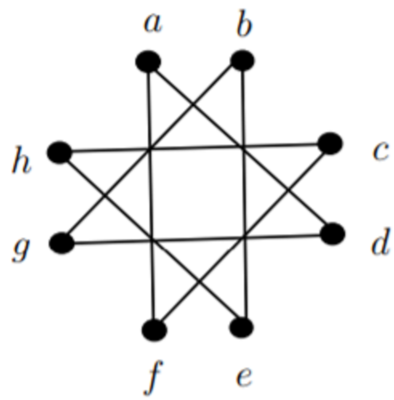
- The **dominating number** of a graph, denoted $\gamma(G)$, is the smallest number of vertices in any dominating set of a graph $G$.
- What is $\gamma(G)$ for the above graph?

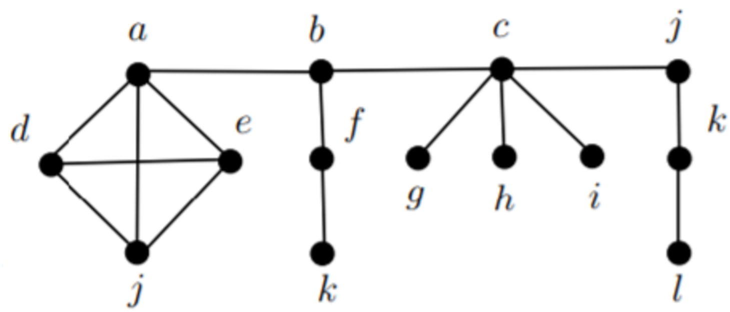
- Recall that the complete graph on $n$ vertices, $K_n$, is the graph with $n$ vertices such that each vertex is adjacent to every other vertex (they are all connected by an edge). What is $\gamma(K_n)$?

- Exercise: Find $\gamma(G)$ for the following graphs ([Smi88]). Look for and discuss patterns that might help come up with algorithms for finding dominating sets. For example, does it help to redraw certain graphs? To break them up into pieces? Make sure you are able to justify why there cannot be a smaller dominating set.






### 1.3 The Five Queens Problem

- In case you're unfamiliar with chess, a chessboard is an $8 \times 8$ board with 64 squares, and a queen can move any number of squares vertically, horizontally, or diagonally in a single move.
- In 1862, de Jaenisch posed the problem of finding the minimum number of queens on a chessboard so that each square was either occupied or under attack by a queen (a square is under attack by a queen if the queen can land on that space in one move).
- We can generalize this problem to any $n \times n$ chessboard.
- If we want to represent this situation using a graph, what are our objects (vertices) and our connections (edges)?

- How can we restate this problem in terms of graph theory?

- Let's start by considering a $1 \times 1$ chessboard. What is the minimum number of queens to satisfy our problem (we'll call the Queen's graph for the $n \times n$ chessboard $Q_n$, so we are looking for $\gamma(Q_1)$)?

- What is $\gamma(Q_2)$? (First draw the chessboard, and then $Q_2$, if it helps.)

- What is $\gamma(Q_3)$?

- What is $\gamma(Q_4)$? (Hint: notice how $Q_3$ sits inside of $Q_4$, i.e., what do we have to add to the $3 \times 3$ chessboard to get the $4 \times 4$ chessboard? Using the fact that you now know $\gamma(Q_3)$, what can you say about an upper bound for $\gamma(Q_4)$? Finally, is there a choice of $\gamma(Q_3)$ queens that would work here? Why or why not?)

- Using the method of the last question, what can you say about an upper bound for $\gamma(Q_{n+1})$ in terms of $\gamma(Q_n)$?

- It has been shown that $\gamma(Q_8) = 5$. Can you find a corresponding dominating set (i.e., a placement of 5 queens on an $8 \times 8$ chessboard that satisfy the problem)? (Note: it is probably too difficult to solve this using $Q_8$. Draw the $8 \times 8$ chessboard and try different placements of queens. This is to show that merely looking at the graph doesn't always make the problem easier, so we would want to develop the theory further.)

- Many of the $\gamma(Q_n)$ are calculated in computer programs, but there is no formula known for any $n$. It is even an open problem to determine if $\gamma(Q_n) \leq \gamma(Q_{n+1})$ for all $n$. Using graph theory may be the key to proving or disproving this claim.

## 1.4 The Kings Graph

- The solution to the Queens Problem for any $n$ is unsolved (for now), but there is a lot more we can say about the Kings Problem.

- A king is able to move one space vertically, horizontally, or diagonally.

- Draw the King's graph, $K(n)$, for $n = 1, 2, 3, 4$ and 5.

- What do you notice about $K(2)$ (which special graph is it, besides $Q_2$)?

- How does $K(3)$ compare to $K(2)$?

- How does $K(n)$ for any $n$ compare to $K(2)$?

- Use this last idea to state (and prove) how many edges are in $K(n)$ for any $n$.

- What is $\gamma(K(2))$?

- What is $\gamma(K(3))$?

- How does $K(5)$ compare to $K(3)$?

- Can you use this fact to say something about $\gamma(K(5))$?

- How do you think $K(7)$ will compare to $K(3)$?

- Using that last idea, what is $\gamma(K(7))$?

- Thinking along these lines, give a formula for $\gamma(K(2n+1))$ for any $n$.

## References

[Smi88] Dayna Brown Smithers, *Graph Theory for the Secondary School Classroom*, Electronic Theses and Dissertations, Paper 1015 (2005).