Monitoring in Dynamic Financial Contracts*

ARASH FAHIM SIMON GERVAIS R. VIJAY KRISHNA

December 24, 2023

Abstract

A principal hires an agent to run a firm. The principal determines not only the compensation and termination time of the agent, but also the *monitoring intensity*. The optimal contract treats pay incentives and monitoring as substitutes. Interestingly, the relationship between firm performance and monitoring intensity is non-monotonic: monitoring intensity increases (decreases) after periods of good performance for firms with low (high) financial slack. When the contract is implemented with financial claims, we show that more productive firms or those with less severe agency problems feature higher stock prices, lower credit yield spreads and, crucially, greater levels of monitoring.

*We thank Inga Deimen, Tristan Gagnon-Bartsch, Glenn Harrison, Oksana Leukhina, Heikki Rantakari, Vasiliki Skreta, Ming Yang, and various seminar audiences for helpful comments. Fahim is at Florida State University, <arashfahim@gmail.com>; Gervais is at Duke University, <sgervais@duke.edu>; Krishna is at Florida State University <rvk3570@gmail.com>.

1. Introduction

Firms hire managers to run them. However, as noted by Adam Smith (1776) and by Berle and Means (1930), any such relationship is beset by a problem as old as time: managers can take actions that benefit them at a cost to the owners of the firm.[1] The owners of the firm can mitigate the *agency problem* by realigning the agent's incentives with their own through well-structured compensation and effective monitoring of the agent. Private equity (PE) investors, for instance, ensure that the management of the firms they invest in is properly incentivised (usually by giving them a sufficiently large equity stake).

Before we proceed, it is important to distinguish between two possible notions of monitoring: *retrospective* and *prospective*. *Retrospective monitoring* involves *auditing* past performance and specifies punitive consequences in the face of malfeasance unearthed.[2,3] *Prospective monitoring*, on the other hand, entails *oversight* that places new barriers to malfeasance; it is purely prophylactic, and imposes no further costs on the agent at all, save for making malfeasance more difficult.[4] In this paper, we shall focus exclusively on prospective monitoring (henceforth, simply 'monitoring'), because it aligns more closely with our conception of governance as a set of preventative measures, guardrails so to speak.

Early work aimed at analysing the agency problem has focused almost exclusively on the design of *incentives*; for example Holmström (1979) shows that a contractual arrangement that links the compensation of the agent to the performance of the project helps mitigate the losses that result from this long-standing problem. However, as emphasised by Milgrom and Roberts (1992, Chapter 7), the optimal contract should *jointly* determine both the sensitivity of the agent's compensation to output (ie, the incentive structure) *and* the firm's *monitoring structure* which, broadly speaking, includes all the mechanisms that are implemented as part of the firm's effort to control the extent of agency problems (Shleifer and Vishny, 1997). The owners of firms, through their choice

[1] Smith (1776, Book 5, Chapter 1, Part 3) writes: "The directors of such companies . . . , being the managers rather of other people's money than of their own, it cannot well be expected that they should watch over it with the same anxious vigilance with which the partners in a private company frequently watch over their own . . . . Negligence and profusion, therefore, must always prevail, more or less, in the management and affairs of such a company."

[2] Malfeasance comes in many flavours and encompasses misuse, mismanagement, misallocation, and misappropriation of firm resources.

[3] Auditing is not always the deterrent one would hope for. Larcker and Tayan (2011) describe the case of Richard Scrushy, CEO of HealthSouth Corp.: "Scrushy and other corporate officers were accused of overstating earnings by $1.4 billion and . . . Scrushy, who received backdated stock, sold back shares . . . [ahead of] regulatory changes that caused the share price to plummet. . . . However, Scrushy was not convicted of account manipulation in a criminal trial; instead he was ordered to pay $2.9 billion in a civil suit", of which he has repaid HealthSouth less than $100 million. The litigation is still ongoing.

[4] Indeed, as we will see below, the agent is completely indifferent to the intensity of monitoring. Monitoring only serves to reduce the reliance on incentives in compensation.

of organisational structure, workplace environment, reporting requirements, and other methods of oversight, can limit their employees' opportunities to act in their own private interest at the expense of their shareholders.[5] We refer to these as the firm's *governance structure*, and use the terms 'monitoring' and 'governance' interchangeably.

The implementation, operation, and management of these governance mechanisms is costly, and such costs must be balanced against those of compensation incentives. In particular, firms *dynamically* trade off their reliance on compensation incentives versus governance mechanisms in their effort to most efficiently ameliorate the agency problem. We analyse this problem by considering a dynamic contracting model in which the agent's compensation structure as well as the firm's corporate governance structure are jointly determined at the optimum, and vary over time as functions of past performance.

Specifically, we consider a continuous-time dynamic contracting environment where a risk-neutral agent manages a risky productive technology. The agent is resource-constrained, and relies on investors (the *principal*) to absorb running losses, should they occur. The cash flow stream produced by the technology is noisy over time, where the noise process is modelled as a Brownian motion. The friction, ie, the source of discrepancy between the two parties' incentives, is that the agent can *unobservably* divert cash flows from the firm for his own private benefit, as in DeMarzo and Sannikov (2006), henceforth DS. The innovation in our model is that the principal can continuously adjust the intensity of monitoring by making it more difficult for the agent to misappropriate the firm's cash flow: by investing in better (and costly) monitoring, the principal can make a more precise attribution of good or bad shocks to chance, which reduces her reliance on early termination and high-powered incentives. This mitigates the extent and scope of the agency problem, and extends the expected life of the firm.

Our contracts can be written recursively in terms of the firm's *financial slack*, which essentially measures the size of negative shocks the firm can withstand before being liquidated or, more succinctly, the "distance to termination." This allows us to characterise the optimal contract and, in the process, establish a number of new results about the firm's optimal monitoring policy. First, as in DS, the optimal contract (Theorem 1 and Proposition 5.1) features payments to the agent when the firm's financial slack is sufficiently high (ie, compensation is backloaded), and termination when it reaches zero. Unique to the present paper, the optimal contract also features a *monitoring schedule* where monitoring intensity and pay-performance sensitivity (which measures how much the value of compensation changes after a random shock to output) are substitutes.[6]

Our second main contribution (Theorem 2) is in characterising the structure of

[5]A recent instance of such concerns is the desire of firms to see workers return to the office post-pandemic; cf, "Bosses Still Aren't Sure Remote Workers Have 'Hustle'," *The Wall Street Journal*, May 23, 2021.

[6]Bengtsson and Ravid (2015) find that in Venture Capital (VC) backed firms, contracts are relatively more high-powered when monitoring is costlier (and hence lower) due to geographical distance.

monitoring intensity, which is non-monotonic in the firm's financial slack; instead, it is single-peaked, first increasing in financial slack and then decreasing to the point where monitoring is minimal when financial slack is high enough for the agent to be paid. Indeed, the firm's monitoring intensity is increasing in the firm's induced *risk aversion*. Because monitoring reduces the uncertainty about the agent's actions, it is more profitable when risk aversion is higher; thus, monitoring intensity is monotone in the firm's risk aversion.

To understand the source of the firm's risk aversion, notice that even though both principal and agent are risk neutral, the agent is protected by limited liability. Because the optimal contract features termination when financial slack becomes zero, this induces the firm to behave as if it were risk averse. Such risk aversion induced by so-called "bankruptcy" or "shut-down" constraints is classically known (Masson, 1972). However, the novel aspect of our setting is that the degree and structure of risk aversion (as a function of financial slack) are *endogenously and optimally* chosen by the principal.

Our continuous-time formulation allows us to identify the two competing forces shaping the firm's risk aversion. On the one hand, as the firm's financial slack increases from its liquidation level (zero), the expected time to liquidation increases, which reduces the risk of liquidation and hence decreases risk aversion. On the other hand, as financial slack increases from zero, the firm's value first increases (because liquidation is less likely, all else being fixed), but then decreases as financial slack becomes high enough that payments are due to the agent. Together, these forces result in the firm's risk aversion being single-peaked in financial slack.

An implication of this structure is that if a firm that had performed well in the past suddenly experiences sustained poor performance, risk aversion (and hence the intensity of monitoring) increases. This is in line with behaviour in VC-backed firms (Kaplan and Strömberg, 2003), where VCs increase oversight and take over some of the decision-making themselves, and therefore gain a more accurate picture of the agent's actions. It also conforms with Vafeas (1999), who finds that in publicly traded firms, the frequency of board meetings increases with such a drop in performance. In contrast, if the firm's bad performance brings it close to termination, its small value can then reduce risk aversion and monitoring; effectively, because the firm has less to protect, it cares less about the agent's potential improprieties. Indeed, a prediction of the financial security implementation of our model is that a firm's stock return will positively correlate with the addition of tighter governance mechanisms when the firm's financial slack is small, and vice versa when the firm's financial slack is high.

Our model also yields rich comparative statics. For example, we show (Theorem 3) that firms with a *greater expected cash flow* (in the absence of any diversion by the agent) or that have *less severe agency problems* (eg, the benefit of misappropriating a dollar of output is lower for the agent) have greater monitoring for every level of financial slack.

Intuitively, when expected cash flow is greater, the marginal returns from monitoring at any level of financial slack are also greater.

In addition, we show that the optimal contract can be implemented using financial securities (stocks, bonds, and cash reserves), as in Biais et al. (2007) (henceforth BMPR). The optimal contract induces the following capital structure: The entrepreneur and the investor each hold some fraction of the equity. The firm issues bonds (as debt) to investors, and makes coupon payments that are proportional to the cash reserves (which represents the financial slack). When cash reserves become sufficiently large, dividends are paid; the payment is such that cash reserves never go above a certain threshold. The equilibrium stock price is a monotone function of the cash reserves and agent compensation is back-loaded and increasing in firm performance; this is consistent with the empirical findings of DeAngelo, DeAngelo, and Stulz (2006) and Kaplan and Rauh (2010). In particular, we find (Theorem 4) that firms with greater expected cash flow or those with less severe agency problems have higher stock and bond prices, lower credit yield spread, and crucially, greater monitoring.

The empirical literature investigating corporate governance has often treated governance as a fixed (but endogenous) firm characteristic. For example, the seminal studies by Gompers, Ishii, and Metrick (2003) and Bebchuk, Cohen, and Ferrell (2009) use measures of corporate governance that do not vary (much, if at all) over time for each firm. As such, their cross-sectional finding that various firm performance measures, such as stock returns, credit yield spreads, return on investment (ROI), and Tobin's q, are positively correlated with measures of corporate governance potentially suffers from at least two deficiencies. First, as pointed out by Himmelberg, Hubbard, and Palia (1999) and by Core, Guay, and Rusticus (2006), because the relationship between corporate governance and performance is endogenous, it is not clear if it is greater corporate governance that *causes* improved performance or if it is the other way around. In fact, the correlation itself is puzzling because, as argued by Demsetz and Lehn (1985), if governance structures are chosen optimally by firms, there ought not to be any correlation between governance and performance. Second, as shown by Wintoki, Linck, and Netter (2012), a firm's needs for better governance will in general change over time, and the choice of monitoring intensity depends on past performance.

Our paper models firms that can constantly change the control they exert on their agents by modifying the environment in which they operate, thus providing a proper investigation of the role of governance, and one that takes into account both its endogenous as well as dynamic nature. In doing so, we provide a mechanism that rationalises both the observed correlation between governance and various measures of firm performance, as well as the dynamics of the tradeoffs between monitoring and pay-sensitivity.

We also make more general methodological contributions that are critical for the derivation of our results, and are also of independent interest. Using techniques from the theory of viscosity solutions (Crandall, Ishii, and Lions, 1992) and from nonlinear differential equations (in particular Schauder estimates, see Gilbarg and Trudinger, 2001), we show that the principal's value function is twice continuously differentiable (C^2) and solves a variational-HJB equation (Theorem 1). Our techniques allow us to establish smooth pasting and high-contact conditions as a Corollary 5.2. Another methodological contribution is the use of the Comparison Principle (Theorem 5) for viscosity solutions of differential equations as the basis for our comparative statics, thereby complementing the techniques introduced by DS and BMPR.[7]

In what follows, we begin by reviewing the literature, both related to monitoring in practise, as well as theoretically. We then lay out the model in Section 3, while Section 4 characterises the class of incentive compatible contracts. Optimal contracts and the regularity of the principal's value function are derived in Section 5, while Section 6 lays out our main comparative statics results. Section 7 studies an implementation of the optimal contract in terms of cash flow and securities. All proofs are in the Appendix.

2. Related Literature

Empirical Literature on Monitoring. Monitoring is an important factor in explaining the existence of venture capitalists (VCs) (Lerner, 1995).[8] Boards of directors at PE-backed firms actively monitor the performance of management, in addition to providing guidance (Gompers, Kaplan, and Mukharlyamov, 2016, Gompers and Kaplan, 2022, Chapter 7). Boards of directors also play a significant role in monitoring of management in publicly traded corporations, as noted for instance in Weisbach (1988) and Vafeas (1999).[9] Importantly, in both PE-backed firms, as well as in publicly traded corporations, the intensity of monitoring varies over time (eg, sustained poor performance leads to increased oversight from the board of directors).

Venture capital activity is sensitive to distance and travel time. Venture capitalists are more likely to sit on boards of firms that are geographically more proximate (Lerner, 1995), the idea being that managers in more distant firms are harder to monitor. VCs incur costs when they monitor and infuse capital, where the monitoring costs include visits to the plant, analysis of data and the choices of the manager, as well the opportunity cost of other activities foregone (Gompers and Lerner, 2006, Chapter 8). Bengtsson and Ravid (2015) show that venture capital contracts account for such monitoring costs by stipulating

[7] Roughly, DS and BMPR provide comparative statics that show how the principal's value function (among other things) changes with parameters. Our focus is on the first and second derivatives of the value function, the latter determining the firm's risk aversion and intensity of monitoring.

[8] There is, of course, a rich and growing literature on monitoring and governance. Here, we merely mention some important studies that relate compensation, incentives, and monitoring.

[9] Shleifer and Vishny (1997) provide an excellent overview of the literature on corporate governance.

contracts that are more high-powered as geographic distance increases, suggesting that monitoring costs increase with distance, and more importantly, *monitoring and pay-sensitivity are substitutes*. Similarly, Bernstein, Giroud, and Townsend (2016) show that with decreased monitoring costs comes greater VC investment and engagement.

Entrepreneurs who are backed by VCs have backloaded compensation (DeAngelo, DeAngelo, and Stulz, 2006) and also find increased oversight after poor performance. If there is sustained good performance, the entrepreneur regains control and monitoring decreases, as documented by Kaplan and Strömberg (2003). Indeed, Branzoli and Fringuellotti (2022) show that the same is true for firms funded by banks in Italy, whereby banks exercise increased oversight and impose additional restrictions on the use of cash and other firm activities when firms underperform relative to expectations. Similarly, Vafeas (1999) shows that the frequency of board meetings (which is a stand-in for the intensity of monitoring) *increases* when the stock price drops.

These striking patterns regarding monitoring and contract structure in firms can be summarised as: (i) pay-sensitivity and monitoring are substitutes, (ii) payments to the entrepreneur are backloaded, and (iii) more stringent standards and monitoring obtain after sufficiently bad outcomes, which ease up after good performance. Notably, this last feature speaks to the dynamics of monitoring over the life-cycle of the firm.

There is also a literature initiated by Gompers, Ishii, and Metrick (2003) that looks at cross-sections of firms, and establishes many regularities between aspects of governance and various measures of firm performance. However, this strand does not speak to the dynamics and evolution of the intensity of governance over time, and its relation to various contractual elements and security prices.

Theoretical Literature: As mentioned above, our paper builds on the seminal analyses of DS and BMPR, who initiate continuous-time methods in the dynamic contracting analysis of the firm and its capital structure, thereby refining and extending the discrete-time framework of Clementi and Hopenhayn (2006), and DeMarzo and Fishman (2007a,b). We add to this literature by allowing the firm to actively monitor the agent.

In a static model, Demougin and Fluet (2001) investigate the possibility for the principal to add to the information contained in observable output by investing in costly signals about the agent's effort. In a similar vein, and more in line with our approach, Milgrom and Roberts (1992) give the principal the ability to adjust the precision with which output is observed and to adjust the agent's compensation contract accordingly. In Section 5.4 we compare our approach to that of Milgrom and Roberts (1992). A new and related literature studies the general (static) problem of allowing the principal to flexibly choose the information she receives, for instance, Li and Yang (2020) and Georgiadis and Szentes (2020); these papers do not consider dynamic settings.

Parigi, Pelizzon, and von Thadden (2015) consider a multi-stage (but not dynamic) model that allows for governance which recreases cash flow. They also allow for managerial effort which increases output. In all these models, as in ours, the cost that the principal incurs to increase the quality of the feedback she gets from the output serves to reduce agency costs by lowering the agent's information rent, a point that Tirole (2006, p341) also makes. However, in contrast to ours, the static nature of all these models leaves them silent about the joint evolution of contracts, monitoring, and security prices.

Closer still are the dynamic-contracting models of Piskorski and Westerfield (2016), Orlov (2018), Varas, Marinovic, and Skrzypacz (2020), Chen, Sun, and Xiao (2020), and Dai, Wang, and Yang (2021) who all consider continuous-time settings in which the firm can, in addition to observing output, change the extent of the moral hazard problem with its agent. The main difference between these models and ours is the fact their monitoring technology is *retrospective* in that it is about auditing past malfeasance by the agent, as opposed to being *prospective*, ie, setting guardrails before the agent chooses his action. This distinction, first made by Holmström and Tirole (1993) and later amplified in Tirole (2006, p.334), means that the firm, instead of reacting to performance shocks by investigating them, implements an ex ante governance system ensuring that it will understand the source of these shocks as it experiences them. In terms of results, our paper is closest to Piskorski and Westerfield (2016) in that the optimal monitoring structure they find is single-peaked in continuation utility. However, their conception of monitoring is akin to auditing, and entails "catching a lie" at any given instant. In contrast to our work, they do not characterise the evolution of the firm's risk aversion, which is central to setting the firm's optimal governance over time. Our analysis goes beyond theirs (and uses entirely different techniques) because of our comparative statics results, and (via the implementation) for the firm's equity prices and other securities, and their relation to governance.

3. Model

Time is continuous, denoted by $t \in [0, \infty)$. There is a risk-neutral *principal* with deep pockets and a discount rate $r > 0$, who is constrained by time or skill and is therefore unable to run a project. An entrepreneur (*agent*) with no wealth or income runs the project; the agent has discount rate $\gamma > r$, and is protected by limited liability.[10] The principal covers all operating losses. For simplicity, we assume the agent's outside option is 0 and that the project has a liquidation (or scrap) value of 0.

The project produces a cumulative cash flow $Y_t \in \mathbb{R}$, where Y_t is given by $Y_t =$

[10]The assumption that $\gamma > r$ reflects the fact that the intertemporal marginal rate of substitution for a wealth-constrained agent is greater that r. If $\gamma = r$, it is optimal for the principal to postpone consumption to infinity. In particular, an optimal contract no longer exists, although approximately optimal contracts (of the kind described below) exist.

$\mu t + \sigma_t B_t$, where B_t is a standard Brownian motion, $\mu > 0$ is the expected rate of return, and $\sigma_t > 0$ is a process chosen by the principal, that we interpret below.

The cumulative cash flow up to time t, Y_t, is not observable by the principal. Instead, the agent reports the process $\hat{Y} = (\hat{Y}_t)_{t \geqslant 0}$ to the principal; $Y_t - \hat{Y}_t$ is the amount of output diverted by the agent for personal consumption. The benefit (to the agent) of diverting $Y_t - \hat{Y}_t$ (up to time t) is $\lambda(Y_t - \hat{Y}_t)$, where $\lambda \in (0, 1]$, ie, there may be some deadweight loss from the diversion. A larger λ naturally reflects a more severe agency problem, because the agent gets a greater benefit from diverting an additional dollar.

The agent's cumulative compensation is denoted by C_t; limited liability requires this process to be non-decreasing. For simplicity, we assume that the agent cannot save privately.[11]

The principal chooses the monitoring intensity $\sigma(t) \in \Sigma := \{\sigma_{(0)}, \ldots, \sigma_{(n)}\}$ as a function of the history of reports $\{\hat{Y}_s : 0 \leqslant s \leqslant t\}$, where $\sigma_{(i)} > \sigma_{(i+1)}$ for $i = 0, \ldots, n-1$. The principal's choice of monitoring, $\sigma(t)$, entails a running cost per unit of time, given by $\rho : \Sigma \to \mathbb{R}_+$; we denote $\rho(\sigma_{(i)})$ by ρ_i.[12] We assume, without loss of generality, that $\rho_0 = 0$, ie, the least amount of monitoring is costless, and that ρ is decreasing in σ, ie, $\rho_j < \rho_{j+1}$ for all $j = 0, \ldots, n-1$.[13]

We interpret $\sigma(t)$ as determining the *intensity of monitoring* by the principal at time t, with the understanding that a lower $\sigma(t)$ corresponds to greater monitoring. Monitoring makes it more difficult for the agent to benefit from misappropriation, as the lower noise makes it easier for the principal to attribute variations in cash flow to possible cash-flow diversion by the agent. This is plausible, for instance, when cash flow comes from multiple sources (say $Y_t^k = \mu_k t + \sigma_k B_t$, for $k = 1, \ldots, m$), and monitoring amounts to *directly observing* the cash flow from some subset of these sources, thereby reducing overall uncertainty, but where the cost of observing multiple sources of cash flow is increasing in the number of sources observed. This can happen, for example, if the principal could visit a particular field site of the firm more often, or install more devices to keep track of cash-flow at each step of the production and payment process to ensure compliance by the agent. All of these measures are costly, but they all improve the principal's information about agent performance, or more precisely, about the amount of possible misappropriation by the agent. Of course, with this interpretation, output is the residual amount under the control of the agent.

Monitoring in our model corresponds to the firm investing in internal controls and

[11] Savings can be allowed and treated in the same way as in DeMarzo and Sannikov (2006).

[12] We assume for simplicity that Σ is finite. Our results go through unchanged when Σ is an interval bounded away from zero, failing which as long as ρ_0 is sufficiently high, ie, as long as the cost of eliminating the agency problem is sufficiently high.

[13] The model can be extended to accommodate fixed *switching costs* of changing the intensity of monitoring. This does not affect any of our qualitative results.

structures that allow it to more precisely attribute production to its various factors. In a way, this determines the extent to which the firm is able to measure its internal processes and deter future malfeasance or misappropriation.

Alternative Formulation: Following Milgrom and Roberts (1992, Chapter 7), we conceive of monitoring as increasing the attribution of good or bad shocks to chance, while leaving the agent's payoffs from misappropriation unchanged. An alternative to our model of monitoring is one where the principal can affect λ, the private benefit of stealing a dollar, while leaving the accuracy of signals unchanged. That is, monitoring now reduces the agent's benefit from misappropriation. In Appendix F, we show that this view of monitoring, which does not affect the volatility of the output, is nonetheless isomorphic to the model describe above in terms of the optimal contract and its implementation via securities (see Section 7 below). The central point here is that regardless of how one models the specifics of monitoring, it is always concerned with the reduction of agency costs, ie, managerial rents.

We make two remarks about monitoring in our framework.

Remark 3.1. (i) It is worth emphasising that monitoring is voluntary, and is a choice made optimally by the principal. Our model reduces to that of DS or BMPR when monitoring is ruled out by fiat. However, because the principal in our model can always choose *not* to monitor, she can always do at least as well by monitoring as without. Thus, the value function of DS (with volatility $\sigma_{(0)}$) provides a lower bound for the principal's value function. Similarly, the case of costless monitoring (where $\rho(\sigma) = 0$ for all $\sigma \in \Sigma$) is an upper bound for the principal's problem (because the agency rents are lower), and corresponds to the value function analysed in DS, but with the lowest possible volatility.

(ii) We assume that the principal can *commit* to a monitoring schedule. While monitoring schedules are not, much like other non-capital investments, typically part of contracts, boards of directors have reputational concerns when considering how and when to monitor agents. Also, non-performing boards can be replaced (but see Gompers, Ishii, and Metrick (2003) on the relative difficulty of achieveing this across firms).

3.1. Contracts

The principal conditions his actions on *reports* made by the agent. During the operation of the firm, the agent reports the cash flow $\hat{Y}_t$.

A *contract* is a tuple $\Phi = \big(C = (C_t), \tau, \sigma = (\sigma_t)\big)$ that specifies, contingent on the report process $\hat{Y}$, the *cumulative payment* C_t made to the agent up to time t which is a non-decreasing process, the (stochastic) *termination time* τ, as well as the *monitoring intensity* $\sigma_t \in \Sigma$. The contract is contingent on the entire path of reported cash flows

$(\hat{Y}_t)$. Note that any signal observed by the principal is also observed by the agent. The principal offers the agent a contract at time $t = 0$, and fully commits to this contract. The agent can leave the contract at any time to an outside option normalized to 0.

We assume that $\hat{Y}_t$ is continuous, $\hat{Y}_t \leqslant Y_t$ (ie, the agent can never over-report cumulative output), and $Y_t - \hat{Y}_t$ is absolutely continuous with respect to Lebesgue measure.[14] This is reasonable because discontinuous reports or reports processes whose quadratic variation is different from that of (the unobserved) Y_t is certain evidence that the agent is lying, and will be punished immediately.

It is clear that the *only* inefficiency that arises from the agency problem is in the termination of the project, ie, when $\tau < \infty$ with positive probability. Indeed, the first-best, full information solution is to run the project forever (so $\tau = \infty$), while paying the agent whatever he is owed right away, and then paying him nothing more.

3.2. Payoffs and Principal's Problem

Let $\hat{w}(\hat{Y};\Phi)$ be the agent's utility from choosing the reporting strategy $\hat{Y}$ under the contract Φ. His utility when choosing an *optimal* reporting strategy is

$$w(\Phi) = \sup_{\hat{Y}} \hat{w}(\hat{Y};\Phi) = \sup_{\hat{Y}} \mathbf{E}^{\hat{Y},\Phi}\left[\int_0^\tau e^{-\gamma t}\big[\mathrm{d}C_t + \lambda(\mathrm{d}Y_t - \mathrm{d}\hat{Y}_t)\big]\right] \tag{3.1}$$

The contract Φ is *incentive compatible* if $w(\Phi) = \hat{w}(Y,\Phi)$, ie, if *truth-telling* with $\hat{Y} = Y$ is optimal for the agent. The principal's profit, when the agent chooses $\hat{Y}$, is given by

$$\mathbf{E}^{\hat{Y},\Phi}\left[\int_0^\tau e^{-rt}\big[\mathrm{d}\hat{Y}_t - \rho(\sigma_t)\,\mathrm{d}t - \mathrm{d}C_t\big]\right] \tag{3.2}$$

Observe that the choice of monitoring strategy $\sigma = (\sigma_t)$ only affects the volatility of the driving uncertainty, B_t, and hence doesn't affect its expectation. Nonetheless, it matters crucially because it affects the agent's reporting strategy, the payment structure, and the termination time τ, all of which are of central concern to the principal.

Given an initial amount of utility w_0 promised to the agent, the principal's *Contracting Problem* is

$$F(w_0) := \max_{\Phi} \mathbf{E}^{Y,\Phi}\left[\int_0^\tau e^{-rt}\big[\mathrm{d}Y_t - \mathrm{d}C_t - \rho(\sigma_t)\,\mathrm{d}t\big]\right] \tag{3.3}$$

subject to (i) $\Phi = (C,\tau,\sigma)$ being incentive compatible, and (ii) satisfying the *promise-*

[14] Thus, the process $(\hat{Y}_t)$ has the same quadratic variation as (Y_t), namely $\int_0^t \sigma_s^2\,\mathrm{d}s$, and the drift of $\hat{Y}$, like that of Y, is absolutely continuous with respect to Lebesgue measure. These are consequences of Girsanov's Theorem.

keeping constraint $w(\Phi) = \hat{w}(Y; \Phi) = w_0$.

4. Incentive-Compatible Contracts

Consider a contract $\Phi = (C, \tau, \sigma)$ that conditions on the reporting process $\hat{Y}$ by the agent. The agent maximizes his utility given the contract. As noted above, the contract Φ is incentive-compatible if the agent's optimal reporting strategy is being truthful.

It is clear from the Revelation Principle that any contract $\Phi = (C, \tau, \sigma)$ in which the agent decides to use a reporting strategy $\hat{Y} = (\hat{Y}_t)$ is payoff equivalent to one where the principal just increases consumption to $C'_t = C_t + \lambda(\mathrm{d}Y_t - \mathrm{d}\hat{Y}_t)$, and where the agent does not divert any cash flows. Thus, we have the following result.

Lemma 4.1. Given a contract $\Phi = (C, \tau, \sigma)$ where the agent optimally reports $\hat{Y} \neq Y$, there exists another, incentive-compatible contract $\Phi' = (C', \tau', \sigma')$ where the agent reports truthfully, and that leaves both principal and agent at least as well off in payoff terms.

Intuitively, because the agent cannot save, any diversion can be simulated by the principal and deferred to a later date by compounding at the agent's discount rate γ. A formal proof of this assertion, in a slightly more general form can be found in DeMarzo and Sannikov (2006, Lemma 1).[15] With the observation from Lemma 4.1 that it is without loss of generality to consider contracts that optimally induce zero cash-flow diversion, we now proceed to a characterization of incentive-compatible contracts.

As in discrete-time principal-agent models, it is useful to understand the evolution of the agent's continuation utility. Fix a contract $\Phi = (C, \tau, \sigma)$ and a reporting strategy $\hat{Y}$, and let W_t be the expected utility from time t onwards. Then W_t is given by

$$W_t = \mathbf{E}_t^{\hat{Y},\Phi}\left[\int_t^{\tau} e^{-\gamma(s-t)}\left[\mathrm{d}C_s + \lambda(\mathrm{d}Y_s - \mathrm{d}\hat{Y}_s)\right]\right] \tag{4.1}$$

While the continuation promised utility in [4.1] is entirely forward looking, W_t can, in fact, be written as a diffusion process, whereby increments of promised utility depend only on the current report, current output, the current level of promised utility, and the exogenous noise. This is the central insight of Sannikov (2008) and DS, and it greatly facilitates further analysis of incentive compatibility and the optimal contract. The next lemma makes this precise.

Lemma 4.2. Let (W_t) be as in [4.1] and fix a reporting strategy $\hat{Y}$. Then there exists a

[15] Their proof does not rely on the fact that σ is constant, and so is also valid in our setting.

($\hat{Y}$-measurable) process $Z = (Z_t)$ such that

$$\mathrm{d}W_t = \gamma W_t \mathrm{d}t - \mathrm{d}C_t + \lambda(\mathrm{d}Y_t - \mathrm{d}\hat{Y}_t) + Z_t\sigma_t^{-1}\big[\mathrm{d}Y_t - \mu\mathrm{d}t\big] \tag{4.2}$$

with terminal condition $\lim_{s\to\tau} \mathbf{E}_t^{\hat{Y},\Phi}[e^{-r(s-t)}W_s] = 0$ almost surely.

It is useful to think of continuation utility W_t as a stock that grows at the rate γ. Thus, the increment to promised utility, $\mathrm{d}W_t$ is the interest paid on the stock W_t, net of the payment from the principal, and the amount stolen (which comes from the reported output) plus a random term. The process Z_t is the *sensitivity* of the increment $\mathrm{d}W_t$ to the (random) noise term $\mathrm{d}B_t$, which is output net of the known drift. The proof of the lemma is found in Appendix D.

Notice that given a contract Φ and a reporting strategy $\hat{Y}$, W_t in [**4.1**] and [**4.2**] is how the agent perceives his promised utility. The principal cannot see the true process Y_t, and so requires that the agent report truthfully, ie, that the contract be incentive-compatible. Given the diffusion representation of promised utility in [**4.1**], it is now relatively straightforward to characterize incentive-compatible contracts.

Lemma 4.3. Truth telling, ie $\hat{Y}_t = Y_t$, is incentive-compatible if and only if $Z_t \geqslant \lambda\sigma_t$ for all $t \leqslant \tau$.

The intuition behind this characterization is exactly as in DeMarzo and Sannikov (2006). The benefit to diversion is $\lambda(\mathrm{d}Y_t - \mathrm{d}\hat{Y}_t)$, while the cost, as seen from [**4.2**] is $Z_t\sigma_t^{-1}(\mathrm{d}Y_t - \mathrm{d}\hat{Y}_t)$, because $\mathrm{d}\hat{Y}_t - \mu\mathrm{d}t = \sigma_t\mathrm{d}B_t - \mathrm{d}(Y_t - \hat{Y}_t)$. Incentive compatibility is therefore the condition that the costs of misreporting are greater than the benefits.[16]

All that matters to the agent is his promised utility, and its evolution given C, Z, and σ. It follows from Theorem 1 of Sannikov (2012) that promised utility W_t is a summary statistic of the entire history $(Y_s)_{s\leqslant t}$ at each time t.[17] Thus, we consider contracts that contain elements C, Z, σ, and τ, that are *deterministic* functions of promised utility. This allows us to formulate the principal's problem recursively, as we do next.

5. Optimal Contracts

We now use the dynamic programming principle to derive the principal's optimal contract. As noted above, instead of contracts that depend on the entire path of reported output $\hat{Y}$,

[16]If the principal cannot control σ_t, ie, his monitoring intensity is constant over time, then we recover the characterization provided by DeMarzo and Sannikov (2006), namely that $\beta_t \geqslant \lambda$ where $\beta_t := Z_t\sigma^{-1}$ is the sensitivity in DeMarzo and Sannikov (2006).

[17]More precisely, there is a one-to-one correspondence between incentive compatible contracts $\Phi = (C, \tau, \sigma)$ and controlled processes W as in [**4.2**] with *Markovian* controls (C, σ, Z, τ), where $Z_t \geqslant \lambda\sigma_t$ for all $t \leqslant \tau$.

we may restrict attention to recursive contracts that are Markovian in promised utility W_t. Clearly, the agent's outside option of 0 dictates that $W_t \geqslant 0$ for all t. Recall that $F(w) \mapsto \mathbb{R}_+ \to \mathbb{R}$, as defined in [3.3], denotes the principal's value function, which is the largest profit the principal can obtain from all recursive contracts that provide the agent with $w \geqslant 0$ utiles.

In this section, we first show that F is a smooth and concave solution to a variational inequality. We use this characterisation to establish the structure of the optimal contract, ie, how and when payments are made, when the firm is terminated, and how the intensity of monitoring varies over time.

5.1. Regularity of the Value Function

Suppose the agent is initially promised $w \geqslant 0$ utiles. At any instant, if the principal chooses to not pay the agent (so $\mathrm{d}C_t = 0$), it must be that the other controls for the principal (namely, Z and σ) are locally optimal, and satisfy the dynamic programming principle, ie, the HJB equation in the form

[5.1]
$$rF(w) = \mu + \gamma w F'(w) + \max_{z \geqslant \lambda\sigma, \sigma\in\Sigma} \left[\tfrac{1}{2} z^2 F''(w) - \rho(\sigma)\right]$$

which is a necessary condition for optimality. Notice also that the principal can compensate the agent via promised utility or by immediate payments. It must always be the case that compensating via promised utility is at least as profitable, at the margin, as immediately paying the agent. Formally, this amounts to requiring that $F'(w) \geqslant -1$ for all $w \geqslant 0$.[18] These two requirements can be combined to form a *variational inequality* for the value function F:

[5.2]
$$\min\left[rF(w) - \mu - \gamma w F'(w) - \max_{\substack{z \geqslant \lambda\sigma \\ \sigma\in\Sigma}} \left[\tfrac{1}{2} z^2 F''(w) - \rho(\sigma)\right], F'(w) + 1\right] = 0$$

Our first result shows that F, defined from the "sequence problem" in [3.3], is a regular and concave solution of the variational inequality in [5.2].

Theorem 1. *The value function F defined in* [3.3] *is a* C^2 *solution of* [5.2] *with $F(0) = 0$. Moreover, F is concave.*

Notice that F is bounded above by $\mu/r - w$ (the first best payoff for the principal), and bounded below by $-w$ (where the principal pays the agent immediately and terminates the firm). Because F is concave, it follows that F must be asymptotically affine,

[18] Equivalently, the principal can always pay a lump sum of $\delta C > 0$ to the agent, and re-start the contract at $w - \delta C$. Because this is always feasible, it must be that $F(w) \geqslant F(w - \delta C) - \delta C$, ie, $F'(w) \geqslant -1$.

with $\lim_{w\to\infty} F'(w) = -1$; see Figure 1. These bounds are useful in establishing the Comparison Principle in Theorem 5, which is central to our comparative statics results, as well as in establishing the regularity of the value function F.

The difficulty in establishing regularity, as compared to DS, is the presence of the volatility controls. Because the notion of viscosity solution used in the proof is not required for our main arguments, we provide all the technical details of the proof of Theorem 1 in Appendix H. We first show in Appendix H.1 that F is a continuous *viscosity* solution of [**5.2**], which also allows us to also establish the concavity of F. Next, using techniques from the theory of nonlinear differential equations (Schauder theory), we show in Appendix H.2 that F is also $\mathbf{C}^2$ everywhere. It also follows from the Comparison Principle in Theorem 5 below that F is the unique viscosity (and hence $\mathbf{C}^2$) solution to [**5.2**]. Interestingly, our approach leverages several properties of the value function *without* monitoring (as in DS), which for different levels of volatility provides both upper and lower bounds for F.

To see the concavity of F, notice that if F were not concave on some interval, the principal can randomise between the end points of the interval achieve a higher payoff. However, the proof of Theorem 1 does not rely on such randomisations, and instead uses the fact that F is a viscosity solution to [**5.2**]. Knowing that F is smooth and concave now allows us to deduce the structure of the optimal contract.

5.2. Optimal Payment and Termination

An immediate economic consequence of Theorem 1 is that monitoring and pay-sensitivity are substitutes. To see this, notice that because F is concave and $\mathbf{C}^2$, the optimal pay-sensitivity $z = \lambda\sigma$ in [**5.2**]. Thus, pay-sensitivity is high whenever monitoring is low, and vice versa, as described in Bengtsson and Ravid (2015). We now describe the optimal structure of payment and termination. A useful implication of this structure is that payments are backloaded (see Proposition 5.1).

Let $w^\star := \inf\{w : F'(w) = -1\}$ be the smallest level of promised utility such that the principal is indifferent between compensating the agent via payment or promises; we call $w^\star$ the *payment boundary*. Because F is concave and twice differentiable, $w^\star$ is well defined. Because the agent discounts the future more than the principal, it is optimal to immediately pay the agent $W_t - w^\star$ utiles whenever $W_t \geqslant w^\star$, and re-start the contract at $w^\star$. This is exactly as in DeMarzo and Sannikov (2006), in spite of the additional (monitoring) instruments available to the principal. The intuition behind this result is that because $F'(w) \geqslant -1$, the principal wants to backload payments (which arise as the agent's information rents) as much as possible.[19] This allows the principal to use

[19]This property is also seen in the discrete time literature on dynamic contracting; see, for instance, De-

promised utility as a stock of carrots which she can add to when performance is good, and deplete when performance is bad, and pay the agent when the stock of carrots is sufficiently high.[20]

Notice that running the project without the agent diverting all the output entails giving up some information rents, but to incentivise the agent, it must be the case that promised utility can further decrease. This is impossible at $w = 0$ (because of the agent's limited liability and outside option of 0), and hence it is optimal to choose the random termination time τ as $\tau := \inf\{t : W_t = 0\}$. Thus, the optimal contract should (i) pay the agent when $W_t \geqslant w^\star$, and (ii) terminate him when $W_t = 0$. The next proposition summarises our findings.

Proposition 5.1. There exists a $w^\star \in (0, \infty)$ such that under the optimal profit-maximizing and incentive-compatible contract that delivers $w \in [0, w^\star]$ to the agent, promised utility W_t, with $W_0 = w$, evolves as

$$\text{[5.3]} \qquad \mathrm{d}W_t = \gamma W_t \,\mathrm{d}t - \mathrm{d}C_t + \lambda\big(\underbrace{\mathrm{d}Y_t - \mu \,\mathrm{d}t}_{=\sigma_t \,\mathrm{d}B_t}\big)$$

and where cumulative payment C_t satisfies

$$\text{[5.4]} \qquad C_t = \int_0^t \mathbf{1}(W_s = w^\star) \,\mathrm{d}C_s$$

for all $t \in [0, \tau]$. The termination time $\tau = \inf\{t \geqslant 0 : W_t = 0\} < \infty$ a.s. and $W_t = 0$ for $t \geqslant \tau$. The payment process C_t is nondecreasing in time, and payments are made only when W_t hits $w^\star$. If $w_0 > w^\star$, an immediate payment of $w_0 - w^\star$ is made to the agent, and the contract is re-started at $w^\star$.

Condition [5.4] is a *flat-off* condition, which requires that C increase only when W hits the payment boundary $w^\star$, and can equivalently be written as $\int_0^t \mathbf{1}(W_s < w^\star) \,\mathrm{d}C_s = 0$ for all $t \geqslant 0$. The flat-off condition ensures that $W_t \leqslant w^\star$, ie, that W_t is reflected at $w^\star$, and so W_t can never escape the interval $[0, w^\star]$. The proof of Proposition 5.1 is straightforward, except for showing that $w^\star \in (0, \infty)$. This is established in Proposition H.7 in Appendix H, whereby $w^\star$ is finite if, and only if, $\gamma > r$.

Thus, Theorem 1 and Proposition 5.1 allow us to completely describe the local behaviour of F on $[0, w^\star]$ and also determine its structure beyond this interval.

Marzo and Fishman (2007a,b), Clementi and Hopenhayn (2006) for cash-flow diversion models and Krishna, Lopomo, and Taylor (2013) for a dynamic procurement model that features similar backloading of information rents.

[20] Technically, the payment here is more complicated than in discrete time. The payment process is designed to ensure that $W_t \leqslant w^\star$; formally $C = (C_t)$ is a singular process because it is not absolutely continuous with respect to Lebesgue measure.

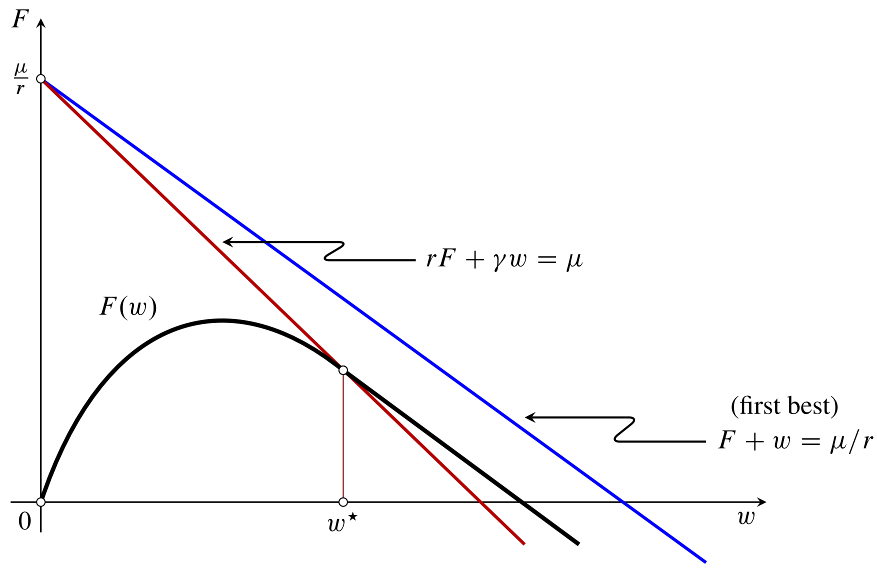

Figure 1: Value function $F(w)$. The line $F = \mu/r - w$ is the full information payoff for the principal, where B_t is observed at no cost, or equivalently, σ can be set to 0 for free. F satisfies the HJB equation [**5.1**] on $[0, w^\star]$ and has slope -1 on $[w^\star, \infty)$.

Corollary 5.2. The principal's value function $F(w)$ satisfies the HJB equation [**5.1**] on $(0, w^\star)$, along with the boundary conditions

$$F(0) = 0, \qquad F'(w^\star) = -1, \qquad F''(w^\star) = 0 \tag{5.5}$$

Moreover, F is affine beyond $w^\star$, ie, $F(w) = F(w^\star) + (w^\star - w)$ for $w \geqslant w^\star$.

The boundary conditions in [**5.5**] are intuitive. As noted above, the project must be terminated when $W_t = 0$, and thus $F(0) = 0$.[21] At $w = w^\star$, the principal is, at the margin, indifferent between using immediate payments and future promises to incentivise the agent; thus, $F'(w) = -1$ at $w = w^\star$, which is the $\mathbf{C}^1$-fit condition in [**5.5**]. To understand the $\mathbf{C}^2$-fit condition, notice that by [**5.1**], $F''(w^\star) = 0$ is equivalent to

$$rF(w^\star) + \gamma w^\star = \mu \tag{5.6}$$

Clearly, it is optimal for the principal to increase $w^\star$ as much as possible as this increases the (expected, discounted) time to termination. The condition [**5.6**] says that the principal will increase $w^\star$ until the sum of expected returns for principal and agent can no longer be satisfied the the technology of the firm. The line $w \mapsto (\mu - \gamma w)/r$ (see Figure 1) represents the division of expected returns between principal and agent, and must equal

[21] Recall that the project has zero scrap value for both principal and agent.

μ; the payment boundary $w^\star$ is the point where the value function F meets this line.[22]

It is now easy to check that $F'(w) = -1$ and $rF(w) > \mu + \gamma w F'(w)$ (ie, $F(w) > (\mu - \gamma w)/r$, as in Figure 1) for $w > w^\star$. Intuitively, payments and promises are two distinct instruments that are never used simultaneously. Corollary 5.2 follows immediately from Theorem 1 and its proof is omitted.

To completely characterise the optimal contract, we need to describe the optimal monitoring choice σ_t. We do this next.

5.3. Optimal Monitoring

Recall from Corollary 5.2 that F satisfies the HJB equation [**5.1**] on $[0, w^\star]$. Let $\sigma_*(w)$ denote the optimal intensity of monitoring, ie, $\sigma_*(w) \in \arg\max_{\sigma \in \Sigma}[\frac{1}{2}\lambda^2\sigma^2 F''(w) - \rho(\sigma)]$. Because $F'' \leqslant 0$, the optimal choice of σ is monotone increasing in F''. In other words, the higher the firm's induced *risk aversion*, namely $-\lambda^2 F''$, the greater the intensity of monitoring.[23] To understand how monitoring varies with w, we need an understanding of how $F''(w)$ varies with w. This is a subtle exercise because F'' is not differentiable at points where there is a switch in σ. Indeed, the shape of F'' is not even obvious in the setting of DS without monitoring, where F is infinitely differentiable. Nonetheless, we have a rich characterisation of $F''(w)$.

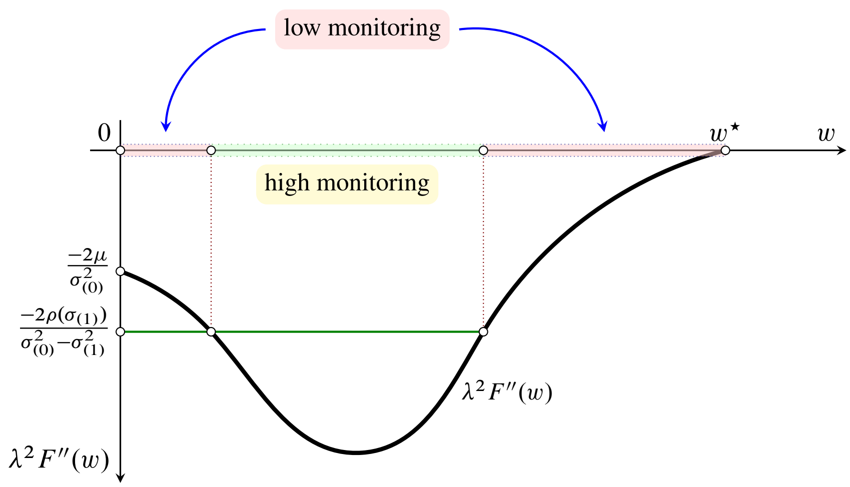

Figure 2: The shape of $\lambda^2 F''$ with two levels of monitoring.

[22]More generally, the $\mathbf{C}^2$-fit condition is a natural requirement for free boundary problems with reflecting boundaries; see, for instance, Dumas (1991).

[23]The expected loss from a lottery with mean w and small variance ε is proportional to $-\lambda^2 F''(w)\varepsilon$. Thus, $-\lambda^2 F''(w)$ represents the firm's induced risk aversion. Intuitively, the *risk* comes from the fact that at any w, the life expectancy of the project is uncertain. Of course, if the agent were not constrained by limited liability, then the first best is achievable and the principal is effectively risk neutral.

Theorem 2. *For each $\mu > 0$ and $\lambda \in (0, 1]$, the function $\lambda^2 F''(w)$ is continuous and piecewise $\mathbf{C}^2$ in w, is single-valleyed, and has a unique local and global minimum. Optimal monitoring $\sigma_*(w)$ is monotone in the firm's risk aversion $-\lambda^2 F''(w)$, and hence is single peaked in w. In particular, pay-sensitivity and monitoring are substitutes.*

In other words, when μ is sufficiently large, optimal monitoring is single-peaked in w; starting at $w = 0$, monitoring initially increases in promised utility, reaches a maximum level of monitoring, and then decreases as w increases, until there is minimal monitoring in a neighbourhood of the payment boundary $w^\star$.

To understand the intuition behind Theorem 2 which describes the shape of $-\lambda^2 F''(w)$, consider the firm's risk aversion at a level w where the optimal monitoring σ_* is locally constant, so that risk aversion is proportionally written as

$$\text{[5.7]} \qquad -\lambda^2 F''(w) \quad \propto \quad \underbrace{r\left[\frac{\mu}{r} - F(w) - w\right]}_{\text{efficiency loss}} + \underbrace{\gamma w\big(F'(w) + r/\gamma\big)}_{\text{expected change in value}}$$

The first term $\mu/r - (F(w) + w)$ is decreasing in w; it represents the efficiency loss, relative to the first best, due to the agency problem. We call w/λ the firm's *financial slack*, because w/λ measures the size of the adverse shocks the firm can withstand before it must be liquidated.[24] With increased financial slack, the effect of the agency problem is more muted, because the principal now has a bigger 'stick' with which to punish the agent, while avoiding termination,[25] which makes it easier (cheaper) to provide incentives.

The second term $\gamma w\big(F'(w) + r/\gamma\big)$ in [5.7] represents the expected change in firm value due to the evolution of w; when μ is sufficiently large (so that $F'(0) > 0$), this term is initially increasing and positive and then decreases in w to $-w^\star(\gamma - r) < 0$. When w is small, the gain in efficiency is more than offset by the expected increase in firm value. However, for sufficiently large w, the payment boundary looms near, and increases in w correspond to greater likelihood of payments to the agent. Of course, at the payment boundary, the value of reduced volatility is 0, so risk aversion drops to 0. Thus, risk aversion initially increases and then decreases, as in Figure 2.[26]

In particular, Theorem 2 says that if the firm has sustained good performance, then monitoring eventually drops to a de minimis level, but if performance falters, the intensity of monitoring will increase. This is in line with the findings of Kaplan and Strömberg (2003) who show that if a firm performs poorly, the investor (VC) obtains greater control,

---

[24]Notice that this liquidation threshold is chosen by the principal, so this measure of financial slack is also a part of the contract.

[25]An increase in w results in a decrease in the Lagrange multiplier for the incentive compatibility constraint.

[26]Because $\sigma_*(w)$ is locally constant, we can differentiate [5.7] to get $-\lambda^2 F'''(w) \propto (\gamma - r)F'(w) + \gamma w F''(w)$. For μ sufficiently large, $F'(w)$ is initially positive and then negative (indeed, $F'''(0) < 0$ if, and only if, $F'(0) > 0$), while $F'' \leqslant 0$ (because F is concave).

but if performance improves sufficiently, the entrepreneur (agent) retains or obtains more control and cash flow rights.[27] Similarly, Vafeas (1999) shows that in publicly traded firms, the frequency of board meetings (which corresponds to monitoring intensity) increases after poor performance.

Naturally, this raises the question, Where (ie, at what level of promised utility) does a small and temporary reduction to σ_t have the greatest effect? When w is near the payment boundary $w^\star$, the agent is about to get paid (with high probability), and so a small reduction in volatility is not very valuable because the probability of affecting this event (payment) is very low; indeed, at $w = w^\star$, any reduction in volatility is worth exactly 0, because it affects neither the expected time to liquidation nor the payment to the agent. Similarly, when w is near 0, the marginal value of reducing volatility is small (but not zero), because reducing volatility increases the time to liquidation (which is valuable), but not by very much given the proximity to the absorbing boundary $w = 0$. It is for intermediate values of w that the value of reducing volatility is highest, because it is here that such a reduction will have its greatest impact, in terms of delaying liquidation.

Finally, we note that for each $\lambda \in (0, 1]$, there exists a $\mu^\dagger(\lambda)$ such that the global minimum of F'' occurs at $w_* > 0$ if, and only if, $\mu > \mu^\dagger(\lambda)$. Intuitively, when $\mu < \mu^\dagger(\lambda)$, the expected cash flow is insufficient to make up for the inefficient termination of the agent, and so $F(w; \mu, \lambda) < 0$ for all $w > 0$. Of course, for μ sufficiently small, the optimal intensity of monitoring is none at all, ie, to set $\sigma_* \equiv \sigma_{(0)}$ for all $t \leqslant \tau$.

5.4. Substitutes vs Complements

In our linear model, pay-sensitivity ($= \lambda\sigma_*(W_t)$) and monitoring are substitutes, as noted in Theorem 2. As discussed above, this has empirical support, at least in financial contracts. In contrast, Milgrom and Roberts (1992, Chapter 7) suggest that monitoring and pay-sensitivity should be *complements*. They consider a static model where output is $y = a + \sigma\varepsilon$, where a is effort, ε is noise, and σ controls the volatility of output. The agent is risk averse and wage contracts are linear in output. A contract delivers wage $w = s + \beta \cdot (y - a)$ where β is the pay-sensitivity. (Linear contracts are chosen by fiat; in general, linear contracts are not optimal in their setting.) As in our model, monitoring reduces the variance of the noise. Their main observation is summarised as the *Monitoring Intensity Principle* (p 226) which states that pay-sensitivity and monitoring are complements: "... When the plan is to make the agent's pay very sensitive to performance [output], it will pay to measure that performance carefully."

The reason for the discrepancy between our model and theirs is that in their setting,

[27] While we do not explicitly model control, a natural interpretation of monitoring is as reducing the amount of freedom the agent has to make decisions *without oversight*.

the optimal action can vary depending on the amount of monitoring (in particular, the variance of the output), while the optimal action in our model is always constant, namely, the agent never diverts the flow of cash. In their model, the principal can induce different actions by changing the pay-sensitivity and the level of monitoring. These, and the optimal action, are all chosen simultaneously at the optimum. Our model simplifies this by keeping the optimal action fixed, so that it is only pay-sensitivity and monitoring that vary, and do so as substitutes. In future work, we aim to study the Monitoring Intensity Principle in a dynamic setting such as Sannikov (2008) where the optimal action can also vary over time.

6. Comparative Statics

We consider two main variations in parameters. First, we consider the impact of a change in μ, the intrinsic profitability of the firm, and next we consider the impact of a change in λ, the severity of the agency problem. Notice that uniform changes to the cost of monitoring, namely reducing $\rho(\cdot)$ by a constant amount, is equivalent to reducing μ as it merely reduces the flow revenue from the project.

A firm with a greater μ has greater expected cash flow and any incentive compatible policy for a lower level of μ will deliver a higher expected profit for a higher level of μ while still being incentive compatible. Similarly, a firm with a lower λ has a *less severe agency problem* because it can adopt the optimal contract of a firm with a higher λ, and still make the same profit, or optimally choose another contract, with a lower pay-sensitivity (Z_t), which generates higher profit.

Proposition 6.1. For a fixed $w > 0$, $F(w; \mu, \lambda)$ is strictly increasing in μ and strictly decreasing in λ. Moreover, the payment boundary $w^\star$ is strictly increasing in both μ and λ.

The monotonicity of F in μ is intuitive, and is also straightforward to prove. That this results in a higher payment boundary $w^\star(\mu)$ is shown in Corollary C.4. Intuitively, when μ is larger, it pays to defer payments, because once the firm has built up enough financial slack, it is less likely to lose this slack, which results in more frequent payments, conditional on reaching the payment boundary.

To see the monotonicity in λ, notice that when λ decreases, the principal can reduce the sensitivity $\lambda\sigma_t$ of the optimal contract. Keeping all other policy variables fixed (namely, $w^\star$ and hence C), it follows that the agent's promised utility process is less sensitive to shocks, and in particular, the time to liquidation strictly increases. This, in and of itself, constitutes an improvement for the principal. Recall that $w^\star(\lambda)$ is where $F(w; \mu, \lambda)$ intersects the line $w \mapsto (\mu - rF)/\gamma$ (also see Figure 1), which is independent of λ. A higher $F(w; \cdot)$ must therefore result in a lower $w^\star(\lambda)$.

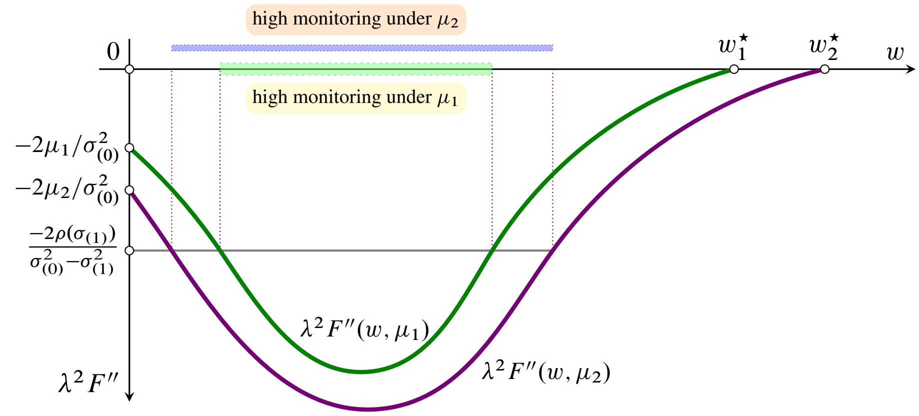

Figure 3: The shape of F'' as a function of μ where $\mu_1 < \mu_2$.

Our comparative statics are easiest to state when fixing the firm's financial slack, which is measured by w/λ. As noted above, financial slack measures, roughly, the amount of negative shocks to cash flow the firm can absorb before it must terminate the agent (for incentive reasons). Clearly, w/λ is not invariant to changes in λ; thus, we state our comparative statics results for fixed levels of financial slack.

Theorem 3. *Consider the firm's policies as a function of μ or λ, and fix the firm's financial slack w_0/λ. Suppose there is an increase in μ or a decrease in λ that keeps financial slack fixed.*[28] *Then, the following hold:*

(i) *The firm's induced risk aversion increases.*
(ii) *The level of monitoring (weakly) increases.*
(iii) *Sensitivity of pay to output (weakly) decreases.*
(iv) *Expected (discounted) life span of the firm, given by $\mathbf{E}^{Y,\Phi}[\int_0^\tau e^{-rt}\,\mathrm{d}t]$, increases.*
(v) *The expected expenditure on monitoring, $\mathbf{E}^{Y,\Phi}[\int_0^\tau e^{-rt}\rho(\sigma_t)\,\mathrm{d}t]$, increases in μ.*

Our analysis also shows that for more valuable firms, monitoring is naturally higher, because the marginal returns from monitoring are higher. Equivalently, monitoring increases because the firm's induced risk aversion increases when their expected cash flow is higher, or when the agency problem is less severe. Intuitively, this is because with a greater μ or smaller λ, the firm is more valuable (see Proposition 6.1), and so the (opportunity) cost of termination is greater.

To understand the intuition behind Theorem 3(i), consider the firm's risk aversion as decomposed in [5.7]. Assuming that $\sigma_*(w)$ is locally constant in μ, differentiate both sides with respect to μ. Then, the marginal change in risk aversion is the sum of $r^{-1} - \partial_\mu F(w;\mu)$ and $\gamma w \partial_\mu F'(w;\mu)$. The first term is always positive because the efficiency loss due to the

[28]Thus, a decrease in λ entails a reduction in w_0 in order to keep financial slack w_0/λ fixed.

agency problem increases in μ (see Lemma C.3) for a fixed level of financial slack. The second term is also positive because the expected change in firm value is increasing in μ (ie, $\partial_\mu F'(w;\mu) > 0$; see Lemma C.5). Intuitively, this is because financial slack (w) and intrinsic profitability (μ) are complements: increased financial slack is more valuable when the firm itself is more valuable. Points (ii) and (iii) now follow immediately.

Point (iv) of Theorem 4 says that with increased monitoring (see point (ii)), the principal need rely less on termination as an incentive device, which has the happy consequence of increasing the expected (discounted) life expectancy of the firm. Finally, point (v) of Theorem 3 formalises our intuition that in our model, greater marginal benefits to monitoring imply a marginal increase in monitoring, and with it, an increase in total monitoring, as measured by the expected expenditure on monitoring. However, this is only true when firms have greater expected cash flow (greater μ). Indeed, numerical calculations show that an increase in λ leads to an increase in expected expenses for some values of financial slack and a decrease for others. This finding highlights the disparate forces than an increase in λ entails. While there is increased monitoring at any level of financial slack, the expected discounted cost of monitoring depends on how long each regime persists, that is, the amount of time spent with a particular level of monitoring. It may be that if the amount of time spent near the payment boundary is sufficiently high (where monitoring is lowest), then the expected cost of monitoring actually goes down. Theorem 3 is proved in Appendix C.[29]

Theorem 3 also suggests two natural dimensions along which to order or sort firms, namely, the expected cash flow of the firm (its μ), or the severity of the agency problem (its λ). This allows one to ask, for instance, Would an agent prefer to be at a firm with more, or less, monitoring? As it turns out, the answer depends on how much bargaining power the agent has, and more crucially, the *source* of the increased governance, which is either greater intrinsic profitability or smaller private benefits for the agent from misappropriation.

It is useful to consider the utility promised to the agent at the time of initialisation, which is determined by the relative bargaining power of principal and agent. At one extreme, if the principal has all the bargaining power, she chooses w_0 such that $F'(w_0) = 0$, ie, an initial promise to maximise her profits. If the agent has all the bargaining power, she chooses the largest w for which $F(w) = K$, where $K \geqslant 0$ is the initial capital outlay required for the project. We denote the agent's choice of promised utility by $w_\sharp$.

Proposition 6.2. If the principal has the all the bargaining power, then w_0 increases in μ and λ. On the other hand, if the agent has all the bargaining power, then $w_\sharp$ increases in

[29] In spite of the fact that Theorem 3 states that the impact of an increase in μ or decrease in λ is qualitatively the same, establishing these claims requires different approaches, primarily because λ and μ affect the boundary conditions for the HJB equation [**5.1**] differently. Appendix C.2 contains the proofs for a change in μ, while Appendix C.3 proves Theorem 4 for the cases where λ changes.

μ but decreases in λ.

Thus, if the agent has all the bargaining power, he welcomes increased monitoring because the firm is more profitable, and is one where he can extract a larger amount of the surplus. On the other hand, if the principal holds all the bargaining power, the reason for increased monitoring matters. Proposition 6.2 says that agents would prefer to be at a firm with greater governance *if* that increased governance is because the firm is intrinsically more profitable, ie, has a higher μ. This is intuitive, because a higher μ corresponds to greater surplus, and some of that additional surplus goes to the agent, via his information rent. On the other hand, if the increased governance is because the agent's benefits from misappropriation are smaller, then the agent is worse off because his information rents, which is the only reason he gets paid, are lower.

7. Optimal Contract via Securities

Our model has implications for *corporate governance* in firms, which we interpret as monitoring of the CEO. This is seen most easily by implementing the optimal contract via securities, where we follow the approach of Biais et al. (2007).

The dynamics of the optimal contract depend on the agent's continuation utility, W_t. As noted before, the expected discounted time to termination is increasing in w, and so W_t is a measure of how far the firm is from termination. That is, when W_t is larger, the firm can withstand larger negative shocks to cash flow before termination.

Let $M_t = W_t/\lambda$ denote the firm's *observable cash reserves*. Because W_t is a measure of the firm's distance from termination, it is natural to interpret M_t as the firm's available *financial slack* (cf DeMarzo et al., 2012, Section IV, henceforth DFHW). Clearly, the dynamics of W_t completely determine the dynamics of cash reserves M_t.

The financiers of the firm (which could be a single principal, or multiple lenders) hold debt in the form of bonds. Bonds distribute a continuous coupon flow of $\mu - \rho(\sigma_*(\lambda M_t)) - (\gamma - r)M_t$ at time t that varies with the level of cash reserves. The first term $\mu - \rho(\sigma_*(\lambda M_t))$ in the coupon payment is merely the firm's expected cash flow net of monitoring costs, while the second term, $(\gamma - r)M_t$, accounts for the different rates of discounting between principal and agent.[30] In addition, the agent gets a non-tradeable fraction λ of the firm's equity,[31] while the remaining equity goes to the financiers.

[30]Performance-sensitive debt payments are seen in VC contracts (Kaplan and Strömberg, 2003) and in step-up bonds where coupon payments decrease in financial slack; Lando and Mortensen (2004) document the prevalence and use of such step-up bonds and also address questions related to their pricing.

[31]The agent is the holder of *unregistered* or *letter* securities that cannot be publicly traded. This is necessary because $\gamma > r$, which means that the market always values the stock more than the agent does. Allowing the agent to trade his stock will result in the agent trading his stock right away. Notice that if $\gamma = r$, this is no longer an issue, although we would then have to place an upper bound on the payment boundary.

We can write the dynamics for M_t as

$$\mathrm{d}M_t = \gamma M_t \,\mathrm{d}t + \big(\underbrace{\mathrm{d}Y_t - \mu \,\mathrm{d}t}_{=\sigma_*(\lambda M_t)\,\mathrm{d}B_t} \big) - \lambda^{-1}\,\mathrm{d}C_t \qquad [7.1]$$

where $\lambda^{-1}C_t$ is the dividend payment.

Notice that in this implementation, the agent controls how payments are disbursed, either as debt payments or as dividends. The firm is liquidated if coupon payments are not made, or when cash reserves hit 0, ie, at the stochastic time $\tau = \inf\{t : M_t = W_t/\lambda = 0\}$. The agent also chooses the amount of monitoring to be performed by the principal, as well as his cash-diversion strategy. The following proposition summarises the implementation.

Proposition 7.1. Suppose the firm has initial cash reserves M_0 and is in operation as long as it makes dividend payments at the rate $[\mu - \rho(\sigma_*(\lambda M_t)) - (\gamma - r)M_t]$. Then, it is optimal for the agent to not divert any cash flows, to recommend monitoring according to σ_* as in Theorem 2, and to make dividend payments when cash reserves reach $m^\star$. In particular, the agent's expected payoff from this policy is $\lambda M_0 = W_0$.

The proof is similar to that of Proposition 2 in DFHW, and hence is omitted. The above implementation is clearly not unique. For instance, following DS, financial slack M_t may obtain via a combination of cash reserves and a line of credit, with termination occurring upon their exhaustion. The main point is that in *any* implementation, financial slack will be proportional to W_t, which measures the distance to termination. The critical distinguishing feature in our model (relative to BMPR, DS, and DFHW, for instance) is that monitoring (or equivalently, governance) is optimally chosen within the contract, and has the effect of increasing the financial slack of the firm. Importantly, investment in monitoring, which is a *non-capital investment*, is non-monotonic in financial slack. This is in contrast with DFHW who show that in a homegenous model, *capital* investment is monotonic in financial slack.

Next, we show that the securities used in the implementation can be priced and that they vary in natural ways in response to changes in the parameters of the model.

7.1. Comparative Statics for Securities

The cash reserves, and stock and bond prices are all processes that are adapted to M_t. For all $t \in [0, \tau]$, the *stock price* S_t satisfies $S_t = \mathbf{E}_t \big[\int_t^\tau e^{-r(s-t)} \lambda^{-1}\,\mathrm{d}C_s\big]$, so that the stock price is the expected discounted value of dividend payments. The *bond price* is denoted by D_t, where $D_t = \mathbf{E}_t \big[\int_t^\tau e^{-r(s-t)} [\mu - \rho(\sigma_*(\lambda M_s)) - (\gamma - r)M_s]\,\mathrm{d}s\big]$ is the expected discounted value of coupon payments. The *credit yield spread* at time $t \in [0, \infty)$ on a consol bond that pays \$1 until the firm is dissolved is given by ζ_t, where ζ_t satisfies

$\int_t^\infty \exp\big(-(r+\zeta_t)(s-t)\big)\,\mathrm{d}s = \mathbf{E}_t\big[\int_t^\tau e^{-r(s-t)}\,\mathrm{d}s\big]$, which implies $\zeta_t := \frac{rT_t}{1-T_t}$, where $T_t := \mathbf{E}_t[e^{-r(\tau-t)}]$ for all $t \in [0,\tau]$. It can be shown that the stock and bond prices and the credit yield spread are all deterministic functions of M_t, ie, are diffusions themselves.

As in Section 6, we consider the impact of a change in μ, the intrinsic profitability of the firm, and the impact of a change in λ, the severity of the agency problem.

Theorem 4. *Consider the firm's policies as a function of μ or λ. Then, for any level of cash reserves, the following hold:*

(i) *The level of governance (weakly) increases in μ and decreases in λ.*
(ii) *Sensitivity of pay to output (weakly) decreases in μ and increases in λ.*
(iii) *Stock prices increase in μ and decrease in λ.*
(iv) *Credit yield spread decreases in μ and increases in λ.*
(v) *The threshold cash reserve $m^\star$ is increasing in μ and decreasing in λ.*

Our analysis shows that for more valuable firms, governance (adjusted for financial slack) is naturally higher, because the marginal returns from monitoring are higher. Stock prices are higher and credit yield spreads are lower, because with increased monitoring, the probability of termination is lower. Finally, observe that stock prices are higher in spite of the fact that the threshold for payment, $m^\star$, is increasing in μ. That is, stock holders have to wait longer for dividend payments, but in spite of this, stock prices are higher, because the fear of termination is correspondingly lower. We emphasise that (apart from part (i)) Theorem 4 is not a corollary of Theorem 3, although it relies on Theorem 3 for the monotonicity of monitoring in cash reserves. Instead, the proof of Theorem 4 relies on a Comparison Principle for boundary-value problems (Proposition A.4) that characterise stock price and credit yield spread.

The correlation between credit yield spreads and corporate governance is documented in Bhojraj and Sengupta (2003) and Ashbaugh-Skaife, Collins, and LaFond (2008). Theorem 4 not only rationalises these (and other) findings, it also exhibits a *mechanism* that explains the observed correlations. Our implementation also has the following testable implications.

(i) *Governance is single-peaked in financial slack and stock price.* This follows immediately from Theorem 4.
(ii) *Governance and pay-sensitivity are substitutes.* Pay sensitivity is U-shaped as a function of cash reserves, and is inversely related to the firm's aversion to volatility.
(iii) *Empirical measure of governance intensity.* We are able to construct a measure of the local intensity of governance can be constructed from three easily obtainable quantities: the stock price, the local volatility of stock price, and the sensitivity of the agent's compensation to stock price movements.

Point (ii) is in line with the work of Fahlenbrach (2009) and Fernandes et al. (2013) who empirically document that the sensitivity of CEO pay to firm performance is higher in firms with weaker corporate governance, as proxied by board independence or by institutional ownership concentration. Point (iii) represents our key contribution to the measurement of governance. It shows that a scaled measure of governance can be written as a simple product of three quantities that are either observable (stock price) or easy to estimate (local volatility of stock price and pay-performance sensitivity). In particular, this result shows how asset prices can be used to infer the strength of a firm's governance.

As mentioned above (and proved in Appendix C.1), stock price $S_t = \mathcal{S}(M_t)$ for some strictly increasing and concave function $\mathcal{S}$. Thus, we can write $\mathcal{S}'(M_t) = \mathcal{S}'(\mathcal{S}^{-1}(S_t))$, and using Itô's Lemma, we obtain

$$\begin{aligned} \mathrm{d}S_t &= \underbrace{\left[\gamma M_t \mathcal{S}'(M_t) + \tfrac{1}{2}\sigma_*^2(\lambda M_t)\mathcal{S}''(M_t)\right]}_{= r\mathcal{S}(M_t) = rS_t} \mathrm{d}t \\ &\quad + \mathcal{S}'(M_t)\left[\sigma_*(\lambda M_t)\,\mathrm{d}B_t - \lambda^{-1}\,\mathrm{d}C_t\right] \\ &= rS_t\,\mathrm{d}t + \underbrace{\mathcal{S}'(\mathcal{S}^{-1}(S_t)) \cdot \sigma_*(\lambda \mathcal{S}^{-1}(S_t))}_{=: \mathsf{V}_t S_t}\,\mathrm{d}B_t - \lambda^{-1}\,\mathrm{d}C_t \end{aligned} \tag{7.2}$$

where we have used the BVP characterization of $\mathcal{S}$ from [C.1] which requires that $\mathrm{d}C_t = 0$ if $M_t < m^\star$, and also noting (from [C.1]) that $\mathcal{S}'(m^\star) = 1$, which is an analog of equation (57) in BMPR.[32]

The dynamics for S_t in [7.2] make it a *local volatility* model for stock prices, where the *local volatility* of the stock price is $\mathsf{V}_t = \mathcal{S}'(\mathcal{S}^{-1}(S_t))\sigma_*(\lambda\mathcal{S}^{-1}(S_t))/S_t$. The local volatility of stock prices can be measured from stock price data using *Dupire's formula*; see Dupire (1994).

Pay-performance sensitivity (PPS) measures the sensitivity of the manager's wealth to changes in the stock price. In our setting, we can write $W_t = \lambda\mathcal{S}^{-1}(S_t)$, so that $\mathrm{PPS}_t = \lambda\big(\mathcal{S}^{-1}(S_t)\big)' = \lambda/\mathcal{S}'(M_t)$. This is also well measured empirically, see for instance Core and Guay (2002) and Brick, Palmon, and Wald (2012). It is easy to see that

$$\mathsf{V}_t \cdot \mathrm{PPS}_t \cdot S_t =: \mathsf{Gov}(t) = \lambda\sigma_*(t) \tag{7.3}$$

Thus, by measuring local volatility of stock price and pay-performance sensitivity at time t, we can find a measure of the intensity of governance using [7.3]. Of course, in practice, both the profitability (the μ) and the volatility of stock prices are affected by many factors, but [7.3] nonetheless, suggests some testable implications. In particular, we see that local volatility V_t and pay-performance sensitivity PPS_t are inversely related, monitoring and PPS are not positively correlated, and PPS increases in stock price. Moreover, option

[32] Note that stock *returns*, defined as $[\mathrm{d}S_t + \lambda^{-1}\mathrm{d}C_t]/S_t$ (ie, capital gains plus dividends), are equal to $r\,\mathrm{d}t$ in expectation. This is unsurprising because both principal and agent are risk neutral, and stocks are rationally priced. Thus, expected stock returns do not vary with governance in our model.

values are highest when PPS is very high, which also happens to be when monitoring is lowest (and so volatility is higher). But, because this is near the payment boundary, future monitoring will be higher, which reduces stock returns, and so PPS is negatively related to future stock returns. This is broadly consistent with Brick, Palmon, and Wald (2012). We note that even in an alternate model where the principal monitors the agent by effecting changes in λ, we would still obtain the same measure $\mathsf{Gov}(t)$ of the intensity of monitoring, as defined in [**7.3**]. We leave to future work further analysis of the $\mathsf{Gov}(t)$ process.

8. Conclusion

In this paper, we provide a dynamic model of monitoring in contracts. We find that monitoring and pay-performance sensitivity are substitutes, that payments are optimally backloaded, and that monitoring intensity increases after sustained poor performance, but decreases after sufficient good performance. These are properties of the optimal contract for a given firm and rationalise stylised facts about VC contracts for firm financing.

Our comparative statics results show that firms with greater expected cash flow or with a lower intrinsic agency problem have more monitoring and are also more valuable. Intuitively, the marginal cost of monitoring is lower in such firms, and this begets greater monitoring. When viewing monitoring as corporate governance, our findings rationalise observed positive correlations between the governance measures and various measures of firm performance such as stock prices, stock returns, credit yield spread, and more. Thus, our model shows how the firm's balance sheet, security prices, and governance structure are related.

We also make several technical contributions. We use Schauder theory to prove the regularity of the value function and that it satisfies the variational form of the HJB equation. We also use the Comparison Principle in the theory of viscosity solutions for our comparative statics as well as to prove that the value function is the unique solution to the variational form of the HJB.

Our model assumes that the agent's actions have linear costs. An extension to convex costs and multiple actions would allow us to speak to many other features of incentive systems. Another extension would be to consider the question of credit risk but with agency and monitoring concerns. We leave these for future work.

Appendices

In Appendix A, we state the Comparison Principles that we use for our comparative statics. Appendix B proves Theorem 2. Appendix C derives the dynamics of securities (stock price and credit yield spread) and proves the comparative statics from Theorems 3 and 4. All further results are in the Online Appendix.

A. Comparison Principles

We state here two general Comparison Principles that will be used both to show the existence of a smooth value function as well as to establish our comparative statics.

A.1. Comparison Principle via Viscosity Solutions

The HJB equation for the value function F takes the abstract form,

$$\mathscr{H}(x, u(x), u'(x), u''(x)) = 0; \qquad u(0) = 0 \quad \text{and} \quad x > 0 \tag{A.1}$$

where $\mathscr{H} : \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}$ is given by

$$\begin{aligned} \mathscr{H}(x, p, q, \Gamma) &:= \min\left\{rp - \gamma x q - H(\Gamma), q + \eta\right\} \\ H(\Gamma) &:= \sup\left\{\tfrac{1}{2}z^2\Gamma + \mu - \rho(\sigma) : \sigma \in \Sigma \ \text{ and } \ \sigma\lambda \leqslant z\right\} \end{aligned} \tag{A.2}$$

and $\eta \in (0, 1]$. The problem [A.1] is a *boundary value problem* with a *gradient constraint*; the gradient constraint comes from the term $q + \eta$ in $\mathscr{H}$, which imposes the constraint $u'(x) + \eta \geqslant 0$ on the solution. Notice that by posing the problem with a gradient constraint, we are effectively reformulating our free boundary problem, where the boundary $w^\star$ was endogenously determined, into a nonlinear differential equation on the domain $[0, \infty)$.

Lemma A.1. The function H in [A.2] is convex, Lipschitz, and strictly increasing.

Note that the function H in [A.2] is easily extended to $\mathbb{R}$ by setting $H(\Gamma) = \infty$ for $\Gamma > 0$. The notion of viscosity solution that we use is now defined.

Definition A.2. A locally bounded lower (respectively upper) semicontinuous function u is a *viscosity supersolution* (respectively *subsolution*) to [A.1], if for all $x_0 > 0$ and all smooth function ϕ such that $u - \phi$ has a local minimum (respectively maximum) at x_0 with $u(x_0) = \phi(x_0)$, we have

$$\mathscr{H}(x_0, \phi(x_0), \phi'(x_0), \phi''(x_0)) \geqslant \ (\leqslant)\, 0 \tag{A.3}$$

with $u(0) \geqslant 0$ (respectively $u(0) \leqslant 0$).

A continuous function u is called a *viscosity solution* to [A.1] if it is both a sub- and a supersolution in the viscosity sense.

We are now ready to state our first Comparison Principle.[33]

Theorem 5 (Comparison Principle)**.** *Let v and V be, respectively, upper and lower semicontinuous viscosity sub- and supersolutions of* [**A.1**] *with $v(x) \leqslant K - \eta x$ for some $K > 0$. Then, $v \leqslant V$.*

Remark A.3. The condition $v(x) \leqslant K - \eta x$ is crucial for the comparison to hold. For instance, suppose $\eta = 1$, and consider the function $u(x) = x$, which satisfies $ru(x) - \gamma x u'(x) - H(u''(x)) = rx - \mu - \gamma x + \rho(\sigma_0) = (r - \gamma)x - (\mu - \rho(\sigma_0)) \leqslant 0$ for all $x \geqslant 0$, because $\rho(\sigma_0) = 0 < \mu$. Thus, u is a subsolution of [**A.1**]. Similarly, we can check that $V(x) = \mu/r - x$ is a supersolution to [**A.1**]. But $u(x) > V(x)$ for all x sufficiently large, which demonstrates the necessity of such a condition.

The proof of Theorem 5 is somewhat involved and uses ideas not required elsewhere. It is therefore deferred to Appendix G.

A.2. Comparison Principle for Piecewise Smooth Functions

We use the following Comparison Principle for the comparative statics of $\mathcal{S}$ and $\mathcal{T}$.

Proposition A.4 (Comparison Principle)**.** Let v (respectively V) be a bounded $\mathbf{C}^1$ and piecewise-$\mathbf{C}^2$ subsolution (respectively supersolution) of the elliptic equation $\mathsf{L}\varphi = -\frac{1}{2}\hat{\sigma}^2(m)\varphi'' - \gamma m\varphi' + r\varphi = 0$ on $(0, m^\star)$, with boundary conditions $\varphi(0) = \alpha$ and $\varphi'(m^\star) = \beta$, ie,

[**A.4**]	$\mathsf{L}v \leqslant 0$;	$v(0) \leqslant \alpha$	and	$v'(m^\star) \leqslant \beta$
[**A.5**]	$\mathsf{L}V \geqslant 0$;	$V(0) \geqslant \alpha$	and	$V'(m^\star) \geqslant \beta$

Then, $v \leqslant V$.

The proof of Proposition A.4 is in Appendix I.

B. Proof of Theorem 2

Proof. We have already shown in Theorem 1 that F is $\mathbf{C}^2$ everywhere and also globally concave. Thus, there exist intervals $A_j \subset [0, w^\star]$ such that σ_j is optimal on A_j. Generically, at any w, exactly one σ_j will be optimal.

[33] We cannot appeal to the "standard" Comparison Principle in Crandall, Ishii, and Lions (1992, Theorem 3.3) because that result only applies to bounded domains, whereas our domain (and solutions) are unbounded. Nonetheless, we exploit the bounds delivered by the first-best and the immediate-liquidation strategies.

Let w_s be a point where there is a regime change, ie, where σ_i and σ_j are both optimal. In particular, suppose σ_i is optimal on $[w_s - \varepsilon, w_s]$ while σ_j is optimal on $[w_s, w_s + \varepsilon]$. This gives us the following differential equation (which is the HJB) in each of the intervals $\frac{1}{2}\theta_k^2 F''(w) + \gamma w F'(w) - rF(w) + \mu - \rho_k = 0$ where $k = i, j$, and $\theta_k = \lambda\sigma_k$. Differentiating the HJB on the relevant intervals, we have $\frac{1}{2}\theta_k^2 F'''(w) + \gamma w F''(w) + (\gamma - r)F'(w) = 0$. Because F' and F'' are continuous, we may take limits as $w \to w_s$ from above and below in [**B**], which implies $\theta_i^2 F'''(w_s-) = \theta_j^2 F'''(w_s+)$. This implies $F'''(w_s-)$ and $F'''(w_s+)$ have the same sign, and so any *interior* minimum occurs only at points where F'' is differentiable.

On the other hand, except at the switching points, we have $\frac{1}{2}\theta_k^2 F^{(4)}(w) + \gamma w F'''(w) + (2\gamma - r)F''(w) = 0$. At any interior critical point w_* of F'', we have $F'''(w_*) = 0$. The concavity of F implies $F''(w) < 0$ for $w < w^\star$, and so we conclude that $F^{(4)}(w_*) > 0$ and therefore, w_* can only be a local minimum.

A local interior minimum exists if, and only if, $F'(0) > 0$, because $F'''(0+) = -(\gamma - r)F'(0)$. But this happens whenever μ is sufficiently large, a fact that is relatively straightforward to establish, by bounding from below the value function from DS. Similarly, it can be shown that when μ is sufficiently small, $F'(0) < 0$. Lemma C.5 establishes that $F'(0;\mu)$ is strictly increasing and continuous in μ. Because $F'(0;\mu) < 0$ for μ very small while $F'(0;\mu) > 0$ for μ very large, the existence of $\mu^\dagger$ follows immediately. □

C. Proofs of Comparative Statics Results

C.1. Dynamics of Securities

We begin by describing, in the fashion of BMPR, some boundary value problems that characterise the stock price and the credit yield spread. Propositions in this section are proved in Appendix E.1.

The stock price S_t satisfies $S_t = \mathbf{E}_t\left[\int_t^\tau e^{-r(s-t)}\lambda^{-1}\,\mathrm{d}C_s\right]$. Because C_t is a deterministic function of M_t, it follows that we can write $S_t = \mathcal{S}(M_t)$. Furthermore, we have the following result.

Proposition C.1. The stock price S_t is given by $s = \mathcal{S}(m)$, where $\mathcal{S}$ is the unique solution to the boundary value problem

$$\text{[C.1]}\qquad \begin{gathered} r\mathcal{S}(m) = \gamma m \mathcal{S}'(m) + \tfrac{1}{2}\sigma_*^2(\lambda m)\mathcal{S}''(m) \\ \mathcal{S}(0) = 0 \quad \text{and} \quad \mathcal{S}'(m^\star) = 1 \end{gathered}$$

and $m \in [0, m^\star]$. Moreover, the stock price $\mathcal{S}(m)$ is a strictly increasing and strictly concave function of m, the level of cash reserves. It is $\mathbf{C}^1$ everywhere, and piecewise $\mathbf{C}^2$, with $\mathcal{S}''$ being discontinuous at points where $\sigma_*(\cdot)$ is discontinuous.

The price of the bond is denoted by D_t, which can also be written as a $D_t = \mathscr{D}(M_t)$ for some deterministic function $\mathscr{D}$. The function $\mathscr{D}$ solves a boundary value problem, as in BMPR..

It is clear that for all $t \in [0, \tau]$, $T_t = \mathbf{E}_t[e^{-r(\tau-t)}] =: \mathscr{T}(M_t)$ where $\mathscr{T}$ is a deterministic function. Moreover, $\mathscr{T}$ is the solution to the boundary value problem

$$\begin{gathered} \mathscr{T}(m) = \gamma m \mathscr{T}'(m) + \tfrac{1}{2}\sigma_*(\lambda m)\mathscr{T}''(m) \\ \mathscr{T}(0) = 1 \quad \text{and} \quad \mathscr{T}'(m^\star) = 0 \end{gathered} \tag{C.2}$$

This leads to the following proposition.

Proposition C.2. The credit yield spread is a strictly positive, strictly decreasing, and strictly convex function of cash reserves.

C.2. Comparative Statics in μ

We analyze here the sensitivity of the value function and other policy variables to the parameter μ.

Lemma C.3. The value function F satisfies

$$\frac{\partial F(w,\mu)}{\partial \mu} = \frac{1 - \mathbf{E}[e^{-r\tau}]}{r} < \frac{1}{r} \tag{C.3}$$

In particular, $F(w, \mu)$ and $\mu - rF(w, \mu)$ are both strictly increasing in μ.

Proof. Let σ^* denote the optimal monitoring strategy and C^* the optimal payment strategy, resulting in the optimal contract $\Phi^* = (C^*, \sigma^*, \tau^*)$. The value function F is defined in [3.3]. In particular, we have

$$F(w,\mu) = \mathbf{E}^{\Phi^*}\left[\int_0^\tau e^{-rs}\big[(\mu - \rho(\sigma_t^*))\,\mathrm{d}s - \,\mathrm{d}C_s^*\big] \mid W_0 = w\right]$$

Notice that $F(w, \mu)$ is bounded above by $\mu/r - w$, and below by $-w$. Therefore, we may apply the envelope theorem from Milgrom and Segal (2002), to conclude that [C.3] holds. It follows immediately that $F(w, \mu)$ is increasing in μ. By [C.3], $1 - r\frac{\partial F(w,\mu)}{\partial \mu} = \mathbf{E}[e^{-r\tau}] > 0$, which completes the proof. □

Corollary C.4. The payment boundary $w^\star(\mu)$ is strictly increasing in μ, and hence so is the cash reserve threshold $m^\star = w^\star/\lambda$.

Proof. Recall the C[1]-pasting condition $F'(w^\star) = -1$, and the C[2]-pasting condition $F''(w^\star) = 0$ in [5.5], which imply that at $w = w^\star$, the HJB equation [5.2] becomes

$rF(w^\star, \mu) = \mu - \gamma w^\star$. Differentiating with respect to μ, we obtain $r\partial_\mu F(w^\star, \mu) + rF'(w^\star, \mu)(\mathrm{d}w^\star/\mathrm{d}\mu) = 1 - \gamma\frac{\mathrm{d}w^\star}{\mathrm{d}\mu}$. But $F'(w^\star, \mu) = -1$, which implies $(\gamma - r)\frac{\mathrm{d}w^\star}{\mathrm{d}\mu} = 1 - r\partial_\mu F(w^\star, \mu) = \mathbf{E}[e^{-r\tau}] \in (0, 1)$ where the last equality is from [C.3]. Because $\gamma > r$, it follows that $\mathrm{d}w^\star/\mathrm{d}\mu > 0$, as claimed. □

To understand the effect of a change in μ on the optimal contract, one needs to understand how it affects F''. The next lemma takes us in that direction.

Lemma C.5. Given $\mu_1 < \mu_2$, $F'(w, \mu_1) \leqslant F'(w, \mu_2)$, where the equality holds if, and only if, both sides are -1. Thus, F' is strictly increasing in μ in the relevant part of the domain.

Proof. Let $\mu_2 > \mu_1$ and $w_i^\star$ be the corresponding payment boundary for $i = 1, 2$. By Corollary C.4, $w_2^\star > w_1^\star$, and so $F'(w_1^\star, \mu_2) > F'(w_1^\star, \mu_1) = -1$. Let $w_\circ$ be the largest $w \in [0, w_1^\star)$ such that $F'(w_\circ, \mu_2) = F'(w_\circ, \mu_1)$, so that $F'(w, \mu_2) > F'(w, \mu_1)$ for all $w \in (w_\circ, w_1^\star)$.

In the HJB equation [5.1], let us define $\Phi(\Gamma) := \max_{\sigma\in\Sigma} \left(\frac{1}{2}\sigma^2\lambda^2\Gamma - \rho(\sigma)\right)$. It is easy to see that Φ is strictly increasing in Γ. Notice that the HJB equation [5.1] can now be written as $\Phi(F''(w_\circ, \mu_1)) = rF(w_\circ, \mu_1) - \mu_1 - \gamma w_\circ F'(w_\circ, \mu_1) > rF(w_\circ, \mu_2) - \mu_2 - \gamma w_\circ F'(w_\circ, \mu_2) = \Phi(F''(w_\circ, \mu_2))$, where the inequality follows from Lemma C.3 (which says $rF - \mu$ is strictly decreasing in μ) and because $F'(w_\circ, \mu_2) = F'(w_\circ, \mu_1)$ by assumption.

The monotonicity of Φ now implies that $F''(w_\circ, \mu_1) > F''(w_\circ, \mu_2)$. Thus, $F''(w_\circ, \mu_1) > F''(w_\circ, \mu_2) = \lim_{w\downarrow w_\circ} \frac{F'(w,\mu_2) - F'(w_\circ,\mu_2)}{w - w_\circ} \geqslant \lim_{w\downarrow w_\circ} \frac{F'(w,\mu_1) - F'(w_\circ,\mu_1)}{w - w_\circ} = F''(w_\circ, \mu_1)$ which is a contradiction. Thus, there is no such $w_\circ \geqslant 0$. On the other hand, for all $w \geqslant w_2^\star$, $F'(w, \mu_2) = F'(w, \mu_1) = -1$, which proves the claim. □

We are now in a position to describe how F'' changes with μ.

Corollary C.6. $F''(\cdot, \mu)$ is locally strictly decreasing in μ on $[0, w^\star]$.

Proof. Consider the HJB equation [5.1] on $[0, w^\star]$, written as $\Phi(F''(w, \mu)) = rF(w, \mu) - \mu - \gamma w F'(w, \mu)$, with boundary conditions in [5.5]. By Lemma C.3, it follows that $rF(w, \mu) - \mu$ is locally strictly decreasing in μ, while Lemma C.5 says $F'(w, \mu)$ increases in μ. The monotonicity of Φ implies $F''(w, \mu)$ is locally strictly decreasing in μ when $w \in [0, w^\star)$. □

Recall that $\sigma_*(w, \mu) = \arg\max_{\sigma\in\Sigma} \left[\frac{1}{2}\sigma^2\lambda^2 F''(w, \mu) - \rho(\sigma)\right]$ denotes the optimal choice of monitoring at w. The behaviour of F'' with respect to μ dictates how optimal monitoring changes with μ.

Proposition C.7. The optimal level of monitoring, as a function of promised utility or of cash reserves, is increasing in μ.

Proof. The objective $\frac{1}{2}\sigma^2\lambda^2 F''(w,\mu) - \rho(\sigma)$ has increasing differences in (σ,μ) if μ is given the reverse order because F'' is monotone decreasing in μ (in the standard order) by Corollary C.6. Therefore, $\sigma_*(w,\mu)$ is decreasing in μ (in the standard order), ie, the level of monitoring increases in μ. It is clear that the same holds as a function of cash reserves, because $m = w/\lambda$ is independent of μ. □

We now show that stock prices are also monotone in μ.

Proposition C.8. Stock price $\mathcal{S}(m,\mu)$ is increasing in μ.

Proof. Consider $\mu_1 < \mu_2$, and let σ_i^* be the optimal policy under μ_i, $m_i^\star$ the cash reserve threshold, and $\mathcal{S}_i$ the corresponding stock price. We have already established in Corollary C.4 that $m^\star$ increases in μ, which implies that $\mathcal{S}_2'(m_1^\star) > 1$. By virtue of being a solution to the boundary value problem (at $\mu = \mu_1$), we have

$$0 = r\mathcal{S}_1(m) - \gamma m\mathcal{S}_1'(m) - \tfrac{1}{2}\sigma_{*1}^2\mathcal{S}_1''(m) \geqslant r\mathcal{S}_1(m) - \gamma m\mathcal{S}_1'(m) - \tfrac{1}{2}\sigma_{*2}^2\mathcal{S}_1''(m) \quad \text{[C.4]}$$

where the inequality is because $\sigma_{*1}(\lambda m) \geqslant \sigma_{*2}(\lambda m)$ (by Proposition C.7) and because $\mathcal{S}_i'' \leqslant 0$ for $i = 1, 2$ by Proposition C.1.

Thus, $\mathcal{S}_1$ is a subsolution to the boundary value problem [**C.1**] under μ_2. We have also noted that $\mathcal{S}_2(0) = \mathcal{S}_1(0)$, and $\mathcal{S}_2'(m_1^\star) > \mathcal{S}_1'(m_1^\star) = 1$, and so by the Comparison Principle in Proposition A.4, it follows that $\mathcal{S}(m,\mu_2) \geqslant \mathcal{S}(m,\mu_1)$ for all m (where they are both defined). □

We now show that the expected discounted hitting time is decreasing in μ.

Proposition C.9. The expected discounted liquidation time $\mathcal{T}(m)$ in [**C.2**] is decreasing in μ.

Proof. Let $\mu_1 < \mu_2$, σ_{*i} be the optimal policy under μ_i, $m_i^\star$ the cash reserve threshold, and $\mathcal{T}_i$ the corresponding expected discounted liquidation time in [**C.2**]. We have already established in Corollary C.4 that $m^\star$ increases in μ, which implies that $\mathcal{T}_2'(m_1^\star) < 0$. Because $\mathcal{T}_1$ solves [**C.2**] when $\mu = \mu_1$, we have $0 = r\mathcal{T}_1(m) - \gamma m\mathcal{T}_1'(m) - \frac{1}{2}\sigma_{*1}(m)^2\mathcal{T}_1''(m) \leqslant r\mathcal{T}_1(m) - \gamma m\mathcal{T}_1'(m) - \frac{1}{2}\sigma_{*2}(m)^2\mathcal{T}_1''(m)$ where the inequality is because $\sigma_{*1}(\lambda m) \geqslant \sigma_{*2}(\lambda m)$ (by Proposition C.7) and because $\mathcal{T}_i'' \geqslant 0$ for $i = 1, 2$ by Lemma E.1.

Thus, $\mathcal{T}_1$ is a supersolution to the boundary value problem [**C.2**] under μ_1. We have also noted that $\mathcal{T}_2(0) = \mathcal{T}_1(0) = 1$, and $\mathcal{T}_2'(m_1^\star) < 0 = \mathcal{T}_1'(m_1^\star)$, and so by the Comparison Principle in Proposition A.4, it follows that $\mathcal{T}(m,\mu_2) \leqslant \mathcal{T}(m,\mu_1)$ for all m. □

Corollary C.10. The credit yield spread Δ is decreasing in μ.

Proof. As noted in the proof of Proposition C.2, $\Delta = \varphi(\mathcal{T}(m,\mu))$, where $\varphi(x) = rx/(1-x)$ is increasing and convex. By Proposition C.9, $\mathcal{T}(m,\mu)$ is decreasing in μ, so it follows that Δ, for a given level of cash reserves, is also decreasing in μ. □

C.3. Comparative Statics in λ

The HJB equation can be written as a variational inequality as follows:

$$\Psi(w, F, F', F'', \lambda) := \min\left[rF(w) - \mu - \gamma w F'(w) - \Phi(F''(w), \lambda), F'(w) + 1\right] = 0 \tag{C.5}$$

with $F(0) = 0$, where $\Phi(\Gamma, \lambda) := \max_{\sigma\in\Sigma}\left(\frac{1}{2}\sigma^2\lambda^2\Gamma - \rho(\sigma)\right)$ as before. We first show that F decreases in λ, and the solution to [C.5] is $F(w, \lambda)$.

Proposition C.11. Let $F(w, \lambda)$ be the solution to the [C.5]. Then, $\lambda_1 < \lambda_2$ implies $F(w, \lambda_1) \geqslant F(w, \lambda_2)$ for all $w \geqslant 0$.

Proof. Recall that for any λ, $F''(w, \lambda) \leqslant 0$ for all $w \geqslant 0$, with a strict inequality when $w \in [0, w^\star)$. Because $\Gamma \leqslant 0$, $\Phi(\Gamma, \lambda)$ is decreasing in λ. Therefore, we have $0 = \Psi\big(F(w, \lambda_2), \lambda_2\big) \geqslant \Psi\big(F(w, \lambda_2), \lambda_1\big)$. Thus, $F(w, \lambda_2)$ is a subsolution to [C.5] $\Psi(F, \lambda_1) = 0$. Because $F(0, \lambda_1) = F(0, \lambda_2)$, we conclude by the Comparison Principle in Theorem 5 that $F(w, \lambda_1) \geqslant F(w, \lambda_2)$ for all $w \geqslant 0$. □

Corollary C.12. The payment boundary $w^\star$ is increasing in λ.

Proof. The payment boundary is the intersection of $F(w, \lambda)$ and the line $w \mapsto \mu r^{-1} - \gamma r^{-1} w$. Because $F(w, \lambda)$ is decreasing in λ, this point of intersection must be lower, and occur at a higher w, ie, $w^\star$ is increasing in λ. □

To consider the effect of a change in λ on the optimal level of monitoring, it is useful to consider the principal's problem as a function of cash reserves. In particular, consider the change of variable $m = w\lambda^{-1}$, which gives us $\hat{F}(m, \lambda) = F(m\lambda, \lambda)$. Then, we obtain the variational inequality

$$\hat{\Psi}(m, \hat{F}, \hat{F}', \hat{F}'', \lambda) := \min\left[r\hat{F}(m) - \mu - \gamma m \hat{F}'(m) - \hat{\Phi}(\hat{F}''(m)), \hat{F}'(m) + \lambda\right] = 0 \tag{C.6}$$

with the boundary condition $\hat{F}(0) = 0$, and where $\hat{\Phi}(\Gamma) := \max_{\sigma\in\Sigma}\left(\frac{1}{2}\sigma^2\Gamma - \rho(\sigma)\right)$. We also let

$$\hat{\sigma}(m) := \arg\max_{\sigma\in\Sigma}\left(\tfrac{1}{2}\sigma^2\hat{F}''(m) - \rho(\sigma)\right) \tag{C.7}$$

denote the optimal choice of monitoring as a function of cash reserves. The advantage of this change in perspective, demonstrated next, is that the nonlinear operator $r\hat{F}(m) - \mu - \gamma m\hat{F}'(m) - \hat{\Phi}(\hat{F}''(m))$ is independent of λ.

In what follows, we suppress the dependence of $\hat{F}$ on λ where this dependence is not emphasized for comparison.

Proposition C.13. Let $\hat{F}(m,\lambda)$ be a solution to [C.6]. Then, $\lambda_1 \leqslant \lambda_2$ implies $\hat{F}(m,\lambda_2) \leqslant \hat{F}(m,\lambda_1)$.

Proof. By assumption, $\hat{\Psi}(m,\hat{F},\hat{F}',\hat{F}'',\lambda_1) = 0$. But we also have

$$\min\left[r\hat{F}(m,\lambda_1) - \mu - \gamma m\hat{F}'(m,\lambda_1) - \hat{\Phi}(\hat{F}''(m,\lambda_1)), \hat{F}'(m,\lambda_1) + \lambda_2\right] \geqslant 0$$

which implies $\hat{F}(m,\lambda_1)$ is a supersolution to the nonlinear differential equation $\hat{\Psi}(\cdot,\lambda_2) = 0$ in [C.6]. Because $\hat{F}(m,\lambda_2)$ is a solution (and hence a subsolution) to $\hat{\Psi}(\cdot,\lambda_2) = 0$, it follows from the Comparison Principle in Theorem 5 that $\hat{F}(m,\lambda_1) \geqslant \hat{F}(m,\lambda_2)$. □

To understand how optimal monitoring $\hat{\sigma}$ and the dividend payment threshold $m^\star = w^\star/\lambda$ vary with λ, we need to understand how $\hat{F}''(m,\lambda)$ changes with λ. The following lemma describes this behaviour.

Lemma C.14. Let $\hat{F}(m,\lambda)$ be a solution to [C.6]. Then, $\lambda_1 \leqslant \lambda_2$ implies $\hat{F}'(m,\lambda_1) \geqslant \hat{F}'(m,\lambda_2)$ and $\hat{F}''(m,\lambda_1) \leqslant \hat{F}''(m,\lambda_2)$.

Proof. Let $G(m) := r\hat{F}(m) - \gamma m\hat{F}'(m) - \mu = \hat{\Phi}(\hat{F}''(m))$ and notice that $\hat{\Phi}(\Gamma) = \max_\sigma\left(\frac{1}{2}\sigma^2\Gamma - \rho(\sigma)\right)$ is a strictly increasing function of Γ. Therefore, $\hat{\Phi}^{-1}$ is a well defined and strictly increasing function. If $G(m)$ is increasing in λ, then it follows that $\hat{F}''$ is also increasing in λ. In addition, $G(m)$ increasing in λ implies, by Proposition C.13, that $G(m) - r\hat{F}(m) + \mu = -\gamma m\hat{F}'(m)$ is increasing in λ, ie, $\hat{F}'(m,\lambda)$ is decreasing in λ. Thus, all that remains is to show that $G(m)$ is increasing in λ.

To see that $G(m)$ increases in λ, observe first that

$$\begin{aligned} G'(m) &= (r-\gamma)\hat{F}'(m) - \gamma m\hat{F}''(m) \\ &= \underbrace{(r-\gamma)(\gamma m)^{-1}r\hat{F}(m)}_{=:\mathscr{K}(m,\lambda)} + \underbrace{(\gamma-r)(\gamma m)^{-1}G(m) - \gamma m\hat{\Phi}^{-1}(G(m))}_{=:\mathscr{G}(m,G)} \\ &= \mathscr{G}(m,G(m)) + \mathscr{K}(m,\lambda) \end{aligned}$$

Thus, G is the solution to the differential equation $G'(m) = \mathscr{G}(m,G(m)) + \mathscr{K}(m,\lambda)$ for each λ. By Proposition C.13, we see that $\mathscr{K}(m,\lambda)$ is increasing in λ. Thus, by the Comparison Theorem for first order differential equations—see, for instance, Birkhoff and Rota (1989, Theorem 8, p.30)—we find that $G(m,\lambda_1) \leqslant G(m,\lambda_2)$, which completes the proof. □

Corollary C.15. The dividend payment threshold $m^\star = w^\star/\lambda$ is decreasing in λ.

Proof. By definition, $m_i^\star$ satisfies $\hat{F}''(m_i^\star, \lambda_i) = 0$ for $i = 1, 2$ where $\lambda_1 < \lambda_2$. By Lemma C.14, $\hat{F}''(m, \lambda)$ is increasing in λ. Therefore, $0 = \hat{F}''(m_2^\star, \lambda_2) \geqslant \hat{F}''(m_2^\star, \lambda_1)$, which implies $m_2^\star \leqslant m_1^\star$. □

We can now describe how optimal monitoring changes with λ.

Proposition C.16. The optimal $\hat{\sigma}(m)$ is increasing in λ. Thus, monitoring is decreasing in λ.

Proof. Notice that $\frac{1}{2}\sigma^2 \hat{F}''(m, \lambda) - \rho(\sigma)$ has increasing differences in (σ, λ) because by Proposition C.13, $\hat{F}''(m, \lambda)$ is increasing in λ. Therefore, by Topkis' Theorem, $\hat{\sigma}(m, \lambda) = \arg\max_{\sigma \in \Sigma} \left[\frac{1}{2}\sigma^2 \hat{F}''(m, \lambda) - \rho(\sigma)\right]$ is also increasing in λ. □

We now show that stock prices are also monotone in λ.

Proposition C.17. The stock price $\mathcal{S}(m, \lambda)$ is decreasing in λ.

Proof. Consider $\lambda_1 < \lambda_2$, and let $\hat{\sigma}_i$ be the optimal policy under λ_i, $m_i^\star$ the cash reserve threshold, and $\mathcal{S}_i$ the corresponding stock price. We have already established in Corollary C.15 that $m^\star$ decreases in λ, which implies that $\mathcal{S}_1'(m_2^\star) > 1$. By virtue of being a solution to the boundary value problem (at $\lambda = \lambda_1$), we have

$$0 = r\mathcal{S}_1(m) - \gamma m \mathcal{S}_1'(m) - \tfrac{1}{2}\hat{\sigma}_1^2 \mathcal{S}_1''(m) \leqslant r\mathcal{S}_1(m) - \gamma m \mathcal{S}_1'(m) - \tfrac{1}{2}\hat{\sigma}_2^2 \mathcal{S}_1''(m) \tag{C.8}$$

where the inequality is because $\hat{\sigma}_1(m) \leqslant \hat{\sigma}_2(m)$ (by Proposition C.16) and because $\mathcal{S}_i'' < 0$ for $i = 1, 2$ by Proposition C.1.

Thus, $\mathcal{S}_1$ is a supersolution to the boundary value problem [C.1] when $\lambda = \lambda_2$. We have also noted that $\mathcal{S}_2(0) = \mathcal{S}_1(0) = 0$, and $\mathcal{S}_1'(m_2^\star) > \mathcal{S}_2'(m_2^\star) = 1$; the Comparison Principle in Proposition A.4 now implies that $\mathcal{S}(m, \lambda_1) \geqslant \mathcal{S}(m, \lambda_2)$ for all m. □

We now show that the expected discounted hitting time is increasing in λ.

Proposition C.18. The expected discounted liquidation time $\mathcal{T}(m)$ in [C.2] is increasing in λ.

Proof. Let $\lambda_1 < \lambda_2$, $\hat{\sigma}_i$ be the optimal policy under λ_i, $m_i^\star$ the cash reserve threshold, and $\mathcal{T}_i$ the corresponding expected discounted liquidation time in [C.2] when $\lambda = \lambda_i$. We have already established in Corollary C.15 that $m^\star$ decreases in λ, which implies that $\mathcal{T}_1'(m_2^\star) < 0$. Because $\mathcal{T}_1$ solves [C.2] when $\lambda = \lambda_1$, we have

$$0 = r\mathcal{T}_1(m) - \gamma m \mathcal{T}_1'(m) - \tfrac{1}{2}\hat{\sigma}_1^2 \mathcal{T}_1''(m) \geqslant r\mathcal{T}_1(m) - \gamma m \mathcal{T}_1'(m) - \tfrac{1}{2}\hat{\sigma}_2^2 \mathcal{T}_1''(m) \tag{C.9}$$

where the inequality is because $\hat{\sigma}_1(m) \leqslant \hat{\sigma}_2(m)$ (by Proposition C.16) and because $\mathcal{T}_i'' > 0$ for $i = 1, 2$ by Lemma E.1.

Thus, $\mathcal{T}_1$ is a subsolution to the boundary value problem [C.2] when $\lambda = \lambda_2$. We have also noted that $\mathcal{T}_2(0) = \mathcal{T}_1(0) = 1$, and $\mathcal{T}_1'(m_2^\star) < 0 = \mathcal{T}_2'(m_2^\star)$, and so by the Comparison Principle from Proposition A.4, it follows that $\mathcal{T}(m, \lambda_1) \leqslant \mathcal{T}(m, \lambda_2)$ for all m. □

Corollary C.19. The credit yield spread Δ is increasing in λ.

Proof. As noted in the proof of Proposition C.2, $\Delta = \varphi(\mathcal{T}(m, \lambda))$, where $\varphi(x) = rx/(1-x)$ is increasing and convex. By Proposition C.18, $\mathcal{T}(m, \lambda)$ is increasing in λ, so it follows that Δ, for a given level of cash reserves, is also increasing in λ. □

C.4. Proof of Theorem 3

We now collate the results above and complete the proof of Theorem 3. Part (i) of Theorem 3 is established in Corollary C.6 for the monotonicity in μ, while the monotonicity in λ is Lemma C.14. That monitoring is increasing in the firm's risk aversion now follows from Proposition C.7 (for μ) and Proposition C.16 (for λ). Part (iii) follows from the concavity of F whereby sensitivity $Z_t = \lambda\sigma(t)$. By part (ii), $\sigma(t)$ decreases in μ and increases in λ, which establishes part (iii).

To see part (iv), let $X_t = \mathbf{E}_t[\int_0^\tau e^{-r(s-t)}\,\mathrm{d}t]$. The arguments above show that we can write $X_t = \mathcal{X}(M_t)$. In particular, given w_0 (equivalently, m_0), $X_0 = 1 - \mathbf{E}_0[e^{-r\tau}] = 1 - \mathcal{T}(m_0)$. By Proposition C.9, X_0 is increasing in μ, while Proposition C.18 shows X_0 is decreasing in λ.

For part (v), suppose $\mu_1 > \mu_2$. By part (iv), it follows that $X_0^{(1)} \geqslant X_0^{(2)}$, where $X_0^{(j)} = \mathbf{E}^{\Phi_j}[\int_0^{\tau_j} e^{rt}\,\mathrm{d}t]$, where $\Phi_j = (C^j, \tau_j, \sigma_{*j})$ is the optimal contract when $\mu = \mu_j$, for $j = 1, 2$. Because Φ_2 is optimal at μ_2, it follows that

$$F(w, \mu_2) = \mathbf{E}^{\Phi_2}\left[\int_0^{\tau_2} e^{-rt}\big(\mu_2 - \rho(\sigma_{*2}(t))\big)\,\mathrm{d}t\right] \geqslant \mathbf{E}^{\Phi_1}\left[\int_0^{\tau_1} e^{-rt}\big(\mu_2 - \rho(\sigma_{*1}(t))\big)\,\mathrm{d}t\right]$$

which is true if, and only if,

$$\mathbf{E}^{\Phi_1}\left[\int_0^{\tau_1} e^{-rt}\rho(\sigma_{*1}(t))\,\mathrm{d}t\right] - \mathbf{E}^{\Phi_2}\left[\int_0^{\tau_2} e^{-rt}\rho(\sigma_{*2}(t))\,\mathrm{d}t\right] \geqslant \mu_2\big(X_0^{(1)} - X_0^{(2)}\big)$$

But $X_0^{(1)} \geqslant X_0^{(2)}$ by part (iv), so the claim is proved.

References

Alvarez, O. (1994). "A Singular Stochastic Control Problem in an Unbounded Domain." *Communications in Partial Differential Equations* 19(11-12), 2075–2089.

Ashbaugh-Skaife, H., D. W. Collins, and R. LaFond (2008). *Corporate Governance, Risk and Cost of Equity*. Working Paper. University of Wisconsin at Madison.

Bebchuk, L. A., A. Cohen, and A. Ferrell (2009). "What Matters in Corporate Governance?" *Review of Financial Studies* 22(2), 783–827.

Bengtsson, O. and S. A. Ravid (2015). "Location Specific Styles and US Venture Capital Contracting." *Quarterly Journal of Finance* 5(3).

Benth, F. E., K. H. Karlsen, and K. Reikvam (2001). "Optimal Portfolio Selection with Consumption and Nonlinear Integro-differential Equations with Gradient Constraint: A Viscosity Solution Approach." *Finance and Stochastics* 5(3), 275–303.

Berle Jr., A. A. and G. C. Means (1930). "Corporations and the Public Investor." *American Economic Review* 20(1), 54–71.

Bernstein, S., X. Giroud, and R. R. Townsend (2016). "The Impact of Venture Capital Monitoring." *Journal of Finance* 71(4), 1591–1622.

Bhojraj, S. and P. Sengupta (2003). "Effect of Corporate Governance on Bond Ratings and Yields: The Role of Institutional Investors and Outside Directors." *Journal of Business* 76(3), 455–475.

Biais, B., T. Mariotti, G. Plantin, and J.-C. Rochet (2007). "Dynamic Security Design: Convergence to Continuous Time and Asset Pricing Implications." *Review of Economic Studies* 74(2), 345–390.

Birkhoff, G. and G.-C. Rota (1989). *Ordinary Differential Equations*. Fourth Edition. NJ, USA: John Wiley & Sons.

Branzoli, N. and F. Fringuellotti (2022). *The Effect of Bank Monitoring on Loan Repayment*. Tech. rep. No. 923. Federal Reserve Bank of New York.

Brick, I. E., O. Palmon, and J. K. Wald (2012). "Too Much Pay-Performance Sensitivity?" *Review of Economics and Statistics* 94(1), 287–303.

Chen, M., P. Sun, and Y. Xiao (2020). "Optimal Monitoring Schedule in Dynamic Contracts." *Operations Research* 68(5), 1285–1314.

Clementi, G. L. and H. A. Hopenhayn (2006). "A Theory of Financing Constraints and Firm Dynamics." *Quarterly Journal of Economics* 121, 229–265.

Core, J. E., W. R. Guay, and T. O. Rusticus (2006). "Does Weak Governance Cause Weak Stock Returns? An Examination of Firm Operating Performance and Investors' Expectations." *Journal of Finance* 61(2), 655–687.

Core, J. and W. R. Guay (2002). "Estimating the Value of Employee Stock Option Portfolios and Their Sensitivities to Price and Volatility." *Journal of Accounting Research* 40(3), 613–630.

Crandall, M. G., H. Ishii, and P.-L. Lions (1992). "User's Guide to Viscosity Solutions of Second-Order Partial Differential Equations." *Bulletin of the American Mathematical Society* 27(1), 1–67.

Dai, L., Y. Wang, and M. Yang (2021). *Dynamic Contracting with Flexible Monitoring*. Working Paper. Shanghai Jiao Tong University.

David, M. H. A., V. G. Panas, and T. Zariphopoulou (1993). "European Option Pricing with Transaction Costs." *SIAM Journal on Control and Optimization* 31(2), 470–493.

DeAngelo, H., L. DeAngelo, and R. M. Stulz (2006). "Dividend Policy and Earned/Contributed Capital Mix: A Test of the Life-Cycle Theory." *Journal of Financial Economics* 81(2), 227–254.

DeMarzo, P. M. and M. J. Fishman (2007a). "Agency and Optimal Investment Dynamics." *Review of Financial Studies* 20(1), 151–188.

DeMarzo, P. M. and M. J. Fishman (2007b). "Optimal Long-Term Financial Contracting." *Review of Financial Studies* 20, 2079–2128.

DeMarzo, P. M., M. J. Fishman, Z. He, and N. Wang (2012). "Dynamic Agency and the q Theory of Investment." *Journal of Finance* 67(6), 2295–2340.

DeMarzo, P. and Y. Sannikov (2006). "Optimal Security Design and Dynamic Capital Structure in a Continuous-Time Agency Model." *Journal of Finance* 61(6), 2681–2724.

Demougin, D. and C. Fluet (2001). "Monitoring versus Incentives." *European Economic Review* 45(9), 1741–1764.

Demsetz, H. and K. Lehn (1985). "The Structure of Corporate Ownership: Causes and Consequences." *Journal of Political Economy* 93(6), 1155–1177.

Dumas, B. (1991). "Super Contact and Related Optimality Conditions." *Journal of Economic Dynamics and Control* 15, 675–685.

Dupire, B. (1994). "Pricing with a Smile." *Risk* 7(1), 18–20.

Fahlenbrach, R. (2009). "Shareholder Rights, Boards, and CEO Compensation." *Review of Finance* 13(1), 81–113.

Fernandes, N., M. Ferreira, P. Matos, and K. Murphy (2013). "Are US CEOs Paid More? New International Evidence." *Review of Financial Studies* 26, 323–367.

Georgiadis, G. and B. Szentes (2020). "Optimal Monitoring Design." *Econometrica* 88(5), 2075–2107.

Gilbarg, D. and N. S. Trudinger (2001). *Elliptic Partial Differential Equations of Second Order*. Berlin: Springer-Verlag.

Gompers, P. A. and S. N. Kaplan (2022). *Advanced Introduction to Private Equity*. Cheltenham, UK: Edward Elgar.

Gompers, P. A., S. N. Kaplan, and V. Mukharlyamov (2016). “What Do Private Equity Firms (Say They) Do?” *Journal of Financial Economics* 121(3), 449–476.

Gompers, P., J. Ishii, and A. Metrick (2003). “Corporate Governance and Equity Prices.” *Quarterly Journal of Economics* 118(1), 107–156.

Gompers, P. and J. Lerner (2006). *The Venture Captial Cycle*. Second Edition. Cambridge, MA: MIT Press.

Graf, S. (1979). “A Measurable Selection Theorem for Compact-Valued Maps.” *Manuscripta Mathematica* 27(4), 341–352.

Himmelberg, C. P., R. G. Hubbard, and D. Palia (1999). “Understanding the Determinants of Managerial Ownership and the Link Between Ownership and Performance.” *Journal of Financial Economics* 53(3), 353–384.

Holmström, B. (1979). “Moral Hazard and Observability.” *Bell Journal of Economics* 10(1), 74–91.

Holmström, B. and J. Tirole (1993). “Market Liquidity and Performance Monitoring.” *Journal of Political Economy* 101(4), 678–709.

Kaplan, S. N. and P. Strömberg (2003). “Financial Contracting Theory Meets the Real World: An Empirical Analysis of Venture Capital Contracts.” *Review of Economic Studies* 70(2), 281–315.

Kaplan, S. and J. Rauh (2010). “Wall Street and Main Street: What Contributes to the Rise in the Highest Incomes?” *Review of Financial Studies* 23(3), 1004–50.

Krishna, R. V., G. Lopomo, and C. R. Taylor (2013). “Stairway to Heaven or Highway to Hell: Liquidity, Sweat Equity, and the Uncertain Path to Ownership.” *RAND Journal of Economics* 44(1), 104–127.

Lando, D. and A. Mortensen (2004). “On the Pricing of Step-Up Bonds in the European Telecom Sector.” *Journal of Credit Risk* 1(1), 71–110.

Larcker, D. and B. Tayan (2011). *Corporate Governance Matters*. Upper Saddle River, NJ: Pearson Education.

Lerner, J. (1995). “Venture Capitalists and the Oversight of Private Firms.” *Journal of Finance* 50, 301–318.

Li, A. and M. Yang (2020). “Optimal Incentive Contract with Endogenous Monitoring Technology.” *Theoretical Economics* 15(3), 1135–1173.

Masson, R. T. (1972). “The Creation of Risk Aversion by Imperfect Capital Markets.” *American Economic Review* 62(1), 77–86.

Milgrom, P. R. and J. Roberts (1992). *Economics, Organizations, and Management*. NJ, USA: Prentice Hall.

Milgrom, P. and I. Segal (2002). “Envelope Theorems for Arbitrary Choice Sets.” *Econometrica* 70(2), 583–601.

Oberman, A. (2007). “The Convex Envelope is the Solution of a Nonlinear Obstacle Problem.” *Proceedings of the American Mathematical Society* 135(6), 1689–1694.

Oksendal, B. (2010). *Stochastic Differential Equations*. Sixth edition. Berlin: Springer-Verlag.

Orlov, D. (2018). *Frequent Monitoring in Dynamic Contracts*. Working Paper. University of Wisconsin at Madison.

Parigi, B. M., L. Pelizzon, and E.-L. von Thadden (2015). *A Corporate Governance Asset Pricing Model: Theory and Evidence*. Tech. rep. University of Mannheim.

Piskorski, T. and M. M. Westerfield (2016). “Optimal Dynamic Contracts with Moral Hazard and Costly Monitoring.” *Journal of Economic Theory* 166, 242–281.

Sannikov, Y. (2008). “A Continuous-Time Version of the Principal-Agent Problem.” *Review of Economic Studies* 75(3), 957–984.

Sannikov, Y. (2012). “Contracts: The Theory of Dynamic Principal-Agent Relationships and the Continuous-Time Approach.” In: *Advances in Economics and Econometrics: Tenth World Congress*. Ed. by D. Acemoglu, M. Arellano, and E. Dekel. Cambridge University Press. Chap. 4, 89–124.

Shleifer, A. and R. W. Vishny (1997). “A Survey of Corporate Governance.” *Journal of Finance* 52(2), 737–783.

Smith, A. (1776). *An Inquiry into the Nature and Causes of the Wealth of Nations*. Strand, London: W. Strahan and T. Cadell.

Soner, H. M. (1986a). “Optimal Control with State-space Constraint I.” *SIAM Journal on Control and Optimization* 24(3), 552–561.

Soner, H. M. (1986b). “Optimal control with state-space constraint. II.” *SIAM Journal on Control and Optimization* 24(6), 1110–1122.

Tirole, J. (2006). *The Theory of Corporate Finance*. Princeton, NJ: Princeton University Press.

Touzi, N. (2013). *Optimal Stochastic Control, Stochastic Target Problems, and Backward SDEs*. New York, NY: Springer.

Vafeas, N. (1999). “Board Meeting Frequency and Firm Performance.” *Journal of Financial Economics* 53(1), 113–142.

Varas, F., I. Marinovic, and A. Skrzypacz (2020). “Random Inspections and Periodic Reviews: Optimal Dynamic Monitoring.” *Review of Economic Studies* 87(6), 2893–2937.

Weisbach, M. S. (1988). “Outside Directors and CEO Turnover.” *Journal of Financial Economics* 20, 431–460.

Wintoki, M. B., J. S. Linck, and J. M. Netter (2012). “Endogeneity and the Dynamics of Internal Corporate Governance.” *Journal of Financial Economics* 105(3), 581–606.

Online Appendix: Not for Publication

D. Proofs from Section 4

Recall that $\hat{Y}_t$ is the process that is observed by the principal, and hence the contract is conditioned on.

Proof of Lemma 4.2. Fix a contract $\Phi = (C, \tau, \sigma)$. Let $\hat{Y}$ be a reporting strategy for the agent. Recall our assumption that $\hat{Y}$ is absolutely continuous with respect to Y. This implies $\mathrm{d}\hat{Y} = (\mu - \Delta_t)\,\mathrm{d}t + \sigma_t\,\mathrm{d}B_t$ where Δ_t is the instantaneous diversion of output. His utility from such a strategy, for all $t \in [0, \tau]$, is

[**D.1**]
$$\begin{aligned} V_t &:= \mathbf{E}_t^{\hat{Y},\sigma}\left[\int_0^\tau e^{-\gamma s}\big(\mathrm{d}C_s + \lambda(\mathrm{d}Y_s - \mathrm{d}\hat{Y}_s)\big)\right] \\ &= \int_0^t e^{-\gamma s}\big(\mathrm{d}C_s + \lambda(\mathrm{d}Y_s - \mathrm{d}\hat{Y}_s)\big) + e^{-\gamma t} W_t \end{aligned}$$

where W_t is the process defined in [**4.1**]. But for a fixed $\hat{Y}$ and contract, we find that (V_t) is a martingale, and so by the Martingale Representation Theorem there exists a process $Z = (Z_t)$ such that $V_t = \int_0^t e^{-\gamma s} Z_s\,\mathrm{d}B_s$. From this and [**D.1**], we find that

[**D.2**]
$$e^{-\gamma t} W_t = \int_0^t e^{-\gamma s} Z_s\,\mathrm{d}B_s - \int_0^t e^{-\gamma s}\big(\mathrm{d}C_s + \lambda(\underbrace{\mathrm{d}Y_s - \mathrm{d}\hat{Y}_s}_{=a_s\sigma_s})\big)$$

Writing this in differential form (and cancelling $e^{-\gamma t}$ throughout), we obtain

[**D.3**]
$$\mathrm{d}W_t = \gamma W_t\,\mathrm{d}t - \big(\mathrm{d}C_t + \lambda(\mathrm{d}Y_s - \mathrm{d}\hat{Y}_t)\big) + Z_t\,\mathrm{d}B_t$$

Noting that $\mathrm{d}B_t = \sigma_t^{-1}[\mathrm{d}Y_t - \mu \mathrm{d}t]$ and substituting in [**D.3**] completes the proof. □

Lemma 4.2 also now lets us characterize incentive compatibility for the agent.

Proof of Lemma 4.3. Suppose the contract is incentive-compatible. By the Comparison Principle for BSDEs Touzi (2013, Theorem 10.4, p159) or equivalently, following Sannikov (2008) and DeMarzo and Sannikov (2006), it is optimal for the agent to minimize the drift of the SDE in [**D.3**]. Using Girsanov's Theorem, we can rewrite [**D.3**] as

[**D.4**]
$$\mathrm{d}W_t = \gamma W_t \mathrm{d}t - \left[\mathrm{d}C_t + \lambda(\mathrm{d}Y_t - \mathrm{d}\hat{Y}_t)\right] + Z_t\left[\mathrm{d}\hat{B}_t^{\Delta} + \sigma_t^{-1}(\mathrm{d}Y_t - \mathrm{d}\hat{Y}_t)\right]$$

where $\mathrm{d}B_t = \mathrm{d}\hat{B}_t^{\Delta} + \sigma_t^{-1}(\mathrm{d}Y_t - \mathrm{d}\hat{Y}_t)$. For truthtelling (ie, $\mathrm{d}Y_t = \mathrm{d}\hat{Y}_t$) to be optimal, it must be that the contract specifies Z_t and σ_t such that for all t, $-\lambda + Z_t\sigma_t^{-1} \geqslant 0$, ie, $Z_t \geqslant \lambda\sigma_t$.

(This is the content of the Comparison Principle for BSDEs.) The sufficiency of this condition follows from the Comparison Principle for BSDEs. Alternatively, the argument in DeMarzo and Sannikov (2006) may be adapted to our setting. □

E. Securities in Section 7

We begin by establishing that the stock price function is well defined and solves the boundary value problem in [C.1]. We then prove Proposition C.1.

E.1. Proofs from Appendix C.1

Proof sketch of Proposition C.1. In contrast to BMPR, our M process has a volatility σ_* that varies with M. But because σ_* is progressively measurable, Itô's lemma still applies. Thus, an Itô expansion for $e^{-rt}\mathcal{S}(M_t)$ and following BMPR delivers the result. □

Following the arguments above for $\mathcal{S}$, it is not hard to show that $T_s = \mathcal{T}(M_s)$ for all $s \in [0, \tau)$, and that $\mathcal{T}$ is the solution to the following boundary value problem in [C.2].

Lemma E.1. The process T_t is given by $T = \mathcal{T}(m)$, where $\mathcal{T}$ is a solution to the boundary value problem [C.2], and $m \in [0, m^\star]$. Moreover, the expected discounted time to dissolution is a strictly decreasing and strictly convex function of m, the level of cash reserves.

Proof. That $\mathcal{T}$ is the solution the boundary value problem in [C.2] follows the same steps as above in the proof of Proposition C.1 above, and so is omitted.

To see that $\mathcal{T}$ is strictly decreasing, fix $m_1 < m_0$, and define the stopping time $\xi := \min\{t : M_t = m_1 \mid M_0 = m_0\}$. Then, $\mathcal{T}(m_0) = \mathbf{E}[e^{-r\xi}\mathcal{T}(m_1)] = \mathcal{T}(m_1)\,\mathbf{E}[e^{-r\xi}] < \mathcal{T}(m_1)$ because $\mathbf{E}[e^{-r\xi}] \in (0, 1)$, which holds because $\mathbf{E}[e^{-r\xi}] < \infty$ almost surely.

To see that $\mathcal{T}$ is strictly convex, consider the boundary value problem in [C.2]. Thus, $\mathcal{T}''(m) = r\mathcal{T}(m) - \gamma m\mathcal{T}'(m)$. Because $\mathcal{T}(m) > 0$ for all $m \in (0, m^\star]$ and $\mathcal{T}$ is decreasing, so $\mathcal{T}'(m) < 0$, it follows that $\mathcal{T}''(m) > 0$, ie, $\mathcal{T}$ is strictly convex. □

Proof of Proposition C.2. Let $\Delta = \varphi(T) := rT/(1-T)$. It is easy to see that φ is strictly increasing and strictly convex. By Lemma E.1, $T = \mathcal{T}(m)$. Therefore, we can write $\Delta = \delta(m)$, where $\delta(m) := \varphi(\mathcal{T}(m))$. Differentiation shows that $\delta'(m) = \varphi'(\cdot)\mathcal{T}'(m) < 0$, which follows because φ is increasing while $\mathcal{T}$ is decreasing. The function δ is a composition of two convex functions, and hence is also convex. Moreover, on every interval where $\hat{\sigma}$ is constant, we find $\delta''(m) = \varphi''(\cdot)[\mathcal{T}'(m)]^2 + \varphi'(\cdot)\mathcal{T}''(m) > 0$, where we again use the strict monotonicity and convexity of both φ and $\mathcal{T}$. □

E.2. Existence of Stock Price Function

Recall that the optimal monitoring strategy is given by

[E.1] $$\sigma_*(w) \in \arg\max_{\sigma\in\Sigma}\left[\frac{\lambda^2\sigma^2(w)}{2}F''(w) - \rho(\sigma)\right]$$

Because Σ is a finite set, the function $\sigma_*(w)$ is piecewise constant. We recall from [7.1] that $\mathrm{d}M_t = \gamma M_t\,\mathrm{d}t - \lambda^{-1}\,\mathrm{d}C_t^* + \lambda\hat{\sigma}(m)\,\mathrm{d}B_t$ is the cash reserve process of the firm (where $\hat{\sigma}(m) = \sigma_*(\lambda m)$ is also defined in [C.7]), and the stock price is given by

[E.2] $$S_t := \lambda^{-1}\,\mathbf{E}_t\left[\int_t^\tau e^{-r(s-t)}\,\mathrm{d}C_s^*\right]$$

Integrating the Itô expansion of $e^{-rt}M_t$ from t to τ, where the dynamics of M_t are from [7.1], and noting $M_\tau = 0$, we see that

[E.3] $$S_t = M_t + (\gamma - r)\,\mathbf{E}_t\left[\int_t^\tau e^{-r(s-t)}M_s\,\mathrm{d}s\right]$$

Because M is a Markov process, it follows from [E.3] that

[E.4] $$\mathbf{E}_t\left[\int_t^\tau e^{-r(s-t)}M_s\,\mathrm{d}s\right] = \mathbf{E}\left[\int_t^\tau e^{-r(s-t)}M_s\,\mathrm{d}s\middle|M_t\right]$$

The following lemma records that the stock price at time t is given by a function of M_t, namely $S_t = \mathcal{S}(M_t)$.

Lemma E.2. Let M_t follow [7.1] and S_t be as in [E.2]. Then, $S_t = \mathcal{S}(M_t)$, where

$$\mathcal{S}(m) := m + (\gamma - r)\,\mathbf{E}\left[\int_t^\tau e^{-r(s-t)}M_s\,\mathrm{d}s\,\middle|\,M_t = m\right]$$

Proof. By [E.4], we have $S_t = \mathcal{S}(M_t)$. □

We shall now show that $\mathcal{S}$ satisfies the boundary value problem [C.1], recapitulated as:

[E.5] $$\begin{gathered} -\tfrac{1}{2}\hat{\sigma}^2(m)\phi'' - \gamma m\phi' + r\phi = 0 \\ \phi(0) = 0 \quad \text{and} \quad \phi'(m^\star) = 1 \end{gathered}$$

where $\hat{\sigma}$ is the optimal level of monitoring (see [C.7]) and $m^\star$ is the payment boundary.

Note that $\inf_{m\geqslant 0}\hat{\sigma}(m) > 0$, and so the boundary value problem in [E.5] is strongly elliptic. Because $\hat{\sigma}$ is piecewise constant, we cannot expect [E.5] to have a C^2 solution. Instead, as we will see below, [E.5] has a solution in a weak sense, namely a C^1 solution that is piecewise $\mathbf{C}^2$ (indeed, piecewise $\mathbf{C}^\infty$); at the points where $\hat{\sigma}$ jumps, the solution $(\mathcal{S}'')$ has distinct left- and right-limits.

Lemma E.3. For any stopping time $\nu \leqslant \tau$, we have

$$\mathcal{S}(m) = \mathbf{E}_m\left[\lambda^{-1}\int_0^\nu e^{-rs}\,\mathrm{d}C_s^* + e^{-r\nu}\mathcal{S}(M_\nu)\right] \tag{E.6}$$

Proof. By definition of S_t in [E.2] and Lemma E.2, we have

$$\begin{aligned}\mathcal{S}(m) &= \lambda^{-1}\,\mathbf{E}_m\left[\int_0^\tau e^{-rs}\,\mathrm{d}C_s^*\right]\\ &= \mathbf{E}_m\left[\lambda^{-1}\int_0^\nu e^{-rs}\,\mathrm{d}C_s^* + \lambda^{-1}e^{-r\nu}\int_\nu^\tau e^{-r(s-\nu)}\,\mathrm{d}C_s^*\right]\\ &= \lambda^{-1}\,\mathbf{E}_m\left[\int_0^\nu e^{-rs}\,\mathrm{d}C_s^*\right] + \lambda^{-1}\,\mathbf{E}_m\left[e^{-r\nu}\,\mathbf{E}_{M_\nu}\left[\int_\nu^\tau e^{-r(s-\nu)}\,\mathrm{d}C_s^*\right]\right]\end{aligned}$$

Because M is time-homogeneous and by Lemma E.2 and the definition of S, we have

$$\lambda^{-1}\,\mathbf{E}_{M_\nu}\left[\int_\nu^\tau e^{-r(s-\nu)}\,\mathrm{d}C_s^*\right] = \mathcal{S}(M_\nu)$$

Therefore, [E.6] holds, completing the proof. □

The following lemma shows that if a solution exists to the boundary value problem in [E.5] with a suitably modified boundary condition at $m^\star$, then the solution must be $\mathcal{S}$.

Lemma E.4. Let v be a solution to

$$\begin{aligned}&-\tfrac{1}{2}\hat{\sigma}^2(m)v'' - \gamma m v' + rv = 0\\ &v(0) = 0 \quad \text{and} \quad v(m^\star) = \mathcal{S}(m^\star)\end{aligned} \tag{E.7}$$

and suppose v is $\mathbf{C}^1$ and piecewise-$\mathbf{C}^2$. Then, $v(m) = \mathcal{S}(m)$.

Proof. Let $\nu := \inf\{t \geqslant 0 : M_t = 0 \text{ or } M_t = m^\star\}$. Because v is $\mathbf{C}^2$ except at finitely many points, and because $\hat{\sigma}$ is bounded away from 0, it follows that v'' is bounded. Thus, by Problem 4.8 (p 57) of Oksendal (2010) we may apply Itô's formula to obtain

$$\begin{aligned}e^{-r\nu}v(M_\nu) = v(m) &+ \int_0^\nu e^{-rs}\Big(\tfrac{1}{2}\hat{\sigma}^2(M_s)v''(M_s) + \gamma M_s v'(M_s) - rv(M_s)\Big)\,\mathrm{d}s\\ &- \lambda^{-1}\int_0^\nu e^{-rs}v'(M_s)\,\mathrm{d}C_s^* + \int_0^\nu e^{-rs}v'(M_s)\hat{\sigma}(M_s)\,\mathrm{d}B_s\end{aligned} \tag{E.8}$$

Because v satisfies the boundary value problem in [**E.7**], the display above simplifies to

[E.9] $$e^{-r\nu}v(M_\nu) = v(m) - \lambda^{-1}\int_0^\nu e^{-rs}v'(M_s)\,\mathrm{d}C_s^* + \int_0^\nu e^{-rs}v'(M_s)\hat{\sigma}(M_s)\,\mathrm{d}B_s$$

Because $\nu \leqslant \tau < \infty$ a.s., and from the boundedness of $\hat{\sigma}$ and v', we conclude that the stochastic integral is zero in expectation. Using the boundary condition $v(M_\nu) = \mathcal{S}(M_\nu)$, we can now write

[E.10] $$v(m) = \mathbf{E}_m[e^{-r\nu}\mathcal{S}(M_\nu)] + \lambda^{-1}\,\mathbf{E}_m\left[\int_0^\nu e^{-rs}v'(X_s)\,\mathrm{d}C_s^*\right]$$

Because C^* is a local time at $m^\star$, ν is (weakly) smaller than the hitting time of M to $m^\star$, and so $\int_0^\nu e^{-rs}v'(M_s)\,\mathrm{d}C_s^* = 0$, so that $v(m) = \mathbf{E}_m[e^{-r\nu}\mathcal{S}(M_\nu)]$. Finally, from Lemma E.3 and the fact that $\int_0^\nu e^{-rs}\,\mathrm{d}C_s^* = 0$, we have $v(m) = \mathbf{E}_m[e^{-r\nu}\mathcal{S}(M_\nu)] = \mathcal{S}(m)$, completing the proof. □

We now show that the boundary value problem in [**E.7**] has a $\mathbf{C}^1$ solution that is piecewise $\mathbf{C}^2$.

Lemma E.5. Assume that $\hat{\sigma}(m)$ is piecewise Lipschitz continuous, bounded, and $\inf_m \hat{\sigma}(m) > 0$. Then, the boundary value problem in E.7 has a C^1 and piecewise C^2 solution.

Proof. Consider the following equivalent version of [**E.7**],

[E.11] $$\begin{gathered} -\tfrac{1}{2}\psi'' - b(m)\psi' + c(m)\psi = 0 \\ \psi(0) = 0 \quad \text{and} \quad \psi(m^\star) = \mathcal{S}(m^\star) \end{gathered}$$

where $b(m) = \gamma m/\hat{\sigma}^2(m)$ and $c(m) = r/\hat{\sigma}^2(m)$. As b and c are bounded functions, the equivalent problem satisfies the assumption of Gilbarg and Trudinger (2001, Corollary 9.18). Therefore, there exists a weak solution, v, in the Sobolev space $W^{2,p}([0, m^\star])$ for all $p \geqslant 1$. By the Sobolev embedding result, Gilbarg and Trudinger (2001, Corollary 7.11), v is $\mathbf{C}^1$. As $\hat{\sigma}$ is piecewise continuous, on any interval of continuity of $\hat{\sigma}$, $b(m)$ and $c(m)$ are continuous, and so by Gilbarg and Trudinger (2001, Theorem 6.14), v is $\mathbf{C}^2$ on such an interval. □

Thus, it follows from Lemma E.4 that $\mathcal{S}$ is $\mathbf{C}^1$ and piecewise $\mathbf{C}^2$. Next, we show that $\mathcal{S}'(m^\star) = 1$.

Proposition E.6. The function $\mathcal{S} : [0, m^\star] \to \mathbb{R}$ satisfies $\mathcal{S}'(m^*) = 1$.

Proof. Because $\mathcal{S}$ is $\mathbf{C}^2$ except at finitely many points, and because $\hat{\sigma}$ is bounded away from 0, it follows that $\mathcal{S}''$ is bounded. Thus, by Problem 4.8 (p 57) of Oksendal (2010) we may still apply Itô's formula on $\mathcal{S}(M_\tau)$ to obtain

$$\begin{aligned} e^{-r\tau}\mathcal{S}(M_\tau) = \mathcal{S}(m) + \int_0^\tau e^{-rs}\Big(\tfrac{1}{2}\hat{\sigma}^2(M_s)\mathcal{S}''(M_s) + \gamma M_s\mathcal{S}'(M_s) - r\mathcal{S}(M_s)\Big)\,\mathrm{d}s \\ - \lambda^{-1}\int_0^\tau e^{-rs}\mathcal{S}'(M_s)\,\mathrm{d}C_s^* + \int_0^\tau e^{-rs}\mathcal{S}'(M_s)\hat{\sigma}^2(M_s)\,\mathrm{d}B_s \end{aligned} \tag{E.12}$$

Because $M_\tau = 0$, $\mathcal{S}(M_\tau) = 0$ and by [**E.5**], we can write

$$\mathcal{S}(m) = \lambda^{-1}\int_0^\tau e^{-rs}\mathcal{S}'(M_s)\,\mathrm{d}C_s^* + \int_0^\tau e^{-rs}\mathcal{S}'(M_s)\hat{\sigma}^2(M_s)\,\mathrm{d}B_s$$

Both $\hat{\sigma}$ and $\mathcal{S}'$ being bounded, after taking expectations, we obtain

$$\mathcal{S}(m) = \lambda^{-1}\,\mathbf{E}_m\left[\int_0^\tau e^{-rs}\mathcal{S}'(M_s)\,\mathrm{d}C_s^*\right] = \lambda^{-1}\mathcal{S}'(m^\star)\,\mathbf{E}_m\left[\int_0^\tau e^{-rs}\,\mathrm{d}C_s^*\right]$$

where the last equality follows from the properties of the local time $\mathrm{d}C^*$. By the definition of S_t in [**E.2**] and using Lemma E.2, we obtain

$$\big[\mathcal{S}'(m^\star) - 1\big]\lambda^{-1}\,\mathbf{E}_m\left[\int_0^\tau e^{-rs}\,\mathrm{d}C_s^*\right] = 0$$

which implies $\mathcal{S}'(m^\star) = 1$, as claimed. □

We are now ready to prove all the elements of Proposition C.1.

Proof. We have already shown that $\mathcal{S}$ is a solution to the boundary value problem [**C.1**], and is piecewise-$\mathbf{C}^2$ and $\mathbf{C}^1$ everywhere. The Comparison Principle in Proposition A.4 shows that it is the *unique* solution. All that remains is to show the strict monotonicity of concavity of $\mathcal{S}$.

To show $\mathcal{S}$ is strictly increasing, we divide the DE in [**C.1**] by $\frac{-2}{\hat{\sigma}^2(m)}$ and multiply it by $\mathcal{G}(m) := \exp\left(\gamma\int_0^m \frac{2n}{\hat{\sigma}^2(n)}\,\mathrm{d}n\right)$. Since $\mathcal{G}'(m) = \mathcal{G}(m)\frac{2\gamma m}{\sigma^2(m)}$, we have

$$(\mathcal{G}\mathcal{S}')' = r\tfrac{2}{\hat{\sigma}^2(m)}\mathcal{G}\mathcal{S} \tag{E.13}$$

Integrating both sides, we obtain

$$\mathcal{G}(m)\mathcal{S}'(m) = \mathcal{S}'(0) + r\int_0^m \tfrac{2}{\hat{\sigma}^2(n)}\mathcal{G}(n)\mathcal{S}(n)\,\mathrm{d}n \tag{E.14}$$

By the definition of $\mathcal{G}(m)$, we have

$$\mathcal{S}'(m) = \exp\Big(-\gamma\int_0^m \tfrac{2n}{\hat{\sigma}^2(n)}\,\mathrm{d}n\Big)\mathcal{S}'(0) + 2r\int_0^m \exp\Big(-\gamma\int_n^m \tfrac{2n}{\hat{\sigma}^2(n)}\,\mathrm{d}n\Big)\mathcal{S}(n)\tfrac{\mathrm{d}n}{\hat{\sigma}^2(n)} \tag{E.15}$$

Because $\mathcal{S}(m) > 0$ for $m > 0$ and $\mathcal{S}(0) = 0$, it follows that $\mathcal{S}'(0) \geqslant 0$. Therefore, [**E.15**] implies that $\mathcal{S}'(m) > 0$ for $m > 0$, and so $\mathcal{S}$ is strictly increasing.

To show the strict concavity, we evaluate $\mathcal{S}''(m)$ by taking the derivative of [**E.15**] on each interval that hosts a regime. Assume that σ_i is the optimal regime between $[0, m_{s_1}]$. Then, $\mathcal{S}(m)$ satisfies $-\frac{1}{2}\hat{\sigma}^2(m)\mathcal{S}'' - \gamma m\mathcal{S}' + r\mathcal{S} = 0$ with $\mathcal{S}(0) = 0$. It follows from [**E.15**] that $\mathcal{S}$ is second order continuously differentiable on $[0, m_{s_1}]$. As a result, $\mathcal{S}'(m)$ satisfies $-\frac{1}{2}\hat{\sigma}_i^2\mathcal{K}'' - \gamma m\mathcal{K}' - (\gamma - r)\mathcal{K} = 0$. By integrating the above equation, we obtain

$$\mathcal{S}''(m) = \exp\Big(-\gamma\tfrac{2m^2}{\hat{\sigma}_i^2}\Big)\Big(\mathcal{S}''(0) - \tfrac{2(\gamma-r)}{\hat{\sigma}_i^2}\int_0^m \exp\Big(\tfrac{2n^2}{\hat{\sigma}_i^2}\Big)\mathcal{S}'(n)\,\mathrm{d}n\Big) \tag{E.16}$$

Since $\mathcal{S}(0) = 0$, from the ODE for $\mathcal{S}$, we deduce that $\mathcal{S}''(0) = 0$. In addition, since $\mathcal{S}' > 0$, we conclude that

$$\mathcal{S}''(m) = -\exp\Big(-\gamma\tfrac{2m^2}{\hat{\sigma}_i^2}\Big)\tfrac{2(\gamma-r)}{\hat{\sigma}_i^2}\int_0^m \exp\Big(\tfrac{2n^2}{\hat{\sigma}_i^2}\Big)\mathcal{S}'(n)\,\mathrm{d}n < 0 \tag{E.17}$$

In particular, at the switching point we have $\mathcal{S}''(m-) < 0$. Now, we assume that the concavity of $\mathcal{S}$ is established for all $m \leqslant m_s$ for some switching point $m_s < m^\star$ and regime j is optimal right before m_s. Consider the interval $[m_s, m_{s'}]$, where $m_{s'}$ is either the switching point right after m_s or is $m^\star$, and assume that regime i is optimal in this interval. Since $\mathcal{S}$ is continuously differentiable, if follows from the DE in [**C.1**] that

$$\sigma_j^2\mathcal{S}''(m_s-) = \sigma_i^2\mathcal{S}''(m_s+) \tag{E.18}$$

and, in particular, $\mathcal{S}''(m_s+) < 0$. By rewriting [**E.16**] on $[m_s, m_{s'}]$, we obtain that

$$\mathcal{S}''(m) = \exp\Big(-\gamma\tfrac{2m^2}{\hat{\sigma}_i^2}\Big)\Big(\mathcal{S}''(m_s+) - \tfrac{2(\gamma-r)}{\hat{\sigma}_i^2}\int_0^m \exp\Big(\tfrac{2n^2}{\hat{\sigma}_i^2}\Big)\mathcal{S}'(n)\,\mathrm{d}n\Big) \tag{E.19}$$

which implies that $\mathcal{S}$ is concave on $[m_s, m_{s'}]$ and completes the proof. □

F. Directly Controlling Agency Costs

We consider here a variant of the model where the principal directly controls λ_t, the agency cost, or more precisely, the agent's marginal benefit from diverting cash, while the volatility of output remains fixed at σ_0.

For concreteness, suppose $\lambda_t = a_t\lambda_0$, and $a_t \in A := \{a_0, a_1, \ldots, a_n\}$, where $a_0 = 1$, and $a_i > a_{i+1}$ for all $i = 1, \ldots, n-1$. The instantaneous cost of choosing $a_t = a_i$ is $\kappa(a_i)$.

As in the main model, we can write the agent's promised utility process as

$$\mathrm{d}W_t = \gamma W_t \,\mathrm{d}t - \mathrm{d}C_t + \lambda_t(\mathrm{d}Y_t - \mathrm{d}\hat{Y}_t) + Z_t \,\mathrm{d}B_t$$

where Z_t is a sensitivity process, just as in [4.2]. Incentive compatibility is now characterised as requiring $Z_t \geqslant \lambda_t\sigma_0 = a_t\lambda_0\sigma_0$.

Now, consider the change of variables as follows: Let $\sigma_i := a_i\sigma_0$ and $\lambda := \lambda_0$, and define the function $\rho(\sigma_i) := \kappa(a_i)$, so that for all $t \geqslant 0$ we have $\lambda_t\sigma_0 = \lambda\sigma_t$. But the right hand side is precisely the model studied in the paper, and the cost of controlling λ_t is exactly the cost of changing σ_t.

Thus, the evolution of promised utility in both models is the same, as are the principal's costs, which implies that the principal's value function is identical in both models. It is now easy to show that the optimal contract, as a function of W_t, is also identical, ie, the payment boundary is identical.

It is useful to see how to implement the optimal contract, given that λ_t is changing over time. We define $M_t = W_t/\lambda_0$. Then, we may write the evolution of M as

$$\mathrm{d}M_t = \gamma M_t \,\mathrm{d}t - \lambda_0^{-1} \,\mathrm{d}C_t + \underbrace{a_t\sigma_0}_{=\sigma_t} \,\mathrm{d}B_t$$

which is exactly as in [7.1]. The stock price is $S_t = \mathbf{E}_t\left[\int_t^\tau e^{-r(s-t)}\lambda_0^{-1} \,\mathrm{d}C_s\right]$, and it is easy to see that this is the same stock price as in the main model where σ_t is controlled. Similarly, we consider bonds that have a coupon payment of $\mu - (\gamma - r)M_t$, so that bond price is $D_t = \mathbf{E}_t\left[\int_t^\tau e^{-r(s-t)}[\mu - (\gamma - r)M_s] \,\mathrm{d}s\right]$. Because the stock and bond prices are the same as in the main model, and the evolution of cash reserves is the same, both stock and bond prices are deterministic functions of cash reserves, and these functions satisfy the same boundary value problems as they do for the main model.

In the implementation, let the agent own a fraction λ_0 of the stock, the principal hold a fraction $1 - \lambda_0$ of the stock, and all the debt, ie, the bonds, so that coupon payments on the bond are paid to the principal. Thus, the properties of the implementation and all subsequent results remain the same.

G. Proof of Comparison Principle in Theorem 5

We will provide here a proof of the Comparison Principle in Theorem 5. We begin with some preliminaries.

The subsolution property for [**A.1**] requires that for any test function ϕ

$$r\phi(x) - \gamma x\phi'(x) - H(\phi''(x)) \leqslant 0 \quad \text{or} \quad \phi'(x) + \eta \geqslant 0 \tag{G.1}$$

while the supersolution requires both inequalities to hold simultaneously, ie,

$$r\phi(x) - \gamma x\phi'(x) - H(\phi''(x)) \geqslant 0 \quad \text{and} \quad \phi'(x) + \eta \geqslant 0 \tag{G.2}$$

where the function H is defined in [**A.2**].

Remark G.1. If u is a classical, C^2, solution of [**A.1**], then it is a viscosity solution. This is because of the classical fact that if $u - \phi$ attains a local minimum at x_0 with $u(x_0) = \phi(x_0)$, then $u'(x_0) = \phi'(x_0)$ and $u''(x_0) \geqslant \phi''(x_0)$. Because $\mathscr{H}$ is nonincreasing in the second order derivative, we have

$$\mathscr{H}(x_0, \phi''(x_0), \phi'(x_0), \phi(x_0)) \geqslant \mathscr{H}(x_0, u''(x_0), u'(x_0), u(x_0)) = 0 \tag{G.3}$$

Similar calculation hold for the subsolution property.

Lemma G.2. Any supersolution V to [**A.1**] is concave and, therefore, continuous, differentiable almost everywhere with right derivative, $V'(x+)$, and left derivative, $V'(x-)$, both existing at all points, In addition, $V'(x+)$ and $V'(x-)$ are nonincreasing in x, and satisfy $V'(x+) \leqslant V'(x-)$.

Proof. Let V be a supersolution to [**A.1**]. To establish the concavity of V, it is sufficient to show, by Oberman (2007, Theorem 1), that for any smooth test function ϕ such that $0 = V(x) - \phi(x) = \min_{y \geqslant 0}(V(y) - \phi(y))$, we have $\phi''(x) \leqslant 0$. The supersolution property in [**G.2**] requires that we have

$$r\phi(x) - \gamma x\phi'(x) - \max_{\sigma \in \Sigma, z \geqslant \lambda\sigma} \left[\tfrac{1}{2} z^2 \phi''(x) + \mu - \rho(\sigma)\right] \geqslant 0 \tag{G.4}$$

If $\phi''(x) > 0$, then $\max_{z \geqslant \lambda\sigma} \left[\frac{1}{2} z^2 \phi''(x) + \mu - \rho(\sigma\right] = \infty$ for any choice of $\sigma \in \Sigma$, which contradicts the posited supersolution property of V. The other properties are classically known for concave functions. □

We introduce the notion of *strict* viscosity solutions.

Definition G.3. A lower (respectively upper) semicontinuous function u is a *strict* super (respectively sub) solution if for all smooth functions ϕ such that $u - \phi$ has a local minimum (respectively maximum) at x_0 with $u(x_0) = \phi(x_0)$, we have $u(0) \geqslant \varepsilon > 0$ (respectively $u(0) \leqslant -\varepsilon < 0$) and $\mathscr{H}(x_0, \phi(x_0), \phi'(x_0), \phi''(x_0)) \geqslant \varepsilon > 0$ (respectively $\mathscr{H}(x_0, \phi(x_0), \phi'(x_0), \phi''(x_0)) \leqslant -\varepsilon < 0$), where $\varepsilon > 0$ is independent of x_0.

To establish the comparison result, we follow a method initiated in Soner (1986a,b) and applied later to singular control problems in Alvarez (1994), David, Panas, and Zariphopoulou (1993), and Benth, Karlsen, and Reikvam (2001), and adapt useful techniques from Benth, Karlsen, and Reikvam (2001). The main tool is the existence on a *strict supersolution*. In our case, the candidate is

$$\bar{V}(x) = (\mu + \varepsilon)/r - (\eta - \varepsilon)x \tag{G.5}$$

Because H is strictly increasing in Γ, it is straightforward to verify the following in a viscosity sense;

$$\begin{cases} r\bar{V} - \mu - \gamma x \bar{V}' - H(\bar{V}'') \geqslant \varepsilon > 0 \\ \bar{V}'(x) + \eta = \varepsilon > 0 \\ \bar{V}(0) > 0 \end{cases} \tag{G.6}$$

$\bar{V}$ is called a strict viscosity supersolution, because all the above inequalities are strict.

Lemma G.4. For any supersolution V of [A.1] and any $\theta \in (0,1]$, $V_\theta := (1-\theta)V + \theta\bar{V}$ is a strict supersolution of [A.1], where $\bar{V}$ is defined in [G.5]

Proof. Let ϕ and $\bar{\phi}$ be second order differentiable test functions for V and $\bar{V}$, respectively, at x_0. Then, $\phi_\theta := (1-\theta)\phi + \theta\bar{\phi}$ yields is a test function for V_θ. Because H is a convex, we have

$$\begin{aligned} &r\phi_\theta - \gamma x_0 \phi_\theta' - H(\phi_\theta'') \\ &\quad \geqslant (1-\theta)\left[r\phi - \gamma x_0 \phi_\theta' - H(\phi'')\right] + \theta\left[r\bar{\phi} - \gamma x_0 \bar{\phi}' - H(\bar{\phi}'')\right] \\ &\quad \geqslant \theta\varepsilon > 0 \end{aligned}$$

where the last inequality above is because $r\phi_\theta(x_0) - \gamma x \phi_\theta' - H(\bar{\phi}''(x_0)) \geqslant \varepsilon > 0$. Similar strict inequalities for the gradient constraint and the boundary condition hold because of linearity of these terms. □

Lemma G.5. Let V be a supersolution to [A.1]. Then, there exists $K \geqslant 0$ such that $V(x) \geqslant K - \eta x$. Furthermore, for any $\varepsilon > 0$ and $\theta \in (0,1)$, $V_\theta := (1-\theta)V + \theta\bar{V} \geqslant \bar{K} - (\eta - \varepsilon)x$ for some $\bar{K} > 0$.

Proof. By Lemma G.2 and [G.2], $V'(x\pm) + \eta \geqslant 0$ for all $x > 0$. By the properties of $V'(x\pm)$ in Lemma G.2, we conclude that V is absolutely continuous with respect to Lebesgue measure and therefore, by the second inequality in [G.2],

$$V(x) = V(0+) + \int_0^x V'(y+)\,\mathrm{d}y \geqslant V(0+) - \eta x$$

To show the bounds for a strict supersolution, notice that by Lemma G.4, $(1-\theta)V + \theta\bar{V}$ is a strict supersolution. Moreover, because $V(x) \geqslant K - \eta x$, it follows that $V_\theta(x) \geqslant \bar{K} - (1-\varepsilon)\eta x$ for some $\bar{K}$, which proves the claim. □

The next two lemmas are crucial in the proof of our Comparison Principle.

Lemma G.6. Let v and V be respectively, upper and lower semicontinuous and further assume that $v(x) - V(y) - \frac{\alpha}{2}|x-y|^2$ attains its local maximum at an interior point (x_*, y_*). Then, there exist real numbers X and Y such that $(\alpha(x_*, y_*), X)$ and $(\alpha(x_*, y_*), Y)$ are a superjet and a subjet of v and V, respectively, and $X \leqslant Y$.

Intuitively, each test function ϕ for a lower-semicontinuous function u at x_0 in Definition A.2 induces a *subjet* given by $(\phi'(x_0), \phi''(x_0))$. For a more formal definition of sub- and superjets, see Crandall, Ishii, and Lions (1992).

Proof. The existence of real numbers X and Y such that $(\alpha(x_*, y_*), X)$ and $(\alpha(x_*, y_*), Y)$ are a superjet and a subjet of v and V, respectively is a direct result of Crandall, Ishii, and Lions (1992, Theorem 3.2). In addition, Crandall, Ishii, and Lions (1992, Theorem 3.2) yields the inequality

$$A := \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leqslant 3\alpha \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} := \hat{A}$$

where the above inequality is understood in the order induced by the cone of positive definite matrices. Specifically, letting $\mathbf{1} = (1,1)$, we have $X - Y = \mathbf{1}A\mathbf{1}^\top \leqslant \mathbf{1}\hat{A}\mathbf{1}^\top = 0$, which completes the proof. □

Lemma G.7. Let $B > 0$ and v and $V : [0,\infty) \to \mathbb{R}$ be, respectively, upper and lower semicontinuous functions. Set $M := \max_{x\in[0,B]}\{v(x) - V(x)\} \geqslant 0$. Further, for $\alpha > 0$, set $M_\alpha := \max_{x,y\in[0,B]^2}\{v(x) - V(y) - \frac{\alpha}{2}(x-y)^2\}$, and let (x_α, y_α) be a maximiser. Then, up to a subsequence, and as $\alpha \to \infty$, we have $(x_\alpha, y_\alpha) \to (\hat{x}, \hat{x})$, $\alpha|x_\alpha - y_\alpha|^2 \to 0$, and $\lim_{\alpha\to\infty} M_\alpha = M$, where $M = v(\hat{x}) - V(\hat{x})$.

Proof. The proof follows from Crandall, Ishii, and Lions (1992, Lemma 3.1). □

We are now ready to prove Theorem 5.

Proof of Theorem 5. We only need to show the comparison result for a subsolution and a strict supersolution. To see this, notice that for any supersolution V, by Lemma G.4, $V_\theta := (1-\theta)V + \theta\bar{V}$, $\theta \in (0,1]$, is a strict supersolution, with $\bar{V}$ as in [**G.5**]. If $v \leqslant V_\theta$ holds for each $\theta \in (0,1]$, by sending $\theta \to 0$, we obtain $v \leqslant V$. Therefore, without loss in generality, we assume that V is a strict supersolution such that $V(x) \geqslant \bar{K} - (\eta - \varepsilon)x$ for some $\bar{K} > 0$ and all $\varepsilon > 0$, as in Lemma G.5.

Set $M := \sup_{x \in [0,\infty)}\{v(x) - V(x)\}$. If $M \leqslant 0$, we obtain $v(x) \leqslant V(x)$ for all $x \geqslant 0$ and the desired inequality. Therefore, for the rest of proof, we assume that $M > 0$ and find a contradiction.

By assumption, $v(x) \leqslant K - \eta x$, which implies $v(x) - V(x) \leqslant K - \bar{K} - \varepsilon x$, and thus, $\limsup_{x\to\infty} v(x) - V(x) = -\infty$. Therefore, $M = \sup_{x\in[0,B]}\{v(x) - V(x)\}$ for some $B \geqslant 0$ sufficiently large such that $v(B) - V(B) < 0$. On the other hand, since the v and V are sub and super solutions, respectively, we have $v(0) \leqslant 0$ and $V(0) \geqslant 0$. Therefore, $v(0) - V(0) \leqslant 0$ and M is attained at an interior point $\hat{x} \in (0, B)$.

Let $M_\alpha := \max_{x,y\in[0,B]^2}\{v(x) - V(y) - \frac{\alpha}{2}(x - y)^2\}$ and denote a maximum point of M_α by (x_α, y_α). By Lemma G.7, we have $\lim_{\alpha\to\infty} x_\alpha = \lim_{\alpha\to\infty} y_\alpha = \hat{x}$ and $\hat{x} \in (0, B)$ is a maximizer of $v(x) - V(x)$.

On the other hand, by Lemma G.6, $(\alpha(x_\alpha - y_\alpha), X_\alpha)$ a superjet for v at x_α and $(\alpha(x_\alpha - y_\alpha), Y_\alpha)$ is a subjet for V at y_α with $X_\alpha \leqslant Y_\alpha$. Note that as a supersolution, V is concave and, therefore, $X_\alpha \leqslant Y_\alpha \leqslant 0$ and we have $H(Y_\alpha) < \infty$. Because H is increasing, we have $H(X_\alpha) \leqslant H(Y_\alpha)$. From the definition of strict viscosity supersolution, we have

[G.7] $$rV(y_\alpha) - \gamma\alpha y_\alpha(x_\alpha - y_\alpha) - H(Y_\alpha) \geqslant \varepsilon > 0$$

[G.8] $$\alpha(x_\alpha - y_\alpha) + \eta > \varepsilon > 0$$

On the other hand, by the definition of subsolution, we have

[G.9] $$\min\left[rv(x_\alpha) - \gamma\alpha x_\alpha(x_\alpha - y_\alpha) - H(X_\alpha), \alpha(x_\alpha - y_\alpha) + \eta\right] \leqslant 0$$

Since $\alpha(x_\alpha - y_\alpha) + \eta > \varepsilon > 0$ holds, the above inequality reduces to

[G.10] $$rv(x_\alpha) - \gamma\alpha x_\alpha(x_\alpha - y_\alpha) - H(X_\alpha) \leqslant 0$$

From [**G.7**] and [**G.10**], we obtain

[G.11] $$r\big(v(x_\alpha) - V(y_\alpha)\big) - \gamma\alpha|x_\alpha - y_\alpha|^2 + H(Y_\alpha) - H(X_\alpha) \leqslant -\varepsilon < 0$$

Because H is increasing, we have $H(Y_\alpha) - H(X_\alpha) \geqslant 0$. Additionally, $M_\alpha = v(x_\alpha) - V(y_\alpha) - \frac{\alpha}{2}|x_\alpha - y_\alpha|^2$. Therefore,

[G.12] $$rM_\alpha + \alpha(r/2 - \gamma)|x_\alpha - y_\alpha|^2 \leqslant -\varepsilon < 0.$$

It follows from Lemma G.7 that $M_\alpha \to M$ and $\alpha|x_\alpha - y_\alpha|^2 \to 0$ as $\alpha \to \infty$, which yields $rM \leqslant -\varepsilon < 0$, which contradicts our assumption that $M > 0$, completing the proof. □

H. Regularity of the Value Function

We begin by showing that F is a continuous and concave solution of [A.1].

H.1. Continuity and Concavity of F

Priori to establishing the main results, we need the following definition.

Definition H.1 (Semicontinuous envelopes)**.** The lower- and uppersemicontinuous envelope of a function ϕ are defined, respectively by

$$\phi_*(x) := \liminf_{x' \to x} \phi(x') \qquad \text{and} \qquad \phi^*(x) := \limsup_{x' \to x} \phi(x') \tag{H.1}$$

If ϕ is locally bounded, then ϕ_* and ϕ^* are locally bounded.

We use semicontinuous envelopes of F in the proof of Lemma H.3. Therefore, we need to first show F is locally bounded. To maintain consistency with the notation in Appendix A.2, we use x instead of w for the promised utility process, which can be written as $\mathrm{d}X_t = \gamma X_t \,\mathrm{d}t - \mathrm{d}C_t - Z_t \,\mathrm{d}B_t$ under truthtelling, and where $Z_t \geqslant \lambda\sigma_t$ ensures incentive compatibility.

Lemma H.2. For all $x \geqslant 0$, we have $-x \leqslant F(x) \leqslant u(x) \leqslant \frac{\mu}{r} - x$, where $u(x)$ is the $\mathbf{C}^2$ solution of

$$\min\left\{ru - \mu - \tfrac{1}{2}\sigma_n^2 u'' - \gamma w u',\ u' + 1\right\} = 0, \qquad u(0) = 0 \tag{H.2}$$

where $\sigma_n = \min \Sigma$. In particular, $F(0) = 0$.

Proof. Because $\mu - \rho(\sigma) \leqslant \mu$ for all $\sigma \in \Sigma$, we have

$$F(x) \leqslant \sup_{\Xi \in \mathscr{IC}} \mathbf{E}_x^{\Xi}\left[\int_0^\tau e^{-rt}\Big(\mu\,\mathrm{d}t - \mathrm{d}C_t\Big)\right] \tag{H.3}$$

Recall that the set $\mathscr{IC}$ consists of all admissible $\Xi = (Z_t, \sigma_t, C_t)$ such that $Z_t \geqslant \sigma_t\lambda$, $\sigma_t \in \Sigma$, and C_t is an increasing cádlág process with $C_{0-} = 0$. In particular, $\mathscr{IC}$ is included inside the set $\mathscr{IC}_n$ of all admissible (Z_t, σ_t, C_t) such that $Z_t \geqslant \sigma_n\lambda$, $\sigma_t = \sigma_n$, and C_t is an increasing cádlág process with $C_{0-} = 0$. Therefore,

$$F(x) \leqslant \sup_{\Xi \in \mathscr{IC}} \mathbf{E}_x^{\Xi}\left[\int_0^\tau e^{-rt}\big(\mu\,\mathrm{d}t - \mathrm{d}C_t\big)\right] \leqslant \sup_{\Xi \in \mathscr{IC}_n} \mathbf{E}_x^{\Xi}\left[\int_0^\tau e^{-rt}\big(\mu\,\mathrm{d}t - \mathrm{d}C_t\big)\right] \tag{H.4}$$

The problem in the right-hand side above is a single regime problem where $\sigma = \sigma_n$, and the cost of monitoring is 0; this is the problem studied in DeMarzo and Sannikov (2006) and the value function for this problem is u. Thus,

$$F(x) \leqslant u(x) = \sup_{\Xi \in \mathscr{I}\mathscr{C}_{\min}} \mathbf{E}_x^{\Xi} \left[\int_0^{\tau} e^{-rt} \big(\mu \, \mathrm{d}t - \mathrm{d}C_t \big) \right] \tag{H.5}$$

which completes the proof. ☐

Lemma H.3. F is a continuous viscosity solution of [**A.1**].

Proof. We shall show that F_* (respectively F^*), the lower- (respectively upper-) semicontinuous envelope of F, is a supersolution (respectively subsolution) of [**A.1**]. By definition, F_* is lower semicontinuous and F^* is upper semicontinuous. By Lemma H.2, we have $-x \leqslant F_* \leqslant F \leqslant F^* \leqslant \mu/r - x$. Therefore, the conditions of the Comparison Principle, Theorem 5, hold and we have $F^* \leqslant F_*$, resulting in the equality $F^* = F_* = F$. In particular, it implies that F is continuous. We start by showing F_* is a supersolution.

(i) Let $x > 0$. Consider a smooth test function ϕ such that $0 = F_*(x) - \phi(x) = \min_{y \geqslant 0}(F_*(y) - \phi(y))$. We shall show that,

$$-H(\phi''(x)) - \gamma x \phi'(x) + r\phi(x) \geqslant 0 \quad \text{and} \quad 1 + \phi'(x) \geqslant 0 \tag{H.6}$$

We first establish that $1 + \phi'(x) \geqslant 0$. By the definition of F_*, there exists a sequence (y_n) such that $\lim_{n \to \infty} F(y_n) = F_*(x)$. By the dynamic programming principle, Touzi (2013, Theorem 3.3), we have

$$F(y_n) \geqslant \sup_{\substack{Z_t \geqslant \sigma_t \lambda \\ \sigma_t \in \Sigma, C}} \mathbf{E}_{y_n}^{\Xi} \left[\int_0^{\nu} e^{-rt} \Big((\mu - \rho(\sigma_t)) \, \mathrm{d}t - \mathrm{d}C_t \Big) + e^{-r\nu} F_*(X_\nu) \right] \tag{H.7}$$

where ν is an arbitrary stopping time bounded by τ and $\Xi = (Z_t, \sigma_t, C_t) \in \mathscr{I}\mathscr{C}$. Note that $F_*(X_\nu) - \phi(X_\nu) \geqslant \min_{y \geqslant 0}(F_*(y) - \phi(y)) = 0$. Therefore, $F_*(X_\nu) \geqslant \phi(X_\nu)$. If we set $\eta_n := F(y_n) - \phi(y_n)$, we can write [**H.7**] as

$$\phi(y_n) + \eta_n \geqslant \sup_{\substack{Z_t \geqslant \sigma(u_t)\lambda \\ u_t \in U, C}} \mathbf{E}_{y_n}^{\Xi} \left[\int_0^{\nu} e^{-rt} \Big((\mu - \rho(\sigma_t)) \, \mathrm{d}t - \mathrm{d}C_t \Big) + e^{-r\nu} \phi(X_\nu) \right] \tag{H.8}$$

By applying Itô's formula to $e^{-r\nu}\phi(X_\nu)$, we obtain

$$\begin{aligned} \eta_n \geqslant \sup_{\substack{Z_t \geqslant \lambda \sigma_t \\ \sigma_t \in \Sigma, C}} \mathbf{E}_{y_n}^{\Xi} \Bigg[\int_0^{\nu} e^{-rt} \Big(&(\mu - \rho(\sigma_t) + \tfrac{1}{2} Z_t^2 \phi''(X_t) \\ &+ \gamma X_t \phi'(X_t) - r\phi(X_t)) \, \mathrm{d}t - (1 + \phi'(X_t)) \, \mathrm{d}C_t \Big) \Bigg] \end{aligned} \tag{H.9}$$

Let us assume, by way of contradiction, that $1 + \phi'(x) < 0$. Then, there exists

ε-neighborhood of x, denoted by $\mathcal{N}_\varepsilon(x)$, such that $1 + \phi(y) < 0$ for $y \in \mathcal{N}_\varepsilon(x)$. We can choose ε sufficiently small so that $0 \notin \mathcal{N}_\varepsilon(x)$. Set $L_\varepsilon := -\sup_{y \in \mathcal{N}_\varepsilon(x)} 1 + \phi(y) < 0$, and $\nu_n := \sqrt{\eta_n} \wedge \inf\{t \geqslant 0 : X_t \notin \mathcal{N}_\varepsilon(x), X_0 = y_n\}$. For all n sufficiently large, we have $y_n \in \mathcal{N}_\varepsilon(x)$ and, therefore, $1 + \phi'(y_n) < 0$. In addition, the definition of ν_n implies that $X_t \in \mathcal{N}_\varepsilon(x)$ and, therefore, $1 + \phi'(X_t) < -L_\varepsilon < 0$ for all $t < \nu_n$. Note that [H.8] holds for any choice of $\Xi = (Z_t, \sigma_t, C_t) \in \mathcal{IC}$ that also satisfies promise keeping, whereby $\mathbf{E}^{\Xi}_{y_n}\left[\int_0^\nu e^{-\gamma t}\,\mathrm{d}C_t\right] = y_n$. In particular, [H.8] holds if we take $\sigma_t \equiv \sigma$ for an arbitrary $\sigma \in \Sigma$, $Z_t \equiv \lambda\sigma$, and $C^n_t \equiv 0$ for $t < \nu_n$ and $C^n_{\nu_n} = -X_{\nu_n}$, provided that C^n satisfies promise keeping.

We can directly verify promise keeping for this specific choice of C^n and $\tau = \nu_n$ by evaluating $\mathbf{E}^{\Xi}_{y_n}[\int_0^\tau e^{-\gamma t}\,\mathrm{d}C^n_t] = \mathbf{E}^{\Xi}_{y_n}[e^{-\gamma\nu_n}X_{\nu_n}]$. Because $C^n_t \equiv 0$ for $t < \nu_n$, $e^{-\gamma\nu_n}X_{\nu_n} = y_n + \lambda\sigma\int_0^{\nu_n} e^{-\gamma s}\,\mathrm{d}B_s$. Therefore, by the martingale property of the stochastic integral, $\mathbf{E}^{\Xi}_{y_n}[\int_0^\tau e^{-\gamma t}\,\mathrm{d}C^n_t] = y_n + \mathbf{E}^{\Xi}_{y_n}[\lambda\sigma\int_0^{\nu_n} e^{-\gamma s}\,\mathrm{d}B_s] = y_n$, and so promise keeping holds.

Therefore,

$$\begin{aligned}\eta_n &\geqslant \mathbf{E}^{\Xi}_{y_n}\left[\int_0^{\nu_n} e^{-rt}\Big(\big(\mu - \rho(\sigma) + \frac{z^2}{2}\phi''(X_t) + \gamma X_t\phi'(X_t) - r\phi(X_t)\big)\,\mathrm{d}t \right.\\ &\qquad\qquad\left. - (1 + \phi'(X_t))\,\mathrm{d}C^n_t\Big)\right]\\ &\geqslant \mathbf{E}^{\Xi}_{y_n}\left[\int_0^{\nu_n} e^{-rt}\Big(\mu - \rho(\sigma) + \frac{z^2}{2}\phi''(X_t) + \gamma X_t\phi'(X_t) - r\phi(X_t)\Big)\,\mathrm{d}t \right.\\ &\qquad\qquad\left. + L_\varepsilon\int_0^{\nu_n} e^{-rt}\,\mathrm{d}C^n_t\right]\end{aligned} \tag{H.10}$$

In the above, we used the bound $L_\varepsilon > 0$ on $-1 - \phi'(y)$ over $\mathcal{N}_\varepsilon(x)$. By the definition of C^n_t, we have $\mathbf{E}^{\Xi}_{y_n}[\int_0^{\nu_n} e^{-rt}\,\mathrm{d}C^n_t] = \mathbf{E}^{\Xi}_{y_n}[e^{-r\nu_n}X_{\nu_n}]$. Because $r < \gamma$, $\mathbf{E}^{\Xi}_{y_n}[e^{-r\nu_n}X_{\nu_n}] \geqslant \mathbf{E}^{\Xi}_{y_n}[e^{-\gamma\nu_n}X_{\nu_n}] = y_n$, and we can write

$$\eta_n \geqslant \mathbf{E}^{\Xi}_{y_n}\left[\int_0^{\nu_n} e^{-rt}\Big(\mu - \rho(\sigma) + \frac{z^2}{2}\phi''(X_t) + \gamma X_t\phi'(X_t) - r\phi(X_t)\Big)\,\mathrm{d}t + y_n L_\varepsilon\right] \tag{H.11}$$

Note that ε is independent of n. By sending $n \to \infty$ on both sides of the above inequality, and applying the dominated convergence theorem on the right-hand side, we obtain $0 \geqslant xL_\varepsilon > 0$, which is a contradiction. Thus, it must be that $1 + \phi'(x) \geqslant 0$.

We now show that $-H(\phi''(x)) - \gamma x\phi'(x) - \mu + r\phi(x) \geqslant 0$. Towards this end, we note that [H.8] holds for the choice of a constant $\sigma_t \equiv \sigma \in \Sigma$, with $Z_t \equiv \lambda\sigma$ and C^n_t as above. Consider the stopping time $\nu = \bar{\nu}_n := \frac{n}{n+1}\nu_n$. Because $\bar{\nu}_n < \nu_n$, $C^n_t \equiv 0$ on $t \leqslant \bar{\nu}_n$ and we can write

$$\eta_n \geqslant \mathbf{E}^{\Xi}_{y_n}\left[\int_0^{\bar{\nu}_n} e^{-rt}\Big(\mu - \rho(\sigma) + \frac{z^2}{2}\phi''(X_t) + \gamma X_t\phi'(X_t) - r\phi(X_t)\Big)\,\mathrm{d}t\right] \tag{H.12}$$

Let us denote by ω a generic path of process X, so that

[**H.13**] $$g(y_n, \omega) := \int_0^{\bar{\nu}_n(\omega)} e^{-rt}\Big((\mu - \rho(\sigma) + \frac{z^2}{2}\phi''(\omega_t) + \gamma\omega_t\phi'(\omega_t) - r\phi(\omega_t)\Big)\,\mathrm{d}t$$

By the mean value theorem (for integrals), for any sample path ω, there exists $\bar{t}(\omega) \in (0, \bar{\nu}_n(\omega))$, such that

[**H.14**] $$g(y_n, \omega) = \bar{\nu}_n(\omega)e^{-r\bar{t}(\omega)}\Big(\bar{\mu}(\beta) + \frac{z^2}{2}\phi''(\omega_{\bar{t}(\omega)}) + \gamma\omega_{\bar{t}(\omega)}\phi'(\omega_{\bar{t}(\omega)}) - r\phi(\omega_{\bar{t}(\omega)})\Big)$$

By a measurable selection theorem such as Graf (1979, Theorem 2.1), the function $\bar{t} : \omega \to (0, \infty)$ can be chosen to be measurable.

Notice that for each ω, $\lim_{n\to\infty}(\eta_n)^{-1/2}g(y_n, \omega)$ exists and is equal to $\mu - \rho(\sigma) + \frac{z^2}{2}\phi''(x) + \gamma x\phi'(x) - r\phi(x)$. Here we have used, $\bar{t}(\omega) < \bar{\nu}_n(\omega)$, and

$$\lim_{n\to\infty} \bar{t}(\omega) \leqslant \lim_{n\to\infty} \bar{\nu}_n(\omega) \leqslant \lim_{n\to\infty} \sqrt{\eta_n} = 0$$

as well as the continuity of sample path ω, $\lim_{n\to\infty} y_n = x$, the continuity of sample paths of X on the initial point y_n, and $\lim_{n\to\infty} \bar{\nu}_n/\sqrt{\eta_n} = 1$. On the other hand, $(\eta_n)^{-1/2}g(y_n, \omega)$ is bounded uniformly on ω. Therefore, by the Dominated Convergence Theorem, we have

[**H.15**] $$\eta_n^{-1/2}\,\mathbf{E}_{y_n}^{\Xi}[g(y_n, \omega)] = \mu - \rho(\sigma) + \frac{z^2}{2}\phi''(x) + \gamma x\phi'(x) - r\phi(x)$$

Dividing both sides of [**H.12**] by $\sqrt{\eta_n}$ and sending $n \to \infty$, we obtain

[**H.16**] $$0 \geqslant \mu - \rho(\sigma) + \tfrac{1}{2}z^2\phi''(x) + \gamma x\phi'(x) - r\phi(x)$$

where $\sigma \in \Sigma$ and $z \geqslant \lambda\sigma$ are arbitrary. Taking supremum over $\sigma \in \Sigma$ and $z \geqslant \lambda\sigma$, we obtain

[**H.17**] $$0 \geqslant \sup_{\substack{\sigma\in\Sigma \\ z\geqslant\lambda\sigma}} \Big\{\frac{z^2}{2}\phi''(x) + \mu - \rho(\sigma)\Big\} + \gamma x\phi'(x) - r\phi(x)$$

or equivalently $-H(\phi''(x)) - \gamma x\phi'(x) + r\phi(x) \geqslant 0$, as desired.

(ii) At the boundary point $x = 0$, we have $F(0) = 0$. We shall show that $F_*(0) = 0$. To see this, notice that $F_* \leqslant F$ implies $F_*(0) \leqslant 0$. Now by Lemma H.2, $F_*(0) = \liminf_{y\to 0} F(y) \geqslant \lim_{y\to 0}(-y) = 0$, which establishes the claim.

(iii) Consider a smooth test function ϕ such that $0 = F^*(x) - \phi(x) = \max_{y\geqslant 0}(F^*(y) - \phi(y))$. Without loss of generality, we can assume that x is the unique maximizer of $F^*(y) - \phi(y)$, $\phi(y) > F^*(y)$ for $y \neq x$. We shall show that at least one of the inequalities below holds:

$$\text{[H.18]} \qquad 0 \geqslant -H(\phi''(x)) - \gamma x \phi'(x) + r\phi(x) \quad \text{or} \quad 0 \geqslant 1 + \phi'(x)$$

We assume that both inequalities above are violated and provide a contradiction with the dynamic programming principle, Touzi (2013, Theorem 3.3), which requires that

$$\text{[H.19]} \qquad F(y_n) \leqslant \sup_{\substack{Z_t \geqslant \sigma(u_t)\lambda \\ u_t \in \mathcal{U} \\ C\text{: nondecreasing}}} \mathbf{E}^{\Xi}_{y_n}\left[\int_0^{\nu} e^{-rt}\Big((\mu - \rho(\sigma_t))\,\mathrm{d}t - \,\mathrm{d}C_t\Big) + e^{-r\nu}F^*(X_\nu)\right]$$

Towards a contradiction, we assume that there exists ε-neighborhood of x, $\mathcal{N}_\varepsilon(x)$, such that for all $y \in \mathcal{N}_\varepsilon(x)$, we have

$$\text{[H.20]} \qquad 1 + \phi'(y) > \varepsilon \ \text{ and} - H(\phi''(y)) - \gamma y \phi'(y) + r\phi(y) > \varepsilon$$

Because $\partial\mathcal{N}_\varepsilon(x)$, the boundary of $\mathcal{N}_\varepsilon(x)$, is a finite set, the maximum of $F^* - \phi$ on $\partial\mathcal{N}_\varepsilon(x)$ is attained and, because x is the unique global maximizer of $F^* - \phi$ on $\mathcal{N}_\varepsilon(x)$, the maximum on $\partial\mathcal{N}_\varepsilon(x)$ is negative. Set $\eta := -\max_{\partial\mathcal{N}_\varepsilon(x)} F^*(y) - \phi(y) > 0$. As in to part (i), we consider a sequence (y_n) such that $\lim_{n\to\infty} y_n = x$ and $\lim_{n\to\infty} F(y_n) = F^*(x) = \phi(x)$. Let $\nu_n := \inf\{t \geqslant 0 : X_t \notin \mathcal{N}_\varepsilon(x),\ X_0 = y_n\}$. Then, by applying Itô's formula on $e^{-r\nu_n}\phi(X_{\nu_n})$

$$\text{[H.21]} \qquad \mathbf{E}^{\Xi}_{y_n}\left[e^{-r\nu_n}\phi(X_{\nu_n})\right] = \phi(y_n) + \mathbf{E}^{\Xi}_{y_n}\left[\int_0^{\nu_n} e^{-rt}\bigg(\Big(\frac{Z_t^\intercal Z_t}{2}\phi''(X_t) + \gamma X_t \phi'(X_t) - r\phi(X_t)\Big)\,\mathrm{d}t - \phi'(X_t)\,\mathrm{d}C_t\bigg)\right]$$

Therefore, for any arbitrary σ_t, Z_t, and C_t with $Z_t \geqslant \lambda\sigma_t$, we have

$$\begin{aligned} \text{[H.22]} \qquad & \mathbf{E}^{\Xi}_{y_n}\left[\int_0^{\nu_n} e^{-rt}\big((\mu - \rho(\sigma_t))\,\mathrm{d}t - \,\mathrm{d}C_t\big) + e^{-r\nu_n}\phi(X_{\nu_n})\right] \\ &= \phi(y_n) + \mathbf{E}^{\Xi}_{y_n}\left[\int_0^{\nu_n} e^{-rt}\bigg(\Big(\mu - \rho(\sigma_t) + \frac{Z_t^\intercal Z_t}{2}\phi''(X_t) + \gamma X_t\phi'(X_t) - r\phi(X_t)\Big)\,\mathrm{d}t - (1 + \phi'(X_t))\,\mathrm{d}P_t\bigg)\right] < \phi(y_n) \end{aligned}$$

By definition of ν_n and using [H.20], we have

$$\text{[H.23]} \qquad \max\left\{\mu - \rho(\sigma_t) + \frac{Z_t^\intercal Z_t}{2}\phi''(X_t) + \gamma X_t\phi'(X_t) - r\phi(X_t), -(1 + \phi'(X_t))\right\} < -\varepsilon$$

Combining the above inequalities with [**H.22**] yields

$$\text{[H.24]} \quad \begin{aligned} &\mathbf{E}^{\Xi}_{y_n}\left[\int_0^{\nu_n} e^{-rt}\Big((\mu-\rho(\sigma_t))\,\mathrm{d}t - \,\mathrm{d}C_t\Big) + e^{-r\nu_n}\phi(X_{\nu_n})\right] \\ &\quad \leqslant \phi(y_n) - \varepsilon\,\mathbf{E}^{\Xi}_{y_n}\left[\int_0^{\nu_n} e^{-rt}\Big(\,\mathrm{d}t + \,\mathrm{d}C_t\Big)\right] < \phi(y_n) \end{aligned}$$

Note that because $X_\nu \in \partial\mathcal{N}_\varepsilon(x)$, we have $\phi(X_{\nu_n}) \geqslant F^*(X_{\nu_n}) + \eta$. Set $\eta_n := \phi(y_n) - F(y_n)$. Therefore, from [**H.22**], we obtain

$$\text{[H.25]} \quad \mathbf{E}^{\Xi}_{y_n}\left[\int_0^{\nu_n} e^{-rt}\Big((\mu-\rho(\sigma_t))\,\mathrm{d}t - \,\mathrm{d}C_t\Big) + e^{-r\nu_n}(F^*(X_{\nu_n}) + \eta)\right] \leqslant F(y_n) + \eta_n$$

Note that η is independent of y_n. Thus, for n sufficiently large, we have $\varepsilon_n := \mathbf{E}^{\Xi}_{y_n}[e^{-r\nu_n}]\eta - \eta_n > 0$ and

$$\text{[H.26]} \quad \mathbf{E}^{\Xi}_{y_n}\left[\int_0^{\nu_n} e^{-rt}\Big((\mu-\rho(\sigma_t))\,\mathrm{d}t - \,\mathrm{d}C_t\Big) + e^{-r\nu_n}F^*(X_{\nu_n})\right] \leqslant F(y_n) - \varepsilon_n$$

Because the σ_t, C_t, and $Z_t \geqslant \lambda\sigma_t$ are arbitrary, we have

$$\text{[H.27]} \quad \begin{aligned} &\sup_{\substack{Z_t \geqslant \lambda\sigma_t \\ \sigma_t \in \Sigma \\ C \text{nondecreasing}}} \mathbf{E}^{\Xi}_{y_n}\left[\int_0^{\nu_n} e^{-rt}\Big((\mu-\rho(\sigma_t))\,\mathrm{d}t - \,\mathrm{d}C_t\Big) + e^{-r\nu_n}F^*(X_{\nu_n})\right] \\ &\quad \leqslant F(y_n) - \varepsilon_n < F(y_n) \end{aligned}$$

which contradicts [**H.19**].

(iv) For the boundary point $x = 0$, we shall show $F^*(0) \leqslant 0$. By Lemma H.2, $F(x) \leqslant u(x)$ for all $x \geqslant 0$. Because u is a continuous function with $u(0) = 0$, we have $F^*(0) \leqslant u(0) = 0$. □

Corollary H.4. The value function F is concave.

Proof. By combining Lemma G.2, Theorem 5, Lemma H.2, and Lemma H.3, we immediately obtain the concavity of F. Note that by Oberman (2007, Theorem 1), all subjets of F are concave. □

H.2. Smoothness of F

In this section, we show that the value function is $\mathbf{C}^2$. First, we need to have an improved version of Lemma H.2.

Lemma H.5. For all $x \geqslant 0$, we have

[**H.28**] $$-x < u_0(x) \leqslant F(x) \leqslant u_n(x) \leqslant \mu/r - x$$

where u_0 and u_n are $\mathbf{C}^2$ solutions of

[**H.29**] $$\min\Big\{ -\tfrac{1}{2}\lambda^2\sigma_j^2 u'' - \gamma x u' - \mu + ru, u' + 1\Big\} = 0, \qquad u(0) = 0,$$

where $j \in \{0, n\}$. In addition, both functions u_0 and u_n are $\mathbf{C}^2$.

Proof. The function u_n is the value function where $\rho(\sigma_n) = 0$. This is the value function from DS, is $\mathbf{C}^2$, and clearly satisfies $F(x) \leqslant u_n(x) \leqslant \mu/r - x$. The function u_0 is DS's value function when $\sigma = \sigma_0$, and so is $\mathbf{C}^2$. It corresponds to the case where no monitoring is allowed, and so it follows that $-x < u_0(x) \leqslant F(x)$, which proves the claim. □

We recall that the value function F is the concave viscosity solution of [**A.1**]. As a result of concavity, the right and left derivatives $F'(x+)$ and $F'(x-)$, exist, are nonincreasing, and satisfy $F'(x-) \geqslant F'(x+) \geqslant -1$. When $F'(x+) > -1$, then F must satisfy

[**H.30**] $$0 = ru - x\gamma u' - H(u'')$$

Lemma H.6. Set $I := \{x : F'(x-) > -1\}$. Then, one of the following holds:
(i) there exists $x^\star \in (0, \infty)$ such that $I = [0, x^\star]$ and $F'(x^\star+) = -1$, or
(ii) $I = [0, \infty)$ and F has a asymptote with slope -1.

Proof. Because $\mu > 0$, we have $F'(0+) > -1$. The concavity of F implies that $x \mapsto F'(x\pm)$ is decreasing and $F'(x-) \geqslant F'(x+)$. The viscosity solution property of F implies that $F'(x+) \geqslant -1$ for all $x \geqslant 0$. If there exist a point, $x^\star$, such that $F'(x^\star+) = -1$, then $F'(x\pm) = -1$ for $x > x^\star$ and thus $I = [0, x^\star]$, and so (i) holds. Otherwise, we have $F'(x+) > -1$ for all $x \geqslant 0$ and thus, $I = [0, \infty)$. Then, by Lemma H.2, F has a asymptote with slope -1, and so (ii) holds. □

In Lemma H.6, if (2) holds, we define $x^\star := \infty$. Recall the definition of function H in [**A.2**] and Lemma A.1. For the sake of the proof of the following theorem, we extend H to $\mathbb{R}$ by

[**H.31**] $$\tilde{H}(\Gamma) := \begin{cases} H(\Gamma) & \Gamma \leqslant 0 \\ H(0) + H'(0-)\Gamma & \Gamma > 0 \end{cases}$$

Inheriting its properties from H, the function $\tilde{H}$ is convex, Lipschitz, and strictly increasing. In particular, $\tilde{H}$ is invertible and we denote $G := \tilde{H}^{-1}$. The function G is therefore concave, Lipschitz, and strictly increasing. We shall use these properties of G in the proof of the following theorem.

Theorem 6. *Let $x^\star$ be as in Lemma H.6 and let $x_0 \in (0, x^\star]$, with $x_0 < \infty$. Then, the second order ODE* [**H.30**] *with boundary condition $u(0) = 0$ and $u(x_0) = F(x_0)$ has a solution in $\mathbf{C}^2([0, x_0])$.*

Proof. We first write equation [**H.30**], as an ODE:

[**H.32**] $$u'' = G(ru - \gamma x u'), \qquad u(0) = 0, \qquad u(x_0) = F(x_0)$$

where $G = \tilde{H}^{-1}$ defined in [**H.31**]. Using Lemma H.3, one can easily verify that F is a viscosity solution of [**H.32**] on $[0, x^\star]$. Note that boundary value problem [**H.32**] is a Dirichlet problem that satisfies the conditions of Crandall, Ishii, and Lions (1992, Theorem 3.3). Therefore, it has a unique continuous viscosity solution. Because a classical solution is also a viscosity solution, it suffices to show that [**H.32**] has a classical solution in $\mathbf{C}^2([0, x_0])$.

To show the existence of a classical solution to [**H.32**], we establish the Schauder estimates, which relies on *a priori* estimates and the Schauder fixed point theorem; see, for example, Gilbarg and Trudinger (2001, Chapter 11) for more details. By Gilbarg and Trudinger (2001, Theorem 11.4), such a classical solution exists if we show that for any $\beta \in [0, 1]$, if u_β is a classical solution of

[**H.33**] $$u_\beta'' = \beta G(ru_\beta - \gamma x u_\beta'), \qquad u_\beta(0) = 0, \qquad u_\beta(x_0) = \beta F(x_0)$$

then $\sup\{|u_\beta(x)| + |u_\beta'(x)| : \beta \in [0, 1], \ \ x \in [0, x_0]\} < K$. In the following, the constant K can be different line by line.

First, we show that $\sup\{|u_\beta(x)| : \beta \in [0, 1], \ \ x \in [0, x_0]\} < K$. Consider the boundary value problems

[**H.34**] $$u_\pm'' = \mp|G(ru_\pm - \gamma x u_\pm')|, \qquad u_\pm(0) = 0, \qquad u_\pm(x_0) = \pm|F(x_0)|$$

Continuous viscosity solutions $u_\pm$ to [**H.34**] exist by Crandall, Ishii, and Lions (1992, Theorem 5.1). More over, by Lipschitz continuity of G, the boundary value problems for $u_\pm$ satisfy comparison principle, i.e., Crandall, Ishii, and Lions (1992, Theorem 3.3). Note that by Crandall, Ishii, and Lions (1992, Theorem 5.1), continuous viscosity solutions $u_\pm$ exist. More over, by the Lipschitz continuity of G, the boundary value problems for $u_\pm$ satisfy the comparison principle, i.e., Crandall, Ishii, and Lions (1992, Theorem 3.3). In particular, because u_β satisfies [**H.33**], we have

[**H.35**] $$0 = -u_\beta'' + \beta G(ru_\beta - \gamma x u_\beta') \geqslant -u_\beta'' - |G(ru_\beta - \gamma x u_\beta')|$$

and $u_\beta(x_0) = \beta F(x_0) \leqslant |F(x_0)|$. Thus, u_β is a subsolution to the boundary value problem in [H.34] for u_+. Applying the comparison principle, we obtain $u_\beta \leqslant u_+$. Similarly,

[H.36] $$0 = -u''_\beta + \beta G(ru_\beta - \gamma x u'_\beta) \leqslant -u''_\beta + |G(ru_\beta - \gamma x u'_\beta)|$$

and $u_\beta(x_0) = \beta F(x_0) \geqslant -|F(x_0)|$ and, therefore, $u_\beta \geqslant u_-$. Because $u_\pm$ are continuous, they are bounded on interval $[0, x_0]$, independently of β and we have $\sup\{|u_\beta(x)| : \beta \in [0,1],\ \ x \in [0, x_0]\} \leqslant \sup_{x \in [0,x_0]} |u_\pm(x)| < \infty$.

Next, we establish uniform bounds for $|u'_\beta|$ on $[0, x_0]$. Let u_β be as described above. By the mean value theorem, there exists $x_\beta \in (0, x_0)$, such that $u'_\beta(x_\beta) = \beta F(x_0)/x_0$.

Integrating [H.33], we obtain

[H.37] $$u'_\beta(x) = u'_\beta(x_\beta) + \beta \int_{x_\beta}^{x} G\big(ru_\beta(y) - \gamma y u'_\beta(y)\big)\,\mathrm{d}y$$

Let I be the interval between x_β and x; we then have

[H.38] $$\begin{aligned} |u'_\beta(x)| &\leqslant |u'_\beta(x_\beta)| + \beta\left|\int_{x_\beta}^{x} G\big(ru_\beta(y) - \gamma y u'_\beta(y)\big)\,\mathrm{d}y\right| \\ &\leqslant \beta|F(x_0)/x_0| + \beta \int_I \big|G\big(ru_\beta(y) - \gamma y u'_\beta(y)\big)\big|\,\mathrm{d}y \end{aligned}$$

Using the elementary inequality $(a+b)^2 \leqslant 2(a^2+b^2)$, we find that

[H.39] $$|u'_\beta(x)|^2 \leqslant 2\beta^2\left(\frac{F^2(x_0)}{x_0^2} + \left(\int_I \big|G\big(ru_\beta(y) - \gamma y u'_\beta(y)\big)\big|\,\mathrm{d}y\right)^2\right)$$

By the Cauchy-Schwartz inequality, for any integrable function $f \in \mathbb{R}^I$, we have

$$\left(\int_I f(y)\,\mathrm{d}y\right)^2 \leqslant |x - x_\beta| \int_I |f(y)|^2\,\mathrm{d}y \leqslant x_0 \int_I |f(y)|^2\,\mathrm{d}y$$

Because $\beta \in [0,1]$, we can write

[H.40] $$|u'_\beta(x)|^2 \leqslant K + K\int_I \Big(G\big(ru_\beta(y) - \gamma y u'_\beta(y)\big)\Big)^2\,\mathrm{d}y$$

where constant K does not depend on β. Because G is Lipschitz, we obtain the bound

[H.41] $$\begin{aligned} \big|G\big(ru_\beta(y) - \gamma y u'_\beta(y)\big)\big| &\leqslant K(G(0) + |ru_\beta(y) - \gamma y u'_\beta(y)|) \\ &\leqslant K(1 + \gamma y|u'_\beta(y)| + r|u_\beta(y)|) \\ &\leqslant K(1 + |u'_\beta(y)|) \end{aligned}$$

for some constant K that does not depend on β. The last inequality above is obtained from the uniform boundedness of u_β. Once again, we apply the inequality $(a+b)^2 \leqslant 2(a^2+b^2)$ to find that $\big(G\big(ru_\beta(y) - \gamma y u'_\beta(y)\big)\big)^2 \leqslant K(1+|u'_\beta(y)|^2)$. Therefore,

$$|u'_\beta(x)|^2 \leqslant K\Big(1 + \int_I |u'_\beta(y)|^2 \,\mathrm{d}y\Big)$$

Now, Grönwall's inequality implies that $|u'_\beta(x)|$ is bounded independently of β. This completes the requirements of Gilbarg and Trudinger (2001, Theorem 13.8), which provides for the existence of a second order continuous differentiable solution to [**H.32**]. Thus, $F \in \mathbf{C}^2([0, x_0])$ for any finite $x_0 \leqslant x^\star$. In particular, if $x^\star = \infty$, $F \in \mathbf{C}^2([0,\infty))$.

It remains to show that for $x^\star < \infty$, F satisfies the $\mathbf{C}^2$-fit (high-contact) property at $x^\star$. First, we show $\mathbf{C}^1$-fit, ie, smooth pasting. Because F satisfies [**H.32**] at $x^\star$, we have

[**H.42**] $$-H(F''(x^\star-)) - \gamma x^\star F'(x^\star-) + rF(x^\star) = 0$$

Consider a test function ϕ with superjet $(\beta, F''(x^\star-))$, where $\beta = [F'(x^\star-) - 1]/2 \leqslant F'(x^\star-)$. (Notice that $\beta < F'(x^\star)$ if and only if $F'(x^\star) > -1$.). By the subsolution property, we have

[**H.43**] $$-H(F''(x^\star-)) - \gamma x^\star \beta + rF(x^\star) \leqslant 0$$

Thus, we have

$$\begin{aligned} 0 &= -H(F''(x^*-)) - \gamma x^\star F'(x^\star-) + rF(x^\star) && \text{by [H.42]} \\ &\leqslant -H(F''(x^*-)) - \gamma x^\star \beta + rF(x^\star) && \text{definition of } \beta \\ &\leqslant 0 && \text{by [H.43]} \end{aligned}$$

which implies $\beta = F'(x^*-) = -1$, and hence $F'(x^*+) = -1$, which establishes the $\mathbf{C}^1$-fit property. Thus, we can rewrite [**H.42**] as

[**H.44**] $$-H(F''(x^\star-)) + \gamma x^\star + rF(x^\star) = 0$$

To establish the $\mathbf{C}^2$-fit property, consider a sequence $\{(\beta_n, x_n)\}$ such that $\beta_n < 0$, $\beta_n \uparrow 0$, $x_n > x^\star$, and $x_n \downarrow x^\star$. Let $\phi_n(x) = F(x_n) + \frac{1}{2}\beta_n(x - x_n)^2 - (x - x_n)$ be a test function which yields a subjet $(-1, \beta_n)$ of F at x_n. By the supersolution property of F, we must have

[**H.45**] $$-H(\beta_n) + \gamma x_n + rF(x_n) \geqslant 0.$$

By sending $n \to \infty$, we obtain

$$-H(0) + \gamma x^\star + rF(x^\star) \geqslant 0. \tag{H.46}$$

Thus, we find

$$\begin{aligned} 0 &= -H(F''(x^\star-)) + \gamma x^\star + rF(x^\star) && \text{by [H.44]} \\ &\geqslant -H(0) + \gamma x^\star + rF(x^\star) \\ &\geqslant 0 && \text{by [H.46]} \end{aligned}$$

where the first inequality is because $F''(x^\star-) \leqslant 0$ and H is strictly increasing. Thus, $F''(x^\star-) = 0$, and in particular, $F''(x^\star) = 0$, ie, F is $\mathbf{C}^2$ at $x^\star$, as claimed. □

Notice that Theorem 6 does not rely on $x^\star < \infty$. In the next result, we provide a necessary and sufficient condition for $x^\star < \infty$.

Proposition H.7. The payment boundary $x^\star$ is finite if and only if $\gamma > r$.

Proof. Assume that $x^\star = \infty$. Thus, for all $x \geqslant 0$, F satisfies

$$-H(F''(x)) - \gamma x F'(x) + rF(x) = 0. \tag{H.47}$$

Recall that F is concave and is bounded by $-x \leqslant F(x) \leqslant K - x$ for some K. This implies that F has a slant asymptote with equation $y = -x + c$ for some $c > 0$. In particular, $\lim_{x\to\infty} \frac{F(x)}{x} = -1$ and $\lim_{x\to\infty} F'(x) = -1$. Dividing [H.47] by x and sending $x \to \infty$, we obtain

$$0 = -\lim_{x\to\infty} \frac{H(F''(x))}{x} + \gamma - r \tag{H.48}$$

where $\lim_{x\to\infty} H(F''(x))/x$ in [H.48] exists because the other two limits do. Therefore, $\gamma = r + \lim_{x\to\infty} H(F''(x))/x$. Because $r \leqslant \gamma$, it must be that $\lim_{x\to\infty} H(F''(x))/x \geqslant 0$. On the other hand, H takes values in $[-\infty, \mu]$, which implies $\lim_{x\to\infty} H(F''(x))/x \leqslant 0$. Therefore, $\lim_{x\to\infty} H(F''(x))/x = 0$ and $\gamma = r$.

If $\gamma = r$, by Lemma H.5 we have $F \leqslant u_n$. It can be shown that when $\gamma = r$, the payment boundary without monitoring (so, the DS model) is at infinity. Thus, $L(x) = \frac{\mu}{r} - x$ is a slant asymptote for u_n with $u_n(x) < L(x)$, and $\sup_{x\geqslant 0}[u_n(x) - L(x)] = 0$.

If F has a payment boundary, $x^\star < \infty$, then, we have $-H(0) + rx^\star + rF(x^\star) = 0$. Because $H(0) = \mu$, $F(x^\star) = \frac{\mu}{r} - x^\star$, and so $(x^\star, F(x^\star))$ lies on $L(x) = \frac{\mu}{r} - x$. Because F is $\mathbf{C}^2$, $F''(x^\star) = 0$, $F'(x^\star) = -1$, and $F' \geqslant -1$, we conclude that $F(x) = L(x)$ for $x \geqslant x^\star$. But $F(x) \leqslant u_n(x) < L(x)$, which is a contradiction. □

I. Proof of Proposition A.4

Proof. Recall that the dynamics of the process M_t is given by the stochastic differential equation $\mathrm{d}M_t = \gamma M_t \,\mathrm{d}t + \hat{\sigma}(M_t)\,\mathrm{d}B_t - \lambda^{-1}\,\mathrm{d}C_t$, where $M_t \leqslant m^\star$ for all $t \geqslant 0$, $\tau = \inf\{t \geqslant 0 : M_t = 0\}$, and $C_t = \int_0^\tau \mathbf{1}(M_s = m^\star)\,\mathrm{d}C_s$.

First consider the subsolution $\mathsf{L}\,v \leqslant 0$. By applying Itô's formula on $v(M_\tau)$ and following the same steps as the proof of Proposition E.6, we obtain

[I.1]
$$e^{-r\tau}v(M_\tau) = v(m) + \int_0^\tau e^{-rs}\underbrace{\Big(\tfrac{1}{2}\hat{\sigma}^2(M_s)v''(M_s) + \gamma M_s v'(M_s) - rv(M_s)\Big)}_{-\,\mathsf{L}\,v\geqslant 0}\,\mathrm{d}s \\ - \lambda^{-1}\int_0^\tau e^{-rs}v'(M_s)\,\mathrm{d}C_s^* + \int_0^\tau e^{-rs}v'(M_s)\hat{\sigma}(M_s)\,\mathrm{d}B_s$$

By the subsolution property of v in [A.4], we have

[I.2]
$$e^{-r\tau}v(M_\tau) \geqslant v(m) - \lambda^{-1}\int_0^\tau e^{-rs}v'(M_s)\,\mathrm{d}C_s^* + \int_0^\tau e^{-rs}v'(M_s)\hat{\sigma}(M_s)\,\mathrm{d}B_s$$

From the boundary condition $v(0) \leqslant \alpha$ in [A.4] and because v' and $\hat{\sigma}$ are bounded, we obtain

$$v(m) \leqslant e^{-r\tau}\alpha + \lambda^{-1}\,\mathsf{E}\left[\int_0^\tau e^{-rs}v'(M_s)\,\mathrm{d}C_s^*\right]$$

By the properties of the payment process C^* (which is a local time) and because $v'(m^\star) \leqslant \beta$ in [A.4], we have

$$v(m) \leqslant e^{-r\tau}\alpha + v'(m^\star)\lambda^{-1}\,\mathsf{E}\left[\int_0^\tau e^{-rs}\,\mathrm{d}C_s^*\right] \leqslant e^{-r\tau}\alpha + \beta\lambda^{-1}\,\mathsf{E}\left[\int_0^\tau e^{-rs}\,\mathrm{d}C_s^*\right]$$

Next, using the fact that V is a supersolution, ie [A.5] holds, and by repeating the arguments above for v, we obtain the inequality

$$V(m) \geqslant e^{-r\tau}\alpha + V'(m^\star)\lambda^{-1}\,\mathsf{E}\left[\int_0^\tau e^{-rs}\,\mathrm{d}C_s^*\right] \geqslant e^{-r\tau}\alpha + \beta\lambda^{-1}\,\mathsf{E}\left[\int_0^\tau e^{-rs}\,\mathrm{d}C_s^*\right]$$

Combining the inequalities establishes the proposition. □