

Infinite-Time Singularities of Lagrangian Mean Curvature Flow

The Final Meeting of the Simons Collaboration on Special Holonomy :(

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joint work with Wei-Bo Su, Chung-Jun Tsai

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Introduction

Infinite-Time Singularities of Lagrangian Mean Curvature Flow

Subject: *Lagrangian mean curvature flow* - the name given to the fact that in a Calabi-Yau manifold M , the class of Lagrangian submanifolds $L \subset M$ is preserved by mean curvature flow: a popular volume-decreasing flow of submanifolds.

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Reason: We hope that Lagrangian mean curvature flow may be used to prove the *Thomas-Yau conjecture*, that Lagrangians can be represented (in a suitable class) by unions of *special Lagrangians* - minimal Lagrangian submanifolds. However, singularities occur, so we must understand them.

Preliminaries

Mean Curvature Flow

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Let N^n be a smooth manifold, and M^m a smooth Riemannian manifold. A family of immersions $F_t : N^n \rightarrow (M^m; \bar{g})$ is a **mean curvature flow** if

$$\frac{dF}{dt} = H;$$

where H is the trace of the vector-valued second fundamental form of the embedding,

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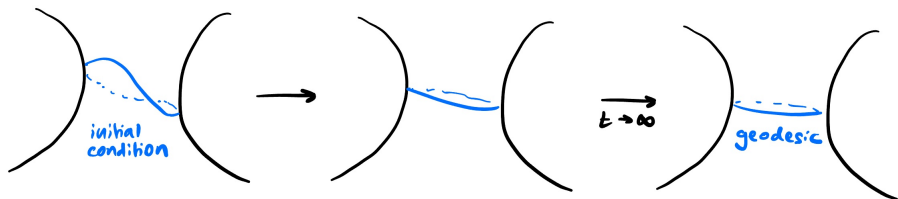
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Since MCF decreases volume, one might hope that the flow exists for all time and converges to a minimal submanifold:



Preliminaries

Examples of Mean Curvature Flow

Shrinking Sphere in \mathbb{R}^n :

$$\frac{dr}{dt} = -\frac{n}{r}$$
$$\Rightarrow r = \sqrt{\frac{R^2 - 2nt}{n}}$$



Preliminaries

Singularities of Mean Curvature Flow

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To analyse the singularity, we may consider Type I and Type II blowups.

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Singularities of Mean Curvature Flow: Type I

Type I: If $(x; T)$ is a singular space-time point for the flow $F_t : N \rightarrow M$, then define Type I rescalings:

$$F_s^i := \lambda_i (F_{\lambda_i^{-2}(s+T)} - x):$$

The differing scalings in space and time ensure that F_s^i is a mean curvature flow.

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If there exists C such that $\max_t |A_j|^2 \leq \frac{C}{T-t}$ for all $t \in [0; T)$ (a Type I singularity), then these rescalings converge subsequentially locally smoothly to a self-similarly shrinking mean curvature flow (Type I blowup or tangent flow).

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Singularities of Mean Curvature Flow: Type II

If the curvature bound $\max_{\Sigma_t} |A|^2 \leq \frac{C}{T-t}$ for all $t \in [0; T)$ doesn't hold (a Type II singularity), then the above process does not necessarily converge (smoothly) to a smooth mean curvature flow.

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If the curvature bound $\max_{x \in M} |A|^2 \leq \frac{C}{T-t}$ for all $t \in [0; T)$ doesn't hold (a Type II singularity), then the above process does not necessarily converge (smoothly) to a smooth mean curvature flow.

By instead carefully choosing a sequence of points $(x_k; t_k) \in (M; T)$ maximising the value of the second fundamental form, and defining the quantity $A_k := |A(x_k; t_k)|$ and the Type II rescalings:

$$F_s^{(x_k; t_k)} := A_k (F_{A_k^{-2} s + t_k} - x_k);$$

we can ensure subsequential local smooth convergence to a smooth mean curvature flow. This is known as a Type II blowup or singularity model.

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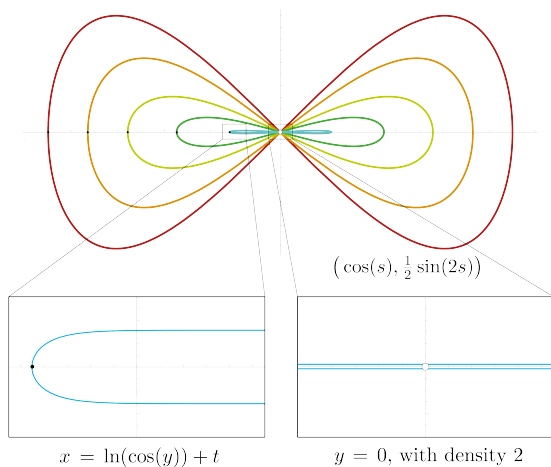
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Remark: For embedded hypersurface mean curvature flow, Type I singularities are expected to be generic. The Type I and Type II Blowups are (as far as we know) always *self-similar solitons* of mean curvature flow.

Preliminaries

Singularities of Mean Curvature Flow: Type II example



Preliminaries

Infinite-time Singularities of MCF

A fundamental question in mean curvature flow is:

If we have long-time existence of the flow, do we have convergence to a minimal submanifold? Or can infinite-time singularities occur?

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There exists a smooth mean curvature flow in \mathbb{R}^3 for $t \in [0; 1)$ forming an finite-time singularity.

Remarks:

This is a non-compact example. Therefore we may ask: are there finite-time singularities in the compact setting?

The construction shows convergence to a multiplicity two plane, but doesn't describe the rate of blowup, or the Type II blowup.

Preliminaries

Lagrangian Submanifolds

Let $(M^{2n}; \bar{g}; J; \omega)$ be a Calabi-Yau manifold. A submanifold $L^n \subset M^{2n}$ is Lagrangian if $J \circ TL \rightarrow TL^{\perp}$ is an isomorphism, or equivalently $\omega|_L = 0$.

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Let $(M^{2n}; \bar{g}; J; \iota; \cdot)$ be a Calabi-Yau manifold. A submanifold $L \subset M^{2n}$ is Lagrangian if $J : TL \rightarrow TL^\perp$ is an isomorphism, or equivalently $\iota|_L = 0$. If V is a normal vector field on L , then there is a corresponding 1-form α_V on L

$$\alpha_V := \iota(V; \cdot) = g(JV; \cdot):$$

α_V closed $\Rightarrow V$ is a Lagrangian variation field.
 α_V exact $\Rightarrow V$ is a Hamiltonian variation field. Lagrangians related by Hamiltonian variations are Hamiltonian isotopic.

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e.g. The mean curvature vector H has a corresponding 1-form, ν_H . ν_H may be shown to be closed, so the mean curvature is a Hamiltonian variation field.

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Given an oriented Lagrangian L , the holomorphic volume form Ω may be used to define a (multivalued) primitive called the **Lagrangian angle**:

$$\Omega|_L = e^i \text{vol}_L; \quad d = \iota(H; \cdot):$$

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Special Lagrangians

L is **special Lagrangian** if it is minimal - this is equivalent to $\bar{\mu} = 0$.
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L is **special Lagrangian** if it is minimal - this is equivalent to $\int_L \theta = 0$.

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e.g. In \mathbb{C}^m with the standard structures $\theta = \sum_{i=1}^m dx^i \wedge dy^i$, $\Omega = dz^1 \wedge \dots \wedge dz^m$, the following are special Lagrangian planes with $\int_L \theta = 0$:

$$\Pi^0 := f(x^1; \dots; x^m) : x^j \in \mathbb{R}g$$

$$\Pi^1 := f(e^{i_1} x^1; \dots; e^{i_m} x^m) : x^j \in \mathbb{R}g$$

If $\sum_{i=1}^m \theta_i = 0$, then Lawlor, Joyce-Imagi-dos Santos \Rightarrow there is a unique 1-parameter family of special Lagrangians L with asymptotes $\Pi^0 \cup \Pi^1$, called the **Lawlor Neck**.

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Lagrangian Mean Curvature Flow

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Theorem (K. Smoczyk)

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If L satisfies the stronger condition $\text{range}(\int_L \omega) < \pi$, it is **almost calibrated**.

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Theorem (K. Smoczyk)

In a Calabi-Yau manifold, the class of closed Lagrangian submanifolds is preserved under MCF.

If H is exact, i.e. if θ is a *single valued function*, we say L is **zero Maslov**. In this case, mean curvature flow is a Hamiltonian variation.

If L satisfies the stronger condition $\text{range}(\theta) < \pi$, it is **almost calibrated**. Both of these conditions are preserved by Lagrangian mean curvature flow:

Theorem

Under Lagrangian mean curvature flow, the Lagrangian angle satisfies the heat equation: $\frac{\partial \theta}{\partial t} = \Delta \theta$.

Preliminaries

Lagrangian Neighbourhood Theorem

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*If $L \subset M$ is Lagrangian, there exist neighbourhoods $V \subset M$ and $U \subset T^*L$ of L and of the zero section $\underline{0}$ respectively, and a symplectomorphism $\Phi : V \xrightarrow{\sim} U$ mapping L to $\underline{0}$.*

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Then, in the Lagrangian neighbourhood:

Nearby Lagrangians to L map to graphs of *closed* 1-forms under Φ

Nearby Lagrangians Hamiltonian isotopic to L map to graphs of *exact* 1-forms under Φ .

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In particular, given a zero-Maslov Lagrangian L with Lagrangian angle θ , the mean curvature flow is a flow of exact Lagrangians $\text{graph}(du_t)$, and may be expressed on the level of potentials:

$$\frac{du_t}{dt} = \text{graph}(du_t):$$

Preliminaries

The Thomas-Yau Conjecture

So we've seen that we *can* study Lagrangian MCF. But why do we want to?

Theorem (Thomas-Yau Uniqueness)

If L_1, L_2 are Hamiltonian isotopic special Lagrangians, then $L_1 = L_2$.

This suggests the question: does every Lagrangian Hamiltonian isotopy class contain a unique special Lagrangian representative?

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There is a class of 'semi-stable' Lagrangians such that a stable almost-calibrated Lagrangian will flow under LMCF with surgeries to a union of special Lagrangians of the same angle.

Important remark: Even if there is a special Lagrangian representative, it may not be smooth. Therefore, we should expect singularities along the flow.

Preliminaries

An interesting example

Consider the special Lagrangian planes \mathbb{C}^m from before, such that $\bigcap_{i=1}^m L_i = \emptyset$. There exists a 'desingularisation' of $\bigcup_{i=1}^m L_i$ - the Lawlor neck

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Can we desingularise $T^0 \setminus T^1$ to a special Lagrangian as in \mathbb{C}^m ? Joyce (2005) - We can 'glue in' a Lawlor neck at small scale at the point x to produce a Lagrangian desingularisation N .

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Consider the special Lagrangian planes $\Pi^0; \Pi \subset \mathbb{C}^m$ from before, such that $\bigcap_{i=1}^m \Pi_i = \emptyset$. There exists a 'desingularisation' of $\Pi^0 \cup \Pi$ - the Lawlor neck L . By choosing a suitable lattice $\Gamma \subset \mathbb{C}^m$, $\Pi^0; \Pi$ descend to tori $T^0; T \subset \mathbb{C}^m/\Gamma$. Define $x := T^0 \cap T$ to be the intersection point.

Can we desingularise $T^0 \cup T$ to a special Lagrangian as in \mathbb{C}^m ? Joyce (2005) - We can 'glue in' a Lawlor neck at small scale ϵ at the point x to produce a *Lagrangian* desingularisation N^ϵ . However, there is an *obstruction* to perturbing this to a special Lagrangian.

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So, what should the special Lagrangian representative of $T^0 \cup T$ be?

There is a harmonic function u on L which can be extended by constants on N^ϵ to give us an approximate harmonic function $w : N^\epsilon \rightarrow \mathbb{R}$. The corresponding Hamiltonian perturbation corresponds to shrinking the neck, indicating that the special Lagrangian representative should be $T^0 \cup T$!

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So we can conjecture that LMCF starting at N^ϵ will converge to $T^0 \cup T$.

Main Result

There exists a Lagrangian mean curvature flow $(L_t)_{t=0}^1$ in the complex torus $\mathbb{C}^m = \mathbb{T}^0 \times \mathbb{T}^{2m-2}$ such that $L_t \rightarrow \mathbb{T}^0 \times \mathbb{T}^{2m-2}$ as $t \rightarrow 1^-$ - i.e. it forms an infinite-time singularity.

Moreover:

The convergence is smooth away from the immersed point.

Any Type II blowup is the Lawlor neck.

The blowup rate of the second fundamental form $|A_j| = O(t^{-\frac{1}{m-2}})$.

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Throughout, $m \geq 3$, and $\epsilon > 0$ is as large as we like.

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- Any Type II blowup is the Lawlor neck.

- The blowup rate of the second fundamental form $|A_j| = O(t^{-\frac{1}{m-2}})$.

Remarks:

- This is the first example to the authors knowledge of an infinite-time singularity of mean curvature flow in the compact setting.

- This gives an example of the 'semistable' case of Thomas-Yau - where a Lagrangian is represented by a union of smooth special Lagrangians of the same angle.

Details of the Proof

Overview

The proof is via a gluing construction, inspired by the works of Joyce on desingularising special Lagrangians with conical singularities, and by work of Brendle-Kapouleas on a parabolic gluing construction for ancient Ricci flow.

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1. Start with $T^0 [T^2]$, and glue in a Lawlor neck at scale $\epsilon(t)$ for some decreasing function $f : [0 ; 1) \rightarrow \mathbb{R}$. Call the desingularised submanifold M^ϵ .

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$$\Delta u = \lambda_N + \epsilon(t) + L^\epsilon[u] + Q^\epsilon[u]:$$

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$$\Delta u = \lambda_N + \epsilon \Delta f + L[u] + Q[u]:$$

3. Understand the theory for the linearised operator, in particular invertibility and estimates.
4. Use the linear theory to construct an iteration scheme to solve the nonlinear equation.

Details of the Proof

Overview

The proof is via a *gluing construction*, inspired by the works of Joyce on desingularising special Lagrangians with conical singularities, and by work of Brendle-Kapouleas on a parabolic gluing construction for ancient Ricci flow.

Overview:

1. Start with $T^0 \sqcup T$, and glue in a Lawlor neck at scale $\epsilon(t)$ for some decreasing function $\epsilon: [\Lambda; 1) \rightarrow \mathbb{R}$. Call the desingularised submanifold N^ϵ .
2. Choose an appropriate Lagrangian neighbourhood $\Phi: U \rightarrow T \setminus N^\epsilon \rightarrow \mathbb{C}^m \setminus \Gamma$, so that LMCF is represented as an equation on a potential function u :

$$\partial_t u = \Delta_{N^\epsilon} u + L[u] + Q[u]:$$

3. Understand the theory for the linearised operator, in particular invertibility and estimates.
4. Use the linear theory to construct an iteration scheme to solve the nonlinear equation.

Details of the proof

1. Pre-gluing

Choose a ball B_{R_2} around x , define $X^o := T^0 \setminus B_{R_2}$ the **outer region**.

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Think of the ball B_{R_2} as lying in C^m , consider the planes $C := \Pi^0 \setminus \Pi \subset C^m$, and the Lawlor neck L asymptotic to C .

We wish to 'interpolate between' L and C in B_{R_2} . We do this using the following 'exact' Lagrangian neighbourhood:

Details of the Proof

1. Pregluing

Proposition

Let Σ be a Legendrian link in $(S^{2m-1}; \omega_{S^{2m-1}})$, and let $C = \Sigma \times (0; 1)$ be the corresponding Lagrangian cone in $(C^m \times \mathbb{R}; \omega_C)$. There exists a Lagrangian neighbourhood $\Phi_C : U_C \rightarrow T^*C = T^*(\Sigma \times (0; 1)) \times C^m \times \mathbb{R}$ such that

$$\Phi_C^* \omega_C = c + d \frac{rs}{2} ;$$

where c is the tautological 1-form on T^*C , $r \in (0; 1)$, and $s \in \mathbb{R} = T_r(0; 1)$.

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Proposition

Let Σ be a Legendrian link in $(S^{2m-1}; \omega_{S^{2m-1}})$, and let $C = \Sigma \times (0; 1)$ be the corresponding Lagrangian cone in $(\mathbb{C}^m \setminus \{0\}; \omega_{\mathbb{C}^m})$. There exists a Lagrangian neighbourhood $\Phi_C : U_C \rightarrow T^*C = T^*(\Sigma \times (0; 1)) \rightarrow \mathbb{C}^m \setminus \{0\}$ such that

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A Lagrangian $L \subset \mathbb{C}^m$ is called *asymptotically conical* with cone C and rate μ if the following holds. Let $\Sigma = C \setminus S^{2m-1}$ be the link of C . Then there exists a compact subset $K \subset L$, a constant $R_1 > 0$, and a diffeomorphism $\psi : \Sigma \times (R_1; 1) \rightarrow L \setminus K$ such that for any non-negative integer k ,

$$|\psi^* \omega_C - \omega_L| = O(r^{-1-k}) \quad \text{as } r \rightarrow 1^- ; \quad (1)$$

Details of the Proof

1. Pregluing

Theorem

Let $L \subset (C^m; !_0)$ be an exact, connected, asymptotically conical Lagrangian submanifold with cone $C = \Sigma \subset (0; 1)$. Then there exists a Lagrangian neighbourhood $\Phi_L : U_L \rightarrow T^*L \simeq C^m$ and a function $\psi_L : U_L \rightarrow \mathbb{R}$ such that

$$\Phi_L^* \omega = \psi_L + d\psi_L$$

Moreover, Φ_L can be chosen so that

$$(\Phi_L^{-1})^*(\psi_L) = \psi_C + e_1(r) + e_2(s) \quad (2)$$

for any $(r; s) \in \psi_C^{-1}(U_L) \subset T^*(\Sigma \subset (R_1; 1))$, and some exact 1-form $e = dE$. Thus, $\psi_L = \psi_C + dE$.

Details of the Proof

1. Pregluing

Let $L \subset (\mathbb{C}^m; \mathbb{R}_1)$ be an exact, connected, asymptotically conical Lagrangian submanifold with $\text{cone } L = \{(0, 1)\}$. Then there exists a Lagrangian neighbourhood $\pi_L : U_L \xrightarrow{\cong} L \cup \mathbb{C}^m$ and a function $\rho_L : U_L \rightarrow \mathbb{R}$ such that

$$\rho_L|_L = \rho_L \circ d\pi_L$$

Moreover, ρ_L can be chosen so that

$$(\pi_L^{-1})^*(\rho_L) = \rho_C(\cdot; r; s) + e_1(\cdot; r) + e_2(\cdot; r) \quad (2)$$

for any $(\cdot; r; s) \in \pi_L^{-1}(U_L) \cap (\mathbb{R}_1; 1)$, and some exact 1-form $\theta = dE$. Thus, $\rho_L = \rho_C \circ d\pi_L$.

Similarly, we get a potential $E := \rho_C(\cdot; r)$ and a Lagrangian neighbourhood π_L for the scaled Lagrangian L .

Details of the Proof

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By the above,

$$c : dE^n(\cdot; r) \rightarrow (\mathbb{R}_1; \mathbb{R}_2) \times \mathbb{C}^m$$

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Moreover $c \circ d(0)$ maps the zero section onto $\mathbb{C} = \mathbb{C}^0 \setminus \{0\}$.

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By the above,

$$c \circ dE''(\cdot; r) : (\mathbb{R}_1; \mathbb{R}_2) \rightarrow \mathbb{C}^m$$

maps the zero section to the Lawlor neck.

Moreover $c \circ d(0)$ maps the zero section onto $\mathbb{C} = \mathbb{R}^0 \setminus \{0\}$.

So if Q'' interpolates between 0 and E'' , then $dQ''(\cdot; (\mathbb{R}_1; \mathbb{R}_2))$ is a Lagrangian interpolating between \mathbb{R}^0 and \mathbb{C} .

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The domain \underline{N} is the union of the three pieces:

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We now build an immersion $\iota : \underline{N} \rightarrow \mathbb{C}^m$ and its Lagrangian neighbourhood:

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- | On Q , $\iota(x) = c \circ dQ|_Q(x) \circ \iota|_Q(x)$

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The immersion is defined by:

- | On P , $\iota(x) = x$
- | On Q , $\iota(x) = \int_C dQ(x) + \iota(x)$
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The immersion is defined by:

- On P , $\iota(x) = x$
- On Q , $\iota(x) = \int_{c(x)} dQ(x)$
- On X^0 , $\iota(x) = x$

We call the image of the immersion $\iota : \underline{N} \rightarrow \mathbb{C}^m$:

The Lagrangian neighbourhood is built of the aforementioned Lagrangian neighbourhoods, patched together:

$$\iota : T \underline{N} \rightarrow \mathbb{C}^m$$

Details of the Proof

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Now, if $u : N \rightarrow]0, 1[\subset \mathbb{R}$, then $\int_N u \, du$ will be a time-dependent family of Lagrangians near $N^\epsilon(t)$.

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By our choice of Lagrangian neighbourhoods, the LMCF equation descends to an equation on the potential u :

$$\partial_t u = \epsilon \Delta u + \epsilon^2 \nabla^2 u;$$

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Now, if $u : \underline{N} \rightarrow [\Lambda; 1) \rightarrow \mathbb{R}$, then $\Psi^{\epsilon} \circ du$ will be a time-dependent family of Lagrangians nearby $N^{\epsilon(t)}$.

By our choice of Lagrangian neighbourhoods, the LMCF equation descends to an equation on the potential u :

$$\partial_t u = \epsilon^2 \Delta u + \epsilon^2 \langle du, du \rangle;$$

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$$(\epsilon) \quad \partial_t u = \epsilon^2 \Delta u + \epsilon^2 \langle du, du \rangle + L^{\epsilon}[u] + Q^{\epsilon}[u];$$

Our aim now is to find functions u and ϵ such that this equation is satisfied.

Details of the Proof

3. Linear Theory

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Problem: L'' has a 2-dimensional eigenspace w'' . We can however invert on the orthogonal complement of this space, working in suitable weighted Banach spaces:

Given $\epsilon > 0$, $2(0; m-2)$, $2(0; \frac{1}{2})$, $2(0; \frac{1}{m+2})$, there exists $\delta > 0$ with the following significance. Given $2P_{\epsilon}^{0;0;+2}$, there exists a unique $u \in P_{\epsilon}^{1;2;+2}$ $\|h\|_{1;w''}$ and $a, b : [0; 1] \rightarrow \mathbb{R}$ such that

$$\begin{cases} @u \in L'' [u] = \epsilon + a(t) + b(t)w''; & t \in [0; 1]; \\ u(x; \epsilon) = 0; & x \in N; \end{cases} \quad (3)$$

and u satisfies the a priori estimate

$$\|u\|_{P_{\epsilon}^{1;2;+2}} \leq C\epsilon \|k\|_{P_{\epsilon}^{0;0;+2}} \quad (4)$$

for some $C > 0$ independent of ϵ .

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Taking a sequence of functions u_k and a sequence of points at which (4) doesn't hold for a sequence $\epsilon_k \rightarrow 1^-$, we take a limit and extract a solution to the heat equation on one of three 'model spaces'. By establishing a Liouville theorem for the heat equation on those spaces, we arrive at a contradiction.

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In a general Calabi-Yau, the difference between the Laplacian Δ and Δ'' has an error of size $O(1)$, which is too large for our scheme to work. This is one reason why we restrict to the torus case.

Details of the Proof

4. Iteration Scheme

Finally, we use the linear existence theory to create a Newton iteration map to find u and v .

Given u , we define $F := N^* + (0) + Q^*[u]$, and use the linear theory to solve

$$L^*[v] = F + a(t) + b(t)w:$$

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$$\mathcal{L}[v] = \mathcal{N}[u] + a(t) + b(t)w$$

Problem: We need $b(t) = 0$ for a fixed point of our iteration. We carefully choose $w(t)$ to minimise b . Integrating against w :

$$b(t) = \frac{1}{\|w\|_{L^2}^2} \int_{\mathbb{N}} (\mathcal{L}[v] - \mathcal{Q}[u])w + \int_{\mathbb{N}} (\mathcal{N}[u] + \mathcal{Q}[u])w$$

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$$b(t) = \frac{1}{\|w\|_{L^2}^2} \int_{\mathbb{N}} (\mathcal{L}[v] - Q[u])w + \int_{\mathbb{N}} (u_N + u(0))w$$

The second integral has dominant term $u(0) + C t^m$, so the solution $u_0 \sim t^{\frac{1}{m-2}}$ to the corresponding ODE will minimise $b(t)$.

Details of the proof

So we define

$$u(t) := u_0(t)^2 + \int_0^t h(s) ds$$

and create an iteration for h :

$$k(t) := h(t) - u(t)$$

so that $b(t)$ vanishes for a fixed point.

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so that $b(t)$ vanishes for a fixed point.

Finally, we prove that $(u; h) \mapsto (v; k)$ is a continuous contraction map, so there exists a fixed point. And we are done!

Thanks for listening!