Infinite-Time Singularities of Lagrangian Mean Curvature Flow

The Final Meeting of the Simons Collaboration on Special Holonomy :(

Albert Wood

joint work with Wei-Bo Su, Chung-Jun Tsai

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Introduction

Infinite-Time Singularities of Lagrangian Mean Curvature Flow

Subject: *Lagrangian mean curvature flow* - the name given to the fact that in a Calabi-Yau manifold M, the class of Lagrangian submanifolds $L \subset M$ is preserved by mean curvature flow: a popular volume-decreasing flow of submanifolds.

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Our Aim: To understand *singularities* of Lagrangian mean curvature flow in the *Thomas-Yau semistable case*, explicitly *infinite-time singularities* corresponding to a decomposition of a Lagrangian L into two special Lagrangians $L_1 \cup L_2$.

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Reason: We hope that Lagrangian mean curvature flow may be used to prove the *Thomas-Yau conjecture*, that Lagrangians can be represented (in a suitable class) by unions of *special Lagrangians* - minimal Lagrangian submanifolds. However, singularities occur, so we must understand them.

Preliminaries

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Let N^n be a smooth manifold, and M^m a smooth Riemannian manifold. A family of immersions $F_t : N^n \to (M^m, \overline{g})$ is a **mean curvature flow** if

$$\frac{dF}{dt} = \vec{H},$$

where $\vec{H}$ is the trace of the vector-valued second fundamental form of the embedding,

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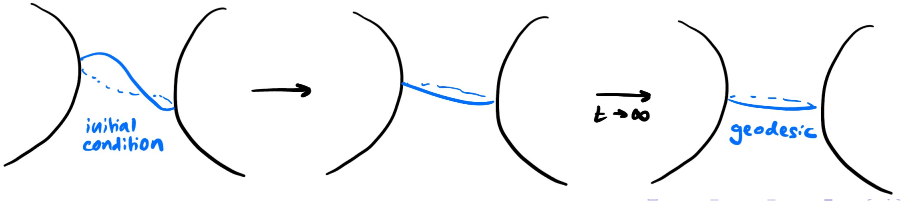
Since MCF decreases volume, one
might hope that the flow exists for all time and converges to a minimal submanifold:

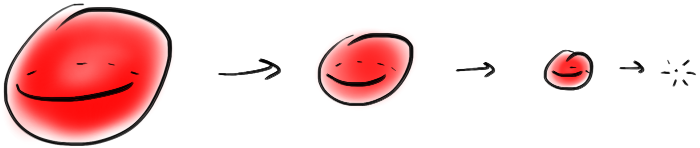


Preliminaries

Examples of Mean Curvature Flow

Shrinking Sphere in $\mathbb{R}^n$:

$$\frac{dr}{dt} = -\frac{n}{r}$$
$$\implies r = \sqrt{R - 2nt}.$$



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Singularities of Mean Curvature Flow

Consider a mean curvature flow $F_t : N \to M$, $t \in [0, T]$ for T the maximal time of existence. If $T < \infty$ then we say F_t has a **finite-time singularity** at T.

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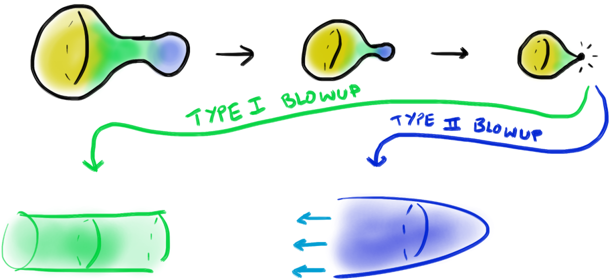
To analyse the singularity, we may consider Type I and Type II blowups.



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Singularities of Mean Curvature Flow: Type I

Type I: If (x, T) is a singular space-time point for the flow $F_t : N \to M$, then define **Type I rescalings**:

$$F_s^{\lambda_i} := \lambda_i(F_{\lambda_i^{-2}s+T} - x).$$

The differing scalings in space and time ensure that $F_s^{\lambda_i}$ is a mean curvature flow.

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Theorem (Huisken)

If there exists C such that $\max_{L_t} |A|^2 \leq \frac{C}{T-t}$ for all $t \in [0, T)$ (a **Type I singularity***), then these rescalings converge subsequentially locally smoothly to a self-similarly shrinking mean curvature flow (a* **Type I blowup** *or* **tangent flow***).*

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Singularities of Mean Curvature Flow: Type II

If the curvature bound $\max_{L_t} |A|^2 \leq \frac{C}{T-t}$ for all $t \in [0, T)$ doesn't hold (a **Type II singularity**), then the above process does not necessarily converge (smoothly) to a smooth mean curvature flow.

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By instead carefully choosing a *sequence* of points $(x_k, t_k) \to (x, T)$ maximising the value of the second fundamental form, and defining the quantity $A_k := |A(x_k, t_k)|$ and the **Type II rescalings**:

$$F_s^{(x_k,t_k)} := A_k(F_{A_k^{-2}s+t_k} - x_k),$$

we can ensure subsequential local smooth convergence to a smooth mean curvature flow. This is known as a **Type II blowup** or **singularity model**.

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Remark: For embedded hypersurface mean curvature flow, Type I singularities are expected to be generic. The Type I and Type II Blowups are (as far as we know) always *self-similar solitons* of mean curvature flow.

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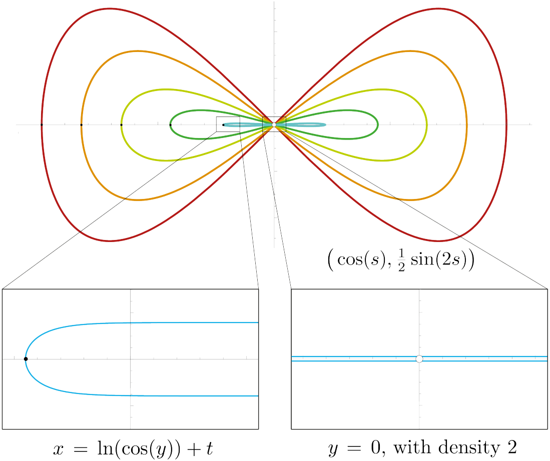
Singularities of Mean Curvature Flow: Type II example



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Infinite-time Singularities of MCF

A fundamental question in mean curvature flow is:

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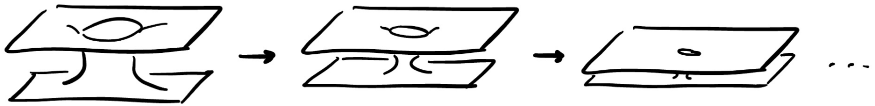
Theorem (Chen-Sun 2024)

There exists a smooth mean curvature flow $F_t \subset \mathbb{R}^3$ for $t \in [0, \infty)$ forming an infinite-time singularity.



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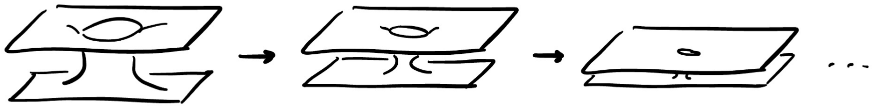

Remarks:

- This is a non-compact example. Therefore we may ask: are there infinite-time singularities in the compact setting?
- The construction shows convergence of F_t to a multiplicity two plane, but doesn't describe the rate of blowup of $|A|$, or the Type II blowup.

Preliminaries

Lagrangian Submanifolds

- Let $(M^{2n}, \overline{g}, J, \omega, \Omega)$ be a Calabi-Yau manifold. A submanifold $L^n \subset M^{2n}$ is **Lagrangian** if $J : TL \to TL^{\perp}$ is an isomorphism , or equivalently if $\omega|_L = 0$.

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- If V is a normal vector field on L, then there is a corresponding 1-form on L:

$$\alpha_V := \omega(V, \cdot) = g(JV, \cdot).$$

α_V closed $\implies$ V is a **Lagrangian variation field**.
α_V exact $\implies$ V is a **Hamiltonian variation field**. Lagrangians related by Hamiltonian variations are **Hamiltonian isotopic**.

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  e.g. The mean curvature vector H has a corresponding 1-form, α_H. α_H may be shown to be closed, so the mean curvature is a Hamiltonian variation field.
- Given an oriented Lagrangian L, the holomorphic volume form Ω may be used to define a (multivalued) primitive called the **Lagrangian angle**:

$$\Omega|_L = e^{i\theta} vol_L, \quad d\theta = \alpha_H.$$

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Special Lagrangians

- L is **special Lagrangian** if it is minimal - this is equivalent to $\theta \equiv \overline{\theta}$.
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e.g. In $\mathbb{C}^m$ with the standard structures $\omega = \sum_{i=1}^m dx^i \wedge dy^i$, $\Omega = dz^1 \wedge \ldots \wedge dz^m$, the following are special Lagrangian planes with $\overline{\theta} = 0$:

$$\Pi^0 := \{(x^1, \ldots, x^m) : x^j \in \mathbb{R}\}$$
$$\Pi^\phi := \{(e^{i\phi_1}x^1, \ldots, e^{i\phi_m}x^m) : x^j \in \mathbb{R}\}$$

If $\sum_{i=1}^m \phi_i = \pi$, then Lawlor, Joyce-Imagi-dos Santos $\implies$ there is a unique 1-parameter family of special Lagrangians εL with asymptotes $\Pi^0 \cup \Pi^\phi$, called the **Lawlor Neck**.

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If L satisfies the stronger condition $\mathrm{range}(\theta) < \pi$, it is **almost calibrated**.

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If L satisfies the stronger condition range$(\theta) < \pi$, it is **almost calibrated**.
Both of these conditions are preserved by Lagrangian mean curvature flow:

Theorem

Under Lagrangian mean curvature flow, the Lagrangian angle satisfies the heat equation: $\frac{\partial \theta}{\partial t} = \Delta \theta$.

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Theorem (Lagrangian Neighbourhood Theorem)

If $L \subset M$ is Lagrangian, there exist neighbourhoods $V \subset M$ and $U \subset T^*L$ of L and of the zero section $\underline{0}$ respectively, and a symplectomorphism $\Phi : V \to U$ mapping L to $\underline{0}$.

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Then, in the Lagrangian neighbourhood:

- Nearby Lagrangians to L map to graphs of *closed* 1-forms under Φ
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In particular, given a zero-Maslov Lagrangian L with Lagrangian angle θ, the mean curvature flow is a flow of exact Lagrangians graph(du_t), and may be expressed on the level of potentials:

$$\frac{du_t}{dt} = \theta(du_t).$$

Preliminaries

The Thomas-Yau Conjecture

So we've seen that we *can* study Lagrangian MCF. But why do we want to?

Theorem (Thomas-Yau Uniqueness)

If L_1, L_2 are Hamiltonian isotopic special Lagrangians, then $L_1 = L_2$.

This suggests the question: does every Lagrangian Hamiltonian isotopy class contain a unique special Lagrangian representative?

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There is a class of 'semi-stable' Lagrangians such that a stable almost-calibrated Lagrangian will flow under LMCF with surgeries to a union of special Lagrangians of the same angle.

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There is a class of 'semi-stable' Lagrangians such that a stable almost-calibrated Lagrangian will flow under LMCF with surgeries to a union of special Lagrangians of the same angle.

Important remark: Even if there is a special Lagrangian representative, it may not be smooth. Therefore, we should expect singularities along the flow.

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An interesting example

- Consider the special Lagrangian planes $\Pi^0, \Pi^\phi \subset \mathbb{C}^m$ from before, such that $\sum_{i=1}^m = \pi$. There exists a 'desingularisation' of $\Pi^0 \cup \Pi^\phi$ - the Lawlor neck L^ϕ.

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So, what should the special Lagrangian representative of $T^0 \cup T^\phi$ be?

- There is a harmonic function β on L^ϕ which can be extended by constants on N^ε to give us an approximate harmonic function $w : N^\varepsilon \to \mathbb{R}$. The corresponding Hamiltonian perturbation corresponds to shrinking the neck, indicating that the special Lagrangian representative should be $T^0 \cup T^\phi$!

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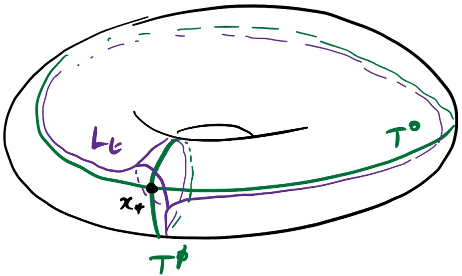
- So we can conjecture that LMCF starting at N^ε will converge to $T^0 \cup T^\phi$.

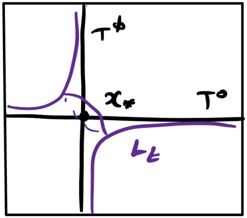
Main Result

Theorem (Su-Tsai-W., 2024)

There exists a Lagrangian mean curvature flow $(L_t)_{t=0}^{\infty}$ in the complex torus $\mathbb{C}^m/\Gamma$ such that $L_t \to T^0 \cup T^\phi$ as $t \to \infty$ - i.e. it forms an infinite-time singularity. Moreover:

- *The convergence is smooth away from the immersed point.*
- *Any Type II blowup is the Lawlor neck.*
- *The blowup rate of the second fundamental form is $|A| = O(t^{\frac{1}{m-2}})$.*





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Theorem (Su-Tsai-W., 2024)

There exists a Lagrangian mean curvature flow $(L_t)_{t=0}^{\infty}$ in the complex torus $\mathbb{C}^m/\Gamma$ such that $L_t \to T^0 \cup T^\phi$ as $t \to \infty$ - i.e. it forms an infinite-time singularity. Moreover:

- *The convergence is smooth away from the immersed point.*
- *Any Type II blowup is the Lawlor neck.*
- *The blowup rate of the second fundamental form is $|A| = O(t^{\frac{1}{m-2}})$.*

Remarks:

- This is the first example to the authors knowledge of an infinite-time singularity of mean curvature flow in the compact setting.
- This gives an example of the 'semistable' case of Thomas-Yau - where a Lagrangian is represented by a union of smooth special Lagrangians of the same angle.

Details of the Proof

Overview

The proof is via a *gluing construction*, inspired by the works of Joyce on desingularising special Lagrangians with conical singularities, and by work of Brendle-Kapouleas on a parabolic gluing construction for ancient Ricci flow.

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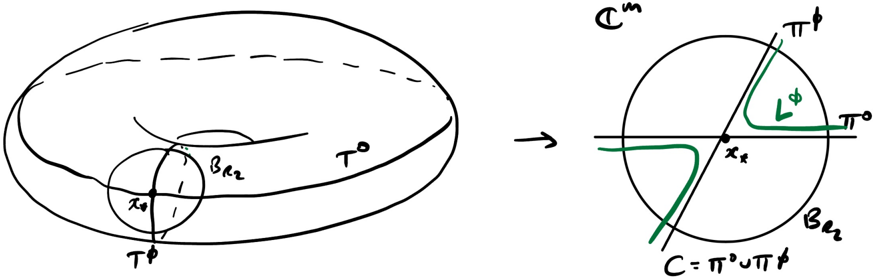

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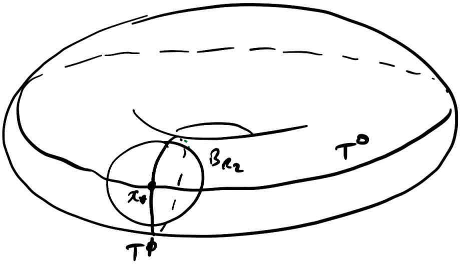

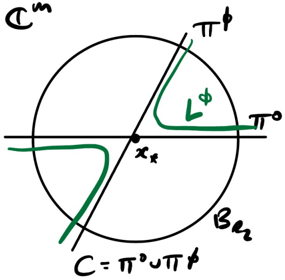

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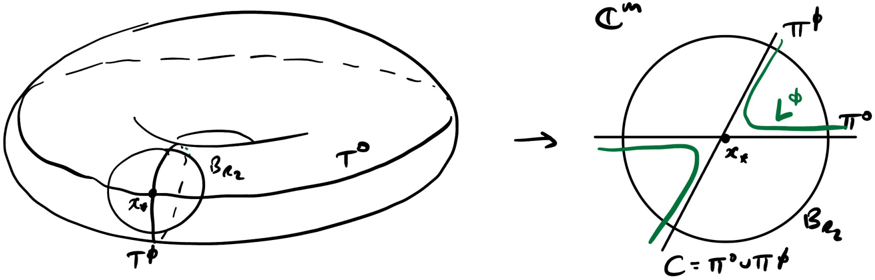

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- Think of the ball B_{R_2} as lying in $\mathbb{C}^m$, consider the planes $C := \Pi^0 \cup \Pi^\phi \subset \mathbb{C}^m$, and the Lawlor neck L^ϕ asymptotic to C.
- We wish to 'interpolate between' L^ϕ and C in B_{R_2}. We do this using the following 'exact' Lagrangian neighbourhood:

Details of the Proof

1. Pregluing

Proposition

Let Σ be a Legendrian link in $(S^{2m-1}, \lambda_0|_{S^{2m-1}})$, and let $C = \Sigma \times (0,\infty)$ be the corresponding Lagrangian cone in $(\mathbb{C}^m \setminus \{0\}, \omega_0)$. There exists a Lagrangian neighbourhood $\Phi_C : U_C \subset T^*C = T^*(\Sigma \times (0,\infty)) \to \mathbb{C}^m \setminus \{0\}$ such that

$$\Phi_C^* \lambda_0 = \lambda_C - d\left(\frac{rs}{2}\right) ,$$

where λ_C is the tautological 1-form on T^*C, $r \in (0,\infty)$, and $s \in \mathbb{R} \cong T_r^*(0,\infty)$.

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A Lagrangian $L \subset \mathbb{C}^m$ is called *asymptotically conical* with cone C and rate γ if the following holds. Let $\Sigma = C \cap S^{2m-1}$ be the link of C. Then there exists a compact subset $K \subset L$, a constant $R_1 > 0$, and a diffeomorphism $\varphi : \Sigma \times (R_1, \infty) \to L \setminus K$ such that for any non-negative integer k,

$$|\nabla^k(\varphi - \iota_C)|(,r) = O(r^{-1-k}) \quad \text{as } r \to \infty \ . \tag{1}$$

Details of the Proof

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Theorem

Let $L \subset (\mathbb{C}^m, \omega_0)$ be an exact, connected, asymptotically conical Lagrangian submanifold with cone $C = \Sigma \times (0, \infty)$. Then there exists a Lagrangian neighbourhood $\Phi_L : U_L \subset T^*L \to \mathbb{C}^m$ and a function $\alpha_L : U_L \to$ such that

$$\Phi_L^* \lambda_0 = \lambda_L - d\alpha_L \ .$$

Moreover, Φ_L can be chosen so that

$$(\Phi_L \circ \varphi_\sharp)(\sigma, r, \varsigma, s) = \Phi_C(\sigma, r, \varsigma + \mathfrak{e}_1(\sigma, r), s + \mathfrak{e}_2(\sigma, r)) \qquad (2)$$

for any $(\sigma, r, \varsigma, s) \in \varphi_\sharp^{-1}(U_L) \subset T^*(\Sigma \times (R_1, \infty))$, and some exact 1-form $\mathfrak{e} = d\mathfrak{E}$. Thus, $\varphi = \Phi_C \circ d\mathfrak{E}$.

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Similarly, we get a potential $\mathfrak{E}^\varepsilon := \varepsilon^2 \mathfrak{E}(\sigma, \varepsilon^{-1} r)$ and a Lagrangian neighbourhood $\Phi_{\varepsilon L}$ for the scaled Lagrangian εL.

Details of the Proof

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- By the above,
  $$\Phi_C \circ \mathrm{d}\mathfrak{E}^{\varepsilon}(\sigma, r) : \Sigma \times (\varepsilon R_1, R_2) \to \mathbb{C}^m$$
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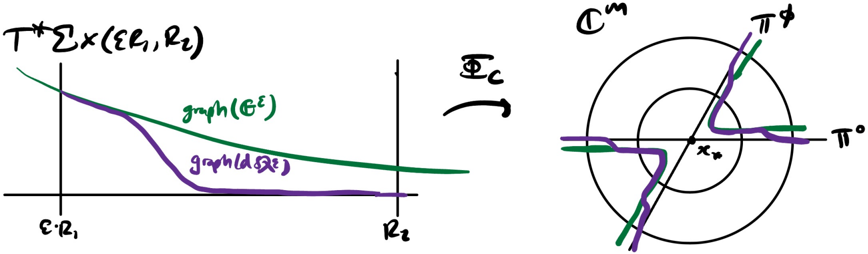
- Moreover $\Phi_C \circ \mathrm{d}(0)$ maps the zero section onto $C = \Pi^0 \cap \Pi^\phi$.
- So if $\mathfrak{Q}^\varepsilon$ interpolates between 0 and $\mathcal{E}^\varepsilon$, then $\Phi \circ d\mathfrak{Q}^\varepsilon(\Sigma \times (\varepsilon R_1, R_2))$ is a Lagrangian interpolating between L^ϕ and C.



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- The Lagrangian neighbourhood is built of the aforementioned Lagrangian neighbourhoods, patched together:

$$\Psi^{\varepsilon} : T^*\underline{N} \to \mathbb{C}^m/\Gamma.$$

Details of the Proof

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- Our aim now is to find functions u and ε such that this equation is satisfied.

Details of the Proof

3. Linear Theory

Problem: $\mathcal{L}^{\varepsilon}$ has a 2-dimensional eigenspace $\langle 1, w^{\varepsilon} \rangle$. We can however invert on the orthogonal complement of this space, working in suitable weighted Banach spaces:

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Theorem

Given $\mu > 0$, $\nu \in (0, m-2)$, $\alpha \in (0, \frac{1}{2})$, $\tau \in (0, \frac{1}{m+2})$, there exists $\Lambda \gg 1$ with the following significance. Given $\psi \in P^{0,0,\alpha}_{\mu,\nu+2,\Lambda}$, there exists a unique $u \in P^{1,2,\alpha}_{\mu,\nu,\Lambda} \cap \langle 1, w^\varepsilon \rangle^\perp$ and $a, b : [\Lambda, \infty) \to \mathbb{R}$ such that

$$\begin{cases} \partial_t u - \mathcal{L}^\varepsilon[u] = \psi + a(t) + b(t) w^\varepsilon, & t \in [\Lambda, \infty), \\ u(x, \Lambda) = 0, & x \in N, \end{cases} \tag{3}$$

and u satisfies the a priori estimate

$$\|u\|_{P^{1,2,\alpha}_{\mu,\nu,\Lambda}} \leq C \|\psi\|_{P^{0,0,\alpha}_{\mu,\nu+2,\Lambda}} \tag{4}$$

for some $C > 0$ independent of t.

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- In a general Calabi-Yau, the difference between the Laplacian and $\mathcal{L}^{\varepsilon}$ has an error of size $O(1)$, which is too large for our scheme to work. This is one reason why we restrict to the torus case.

Details of the Proof

4. Iteration Scheme

- Finally, we use the linear existence theory to create a Newton iteration map, to find u and ε.
- Given u, we define $\psi := \theta_{N^\varepsilon} + \xi(0) + Q^\varepsilon[u]$, and use the linear theory to solve

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- Problem: We need $b(t) = 0$ for a fixed point of our iteration. We carefully choose $\varepsilon(t)$ to minimise b. Integrating against w^ε:

$$b(t) = \frac{1}{|w^\varepsilon|_{L^2}^2}\left(\int_{\underline{N}}(\partial_t v - \mathcal{L}^\varepsilon[v] - \mathcal{Q}^\varepsilon[du])w^\varepsilon + \int_{\underline{N}}(\theta_{N^\varepsilon} + \xi^\varepsilon(0)) \cdot w^\varepsilon\right)$$

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- The second integral has dominant term $\partial_t(\varepsilon^2) + C \cdot \varepsilon^m$, so the solution $\varepsilon_0 \approx t^{\frac{-1}{m-2}}$ to the corresponding ODE will minimise $b(t)$.

Details of the proof

- So we define

$$\varepsilon(t) := \left(\varepsilon_0(t)^{2-m} + \int_\Lambda^t h(s)\right)^{\frac{-1}{m-2}} \approx \varepsilon_0(t)$$

and create an iteration for h:

$$k(t) := h(t) - \varepsilon^{-m} b(t)$$

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  so that $b(t)$ vanishes for a fixed point.
- Finally, we prove that $(u, h) \mapsto (v, k)$ is a continuous contraction map, so there exists a fixed point. And we are done!

Thanks for listening!