1.

The Moduli Space of Graphical Associative Submanifolds

Emily Autumn Windes: University of Oregon

Simons Collaboration Meeting

2.

Outline

1. History / Context
2. Stable forms in dimensions 6 + 7.
3. The Lagrange-multipliers problem
4. Ellipticity and the G_2-cylinder
5. Volume bounds
6. Regularity + an easy compactness result
7. Transversality

3.

History/Context

1. Gromov-Witten invariants: Count J-holomorphic curves
2. Seiberg-Witten invariants: Count solutions (connections) to SW equations.
3. Donaldson-Thomas invariants:
4. Joyce: Count $\mathbb{Q}$-homology 3-spheres in dimension 6.
5. Donaldson-Thomas/Donaldson-Segal☆: Gauge theory in higher dimensions
6. Doan-Walpuski☆/Joyce: Counting associatives in 7-dimensions.

& others...

☆ Calibrated geometry is related to gauge theory

☆ This talk is about dimension 6.

4.

Stable Forms

Definition: $\alpha \in \Lambda^p(\mathbb{R}^n)^*$ is called stable if its $GL(n,\mathbb{R})$-orbit is open.

* Hitchin 2001: Stable forms & special metrics. (Also Bryant & Foscolo)

Facts:

1. If M^n has a stable p-form α, then M has a G-structure where $G = \mathrm{stab}(\alpha_p)$.
2. The existence of a stable p-form $\Rightarrow$ the existence of a stable $(n-p)$-form called its Hitchin dual & denoted by $\hat{\alpha}$.
3. Related to special holonomy where "torsion-free" is expressed in terms of conditions on the forms.

Ex: A G_2-structure φ on a 7-manifold

5.

6 dimensions is special

We have a lot of stable forms

1. 2-forms ω } stabilizer $Sp(6,\mathbb{R})$
2. 4-forms τ }
3. 3-forms ρ
   - stabilizer $SL(3,\mathbb{C})$ ← positive ☆
   - Stabilizer $SL(3,\mathbb{R}) \times SL(3,\mathbb{R})$ ← negative

In dimension 6 only, a stable 3 form => an almost complex structure.

6.

Hitchin Duals

1. $\hat{\omega} = \frac{1}{2}\omega \wedge \omega \qquad \tau = \frac{1}{2}\hat{\tau} \wedge \hat{\tau}$
2. $\hat{\rho}$ is the unique 3-form such that $\rho + i\hat{\rho}$ is a nowhere-vanishing complex volume form.

Ex: M is Calabi-Yau $\Rightarrow$ ω, $\tau = \frac{1}{2}\omega\wedge\omega$, $\rho = Re(\Omega)$, $\hat{\rho} = Im(\Omega)$

Definition: An $SU(3)$-structure on a 6-manifold M is a pair of stable forms

$$(\rho,\tau) \in \Omega^3(M) \times \Omega^4(M)$$

such that

1. $\omega \wedge \rho = 0$ (where $\omega = \hat{\tau}$)
2. $\frac{1}{6}\omega^3 = \frac{1}{4}\rho \wedge \hat{\rho}$

☆ In this case, the stabilizer of the pair is $SU(3)$.

☆ CY <=> $d\omega = 0$
$d\tau = 0$
$d\rho = 0$
$d\hat{\rho} = 0$

7.

7-dimensions

Lemma: $(\rho,\omega)\in\Omega^3(M)\times\Omega^2(M)$ (stable) defines an $SU(3)$-structure with metric g_M if and only if $\varphi=\rho+dt\wedge\omega$ is a G_2-stucture on $\mathbb{R}\times M$ that defines the product metric $g_\varphi=dt^2+g_M$.

☆ On the other hand, it could be the case that $\varphi=\rho+dt\wedge\omega$ is stable (i.e. G_2) but (ρ,ω) is <u>not</u> an $SU(3)$ structure. However, even in this case, $\exists\,\omega'$ such that (ρ,ω') <u>is</u> an $SU(3)$ structure.

☆ $SU(3)$-structures on hypersurfaces in a G_2-manifold were studied by Calabi.

Definition: (ρ,τ) or (ρ,ω) will be called a G_2-pair if $\varphi=\rho+dt\wedge\omega$ is a stable 3-form on $\mathbb{R}\times M$.

☆ $\{SU(3)\text{-structures}\}$ is a deformation-retract of $\{G_2\text{-pairs}\}$

8.

The Lagrange Multipliers Problem

(From Donaldson - Segal)

Let (M,τ,ρ) be a closed 6-manifold equipped with a <u>closed</u> G_2-pair. (τ: 4-form, ρ: 3-form)

Fix:

- A closed 3-manifold P
- $A\in H_3(M;\mathbb{Z})$
- An embedding $\iota_0: P\longrightarrow M$

Let $\mathcal{F}$ = embeddings $\iota: P\to M$, $[\iota]=A$

$\tilde{\mathcal{F}}$ = covering space of $\mathcal{F}$ based at ι_0.

Define:

- $f_\tau:\tilde{\mathcal{F}}\longrightarrow\mathbb{R}$

$$f_\tau(\tilde{\iota})=\int_{[0,1]\times P}\tilde{\iota}^*\tau$$

- $\mathcal{C}_\rho=\{\iota\in\mathcal{F}:\iota^*\rho=0,\ \iota^*\hat{\rho}>0\}$

Goal: Find the critical points of

$$f_\tau\big|_{\tilde{\mathcal{C}}_\rho}$$

9.

Calculus Review (finite-dimensional analogue)

Set-up:

$$\begin{tikzcd} \mathcal{E} \arrow[d, shift right, "\pi"'] \\ M \arrow[u, shift right, "s"'] \arrow[r, "f"] & \mathbb{R} \end{tikzcd}$$

$Z = s^{-1}(0)$, a submanifold

$\mathcal{E}$, a rank k vector bundle

$Ds = \pi_v \circ ds$

Lagrange Multipliers Theorem: $p \in Z$ is a critical point of $f|_Z$ iff $\exists\ \lambda \in \mathcal{E}^*$ such that

$$df_p = \lambda \circ Ds_p$$

Lagrange function: $\Lambda : \mathcal{E}^* \longrightarrow \mathbb{R}$

$$\Lambda(\lambda) = f(\pi(\lambda)) - (\lambda \circ s \circ \pi)(\lambda)$$

Lemma: $\lambda \in \mathcal{E}^*$ is a critical point of $\Lambda \iff \pi(\lambda)$ is a critical point of $f|_Z$

Theorem (Schechter-Xu '13, W.): The Morse homology of Λ is well-defined and equal to the Morse homology of $f|_Z$ (singular homology of Z) with grading shifted by k.

10.

The perturbed SL equations

Set-up

$$\begin{tikzcd} \mathcal{E} = \Omega^3(P) \arrow[d, shift right, "\pi"'] \\ \tilde{\mathcal{F}} \arrow[u, shift right, "c"'] \arrow[r, "f_\tau"] & \mathbb{R} \end{tikzcd}$$

$\widetilde{\mathrm{Emb}}_0$ w/ $\iota^* \hat{\rho} > 0$, $[\iota] = A$

↳ "$\int_{\iota_0}^{\iota} \tau$"

- $c(\iota) = \iota^* \rho$
- $C_p = c^{-1}(0)$
- $\mathcal{E}^* = C^\infty(P)$ $\downarrow$ $\mathcal{F}$

Some notation: If $\alpha \in \Omega^k(M)$ vanishes on a submanifold ιP, let $\alpha_N \in \Omega^{k-1}(P, N^*\iota)$ be defined by

$$\alpha_N(v_1, \dots, v_{k-1}) = \alpha(\iota_* v_1, \dots, \iota_* v_{k-1}, \cdot)$$

Lagrange Functional: $\Lambda : \mathcal{E}^* \longrightarrow \mathbb{R}$

$$\Lambda(\tilde{\lambda}) = \int_{[0,1]\times P} \tilde{\iota}^* \tau + \int_P \lambda\, \iota^* \rho$$

Euler-Lagrange Equations:

1. $\iota^* \rho = 0$
2. $\tau_N + d\lambda \wedge \rho_N = 0$

Harvey-Lawson: $L^3 \subset (M, g, \Omega, J)$ is SL iff

1. $\mathrm{Re}(\Omega)|_L = 0$
2. $\omega|_L = 0$

☆ We should do Morse Theory! But McLean says we need to consider the case where $\lambda \neq$ constant.

11.

Gauge Invariance

☆ Let $\mathscr{G}$ = orientation-preserving diffeomorphisms of P

Let $\tilde{\mathscr{G}}$ = smooth isotopies $[0,1] \longrightarrow \mathrm{Diff}(P)$ starting at the identity.

Let

$$\mathscr{S} = \mathscr{F}/\mathscr{G} \qquad \text{and} \qquad \tilde{\mathscr{S}} = \tilde{\mathscr{F}}/\tilde{\mathscr{G}}$$

☆ The functionals f_τ and Λ are $\tilde{\mathscr{G}}$-invariant.

Their derivatives are $\mathscr{G}$-invariant + horizontal.

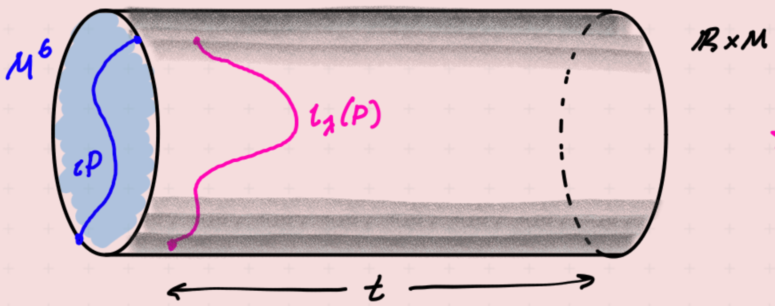
Thus they all descend to submanifolds.

12.

Ellipticity and the G_2-cylinder

Let $\iota_\lambda : P \longrightarrow \mathbb{R} \times M \qquad p \longmapsto (\lambda(p), \iota(p))$.

Lemma: Suppose that (λ, ι) solves the perturbed SL equations. Then $\iota_\lambda(P)$ is an associative submanifold of $\mathbb{R} \times M$ with respect to $\psi = \tau + dt \wedge \rho$.



☆ However, these are not necessarily calibrated/minimal since $\hat{\rho}$, ω, + therefore $\varphi = \hat{\rho} + dt \wedge \omega$ are not closed.

Consequences:

- The operator associated to the perturbed SL equations is self-adjoint + elliptic (after modding out by diffeomorphisms).
- The gradient trajectories are given by Cayley submanifolds in $\mathbb{R}^2 \times M$

13.

Volume Bounds

Definition: Suppose that (X,ψ) is a 7-manifold equipped with a stable 4-form ψ. Then we say that a closed 3-form φ tames ψ if for all $x \in X$, and all ψ-associative, oriented 3-planes $V \subset T_x X$, $\exists\ K > 0$ such that

$$\mathrm{vol}_V \leq K \cdot \varphi'|_V$$

☆ If φ' tames ψ then φ' is stable.

☆ If ιP is ψ-associative, then $\mathrm{Vol}_{g_\psi}(\iota P) \leq K \langle [\varphi'], [\iota] \rangle$ (topological)

Lemma: Suppose $(M, \underset{3}{\rho}, \underset{4}{\tau}, \underset{2}{\omega'}, \underset{3}{\rho'})$ is a 6-manifold with a tamed G_2-pair. Suppose also that (λ, ι) is a graphical associative. Then both the volume of ιP + $\|d\lambda\|_{L^2}$ are bounded. More precisely, $\exists$ a constant $K > 0$ such that

closed

$$\mathrm{Vol}_{g_M}(\iota P) + \|d\lambda\|^2_{L^2} \leq K \langle [\rho'], [\iota] \rangle$$

14.

Regularity

☆ We have to do this because ιP is not minimal!

Theorem: Let $\ell \geq 1$ be an integer, $\alpha \in (0,1)$ a number. Let (X, ψ) be a closed 7-manifold equipped with a stable, $C^{\ell,\alpha}$ 4-form ψ. Let $\iota : P \longrightarrow X$ be an embedding so that ιP is a $C^{2,\alpha}$, ψ-associative submanifold. Then ιP is a $C^{\ell,\alpha}$ submanifold.

In particular, if ψ is smooth then so is ιP.

Proof:

- Locally, an associative looks like the graph of a function $f : U \longrightarrow \mathbb{H}$ which satisfies a modified Harvey-Lawson equation

$U \subset \mathrm{im}\,\mathbb{H}$

This depends on ψ.

$$D(f) - \sigma(f) + \hat{\beta}_{\mathbb{H}}(f) = 0$$

Dirac operator 1^{st}-order Monge-Ampère operator

- Apply "D" to both sides + think of it as a linear 2^{nd}-order PDE with non-smooth coefficients.

- Use Schauder estimates.

15.

Easy Compactness

☆ The following also holds for immersions.

Theorem: Let $\ell \geq 1$ be an integer, $\alpha \in (0,1)$ a number, $p = \frac{3}{1-\alpha}$. Let X be an oriented 7-manifold equipped with a sequence ψ_a of tame stable 4-forms converging to ψ in $C^{\ell,\alpha}$.

Let $\iota_a : P \longrightarrow X$ be a sequence of ψ_a-associative, $W^{3,p}$-embeddings such that

i) $$\frac{|\iota_a(p) - \iota_a(q)|_X}{|\widehat{\iota_a(p) - \iota_a(q)}|_{\iota P}} \geq \frac{1}{C} \qquad C > 0 \text{ a fixed constant.}$$

ii) $$\|\mathrm{II}(\iota_a)\|_{L^p} \leq C$$

$\uparrow$ 2nd fundamental form

Then there exists a subsequence ι_b, an embedding $\iota : P \longrightarrow X$ also satisfying the above, and a sequence of diffeomorphisms ϕ_b, such that

$$\iota_b \circ \phi_b : P \longrightarrow X$$

converges to ι in the $C^{\ell,\alpha}$-topology.

16.

Proof:

- Breuning 2012 : C^1 compactness for immersions with bounded 2nd fundamental form.

- Elliptic regularity

The Local Moduli Space

- $\mathcal{R} \times \mathcal{G}$ acts on $C^\infty(P) \times \mathcal{F}$.

- Given $(\lambda, \tau) \in C^\infty(P) \times \mathcal{F}$, let S be a local slice defined using the exponential map on $\mathcal{M}$.

- Then consider

$W^{k,p}_S$ $\qquad$ $\mathcal{E}^{k-1,p}$ $\qquad$ Sobolev completions.

$$L : S \longrightarrow \Omega^3(P) \times \Omega^3(P, N^*\cdot)$$
$$(\lambda, \tau) \longmapsto (\tau^*\rho,\ \tau_N + d\lambda \wedge \rho_N)$$

perturbed SL equations

- $\mathcal{M}^S = \mathcal{M}^S(A, P; (\rho, \tau)) = L^{-1}(0)$

- $D_{(\lambda,\tau)} =$ Linearization of L at $(\lambda, \tau) \in \mathcal{M}^S$.

- $\mathcal{R}^{l,\alpha} = \{ C^{l,\alpha}\ G_2\text{-pairs } (\rho, \tau) \text{ tamed by } (\rho', \omega') \}$ $\qquad$ (Fredholm operator)

- $\mathcal{R}^{l\alpha}_{reg} = \{ (\rho, \tau) \in \mathcal{R}^{l\alpha}$ such that $D_{(\lambda,\tau)}$ is surjective when $(\lambda, \tau) \in \mathcal{M}^S \}$

- Universal Local Moduli Space: $\mathcal{M}^S(A, P; \mathcal{R}^{l,\alpha})$

18.

Transversality

Theorem: The moduli space $\mathcal{M}(A, P; (\rho, \tau))$ is a set of isolated points whenever $(\rho, \tau) \in \mathcal{R}_{reg}$. $\qquad$ Smooth.

Proof:

- First, work on a slice.

- After taking the appropriate Sobolev completions, view L as a map between Banach spaces and $\mathcal{M}^S \subseteq W^{k,p}_S$

- By regularity, $\mathcal{M}^S$ does not depend on k + p.

- Apply the implicit function theorem and note that the points in $\mathcal{M}^S$ are also isolated in the C^∞-topology.

- Since this holds for all local slices it also holds for $\mathcal{M}$ altogether.

19.

Proposition: For the right choices of ℓ, k, the universal moduli space $\mathcal{M}^S(A, P; \mathcal{R}^\ell)$ is a separable $C^{\ell-k-1}$ Banach submanifold of $W_S^{k,p} \times \mathcal{R}^{\ell,\alpha}$.

Proof: Implicit function theorem + some linear algebra.

Recall...

Definition: If T is a topological space a subset is called residual if it's the intersection of a countable number of open, dense sets.

Sard's Smale Theorem: If $f: X \longrightarrow Y$ between separable Banach manifolds then the set of regular values is residual.

☆ So "residual" is "generic."

20.

Theorem: $\mathcal{R}_{reg}$ is residual in $\mathcal{R}$.

Proof:

1. First, consider the projection $\pi : \mathcal{M}^S(A, P; \mathcal{R}^{\ell,\alpha}) \longrightarrow \mathcal{R}^{\ell,\alpha}$ (Fredholm w/ same index as $D_{(\bar{u},\epsilon)}$)

   Note that $\mathcal{R}_{reg}^{\ell,\alpha}$ = regular values of π + apply Sard-Smale.

   So $\mathcal{R}_{reg}^{\ell,\alpha}$ is dense in $\mathcal{R}^{\ell,\alpha}$ wrt $C^{\ell,\alpha}$ topology.

Taubes' Trick

2. Define $\mathcal{R}_{reg,C} \subset \mathcal{R}$ to be the set of smooth G_2-pairs $(\rho,\tau) \in \mathcal{R}$ such that $D_{(1,\iota)}$ is surjective for all graphical associatives satisfying

i) $$\frac{|\iota(p)-\iota(q)|_M}{|\iota(p)-\iota(q)|_{\iota P}} \geq \frac{1}{C} \qquad \forall\ p,q \in P$$

ii) $$\| \mathrm{II}(\iota_1) \|_{L^p} \leq C \qquad p > 3$$

- Every graphical associative satisfies i) + ii) for some $C > 0$. Therefore
$$\mathcal{R}_{reg} = \bigcap_{C>0} \mathcal{R}_{reg,C}$$
- The easy compactness theorem allows us to prove that $\mathcal{R}_{reg,C}$ is open by proving that its complement is closed.
- Then we also have to show each is dense. This works in exactly the same way as it does for J-holomorphic curves.