Spectrum of singular G_2-instantons

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Based on joint project with Henrique Sá Earp and Thomas Walpuski

More colleagues might join

- Background/motivation.
- results.
- More detail.

Point: Eigenvalues/spectrum of

the link operator P:

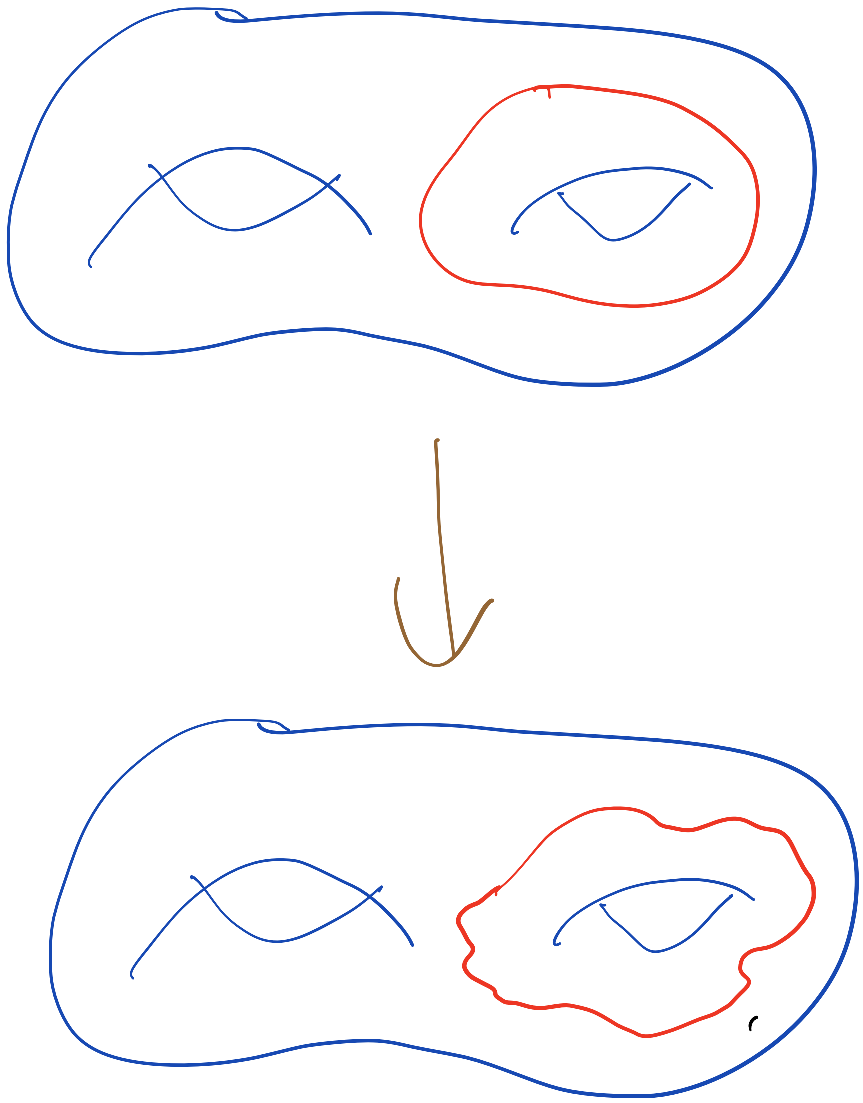
(C^∞) Dirac operator $|_{Dom \to \mathbb{S}^5}$

and corollaries.

Donaldson-Thomas and Donaldson-Segal program.

Walpuski: Gluing construction of G_2-instantons with $1-$dim singularities.

Tian, Tao-Tian: compactness and removability of singularity.
Jacob–Walpuski: building blocks.
Jacob–Sá Earp-Walpuski. Chen-Sun.

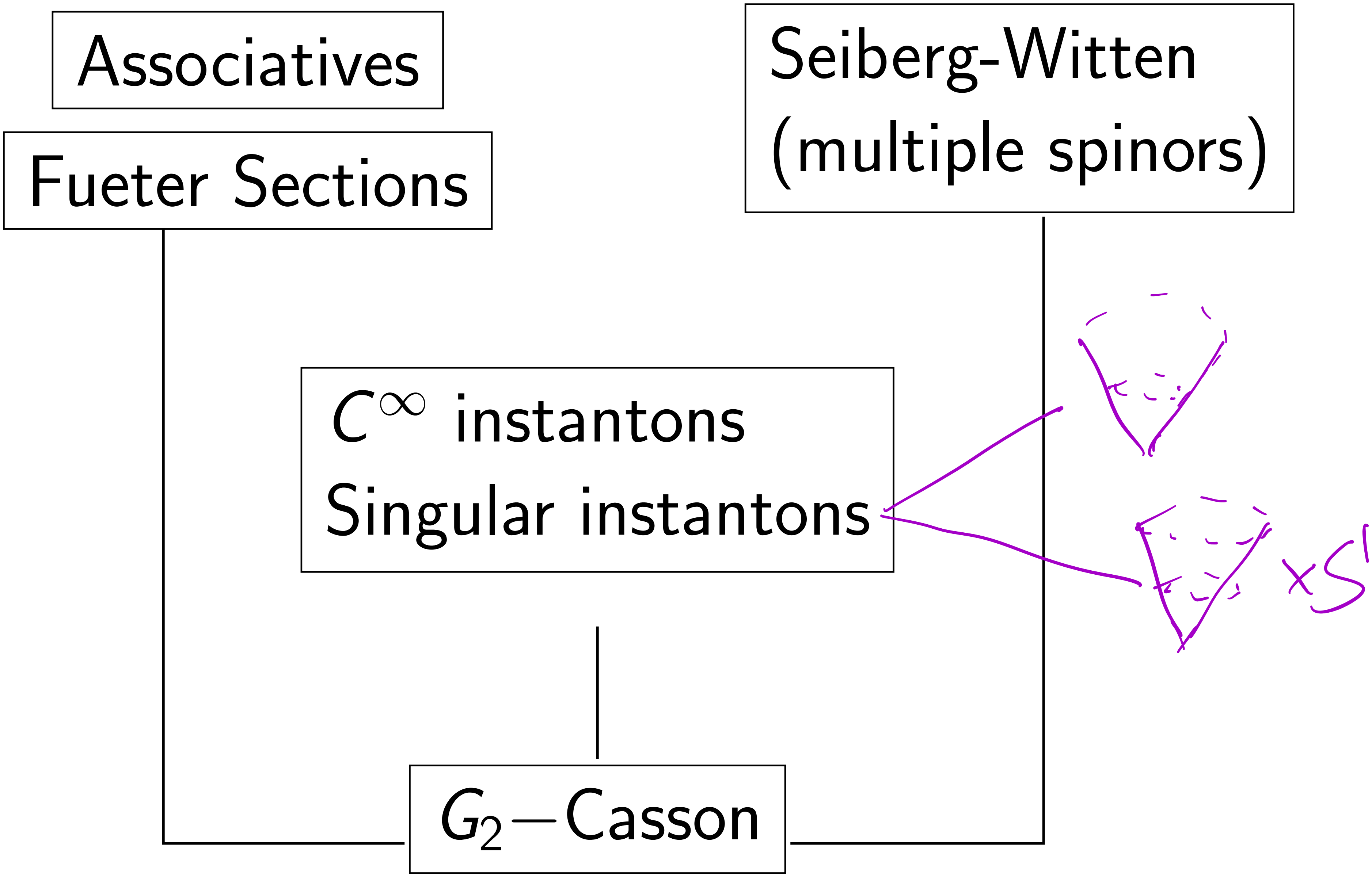


singular locus is an indep variable in the problem.

Recent singular perturbations, dim of singular locus ≥ 1:

Donaldson, Takahashi, He, Parker, He–Walpuski

Donaldson-Segal, Haydys-Walpuski, Doan-Walpuski:


Associatives
Fueter Sections
Seiberg-Witten (multiple spinors)
C^∞ instantons
Singular instantons
$\times S^1$
G_2-Casson

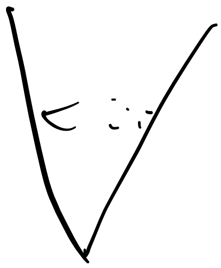
Local model of 1-dim singularities:

proj G_2−instanton singular along $O \times \mathbb{S}^1$	$\Leftarrow$	stable Hermitian Yang-Mills, rank ≥ 2
(A, E)		(A, E)

$$(\mathbb{C}^3 \setminus O) \times \mathbb{S}^1 \to \mathbb{P}^2$$

Gauge theoretic analogue of

closed

 $\times M$ in Rm Geom

minimal surface

Mazzeo – Smale

Serre duality: (Digression).

$$H^1[\mathbb{P}^2, (End E)(l)] = H^1[\mathbb{P}^2, (End E)(-l-3)]$$

Reflection w.r.t $-\frac{3}{2}$

$$\mu \longleftrightarrow -\mu - 3$$

$$\mu^2 + 2\mu - 3 \longleftrightarrow \mu^2 + 4\mu$$

$\mathbb{S}^5, \qquad \eta^{\perp}$

$$Dom = (adE)^{\oplus 4} \bigoplus (D^{\star} \otimes adE)$$

(5−components)

∃ isometries (quaternion str) $I,\ K$ on *Dom* s.t. (model) linearization of instanton with $1-$dimensional singularities is

$$L = \frac{\partial}{\partial s}\cdot I + K\cdot\left(\frac{\partial}{\partial r} - \frac{P}{r}\right)$$

compare $\Delta_{\mathbb{R}^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$

$\mathbb{R}^2 - 0 = \mathrm{cone}(S^1)$

$$\frac{\partial}{\partial s}\cdot I = I\cdot\frac{\partial}{\partial s},\ \text{but } KP = -PK - 3K.$$

Need

Analysis of (singular) linearized operators

Lockhart-McOwen, Melrose-Mendoza (isolated singularities). Mazzeo (edge singularities, general dimension), etc.

Indicial weights of

$$L: \quad \text{Weighted Sobolev Space}_0 \longrightarrow \text{Weighted Sobolev Space}_1$$

is determined by

eigenvalues of P.

$$0 < \nabla^\star \nabla|_{\mathbb{S}^5} \sim \begin{array}{c} adE \\ \Big| \\ \mathbb{S}^5 \end{array}$$

$$\mu^2 + 2\mu - 3 = \lambda.$$

$$S_{\nabla^\star\nabla} = \cup_{\lambda \in Spec \nabla^\star\nabla|_{\mathbb{S}^5}} \{-1+\sqrt{4+\lambda},\ -1-\sqrt{4+\lambda},\ -2+\sqrt{4+\lambda},\ -2-\sqrt{4+\lambda}\}$$

$$\mu^2 + 4\mu = \lambda$$

$$S_{coh} = \{l \in \mathbb{Z} | H^1[\mathbb{P}^2, (EndE)(l)] \neq 0\}.$$

Thm [2019]:

$$SpecP = S_{\nabla^\star \nabla} \cup S_{coh}.$$

Multiplicities and eigen-sections “explicit”.
Symmetric w.r.t $-\frac{3}{2}$.

$$L: \left.\begin{aligned} B &= O(\tfrac{1}{r^p}) \\ \nabla_A B &= O(\tfrac{1}{r^{p+1}}) \end{aligned}\right\} \longrightarrow O(\frac{1}{r^{p+1}})$$

$p \in \mathbb{R}.$

Quadratic nonlinearity

$$F_A = dA + \frac{1}{2}[A, A],$$

$$\frac{1}{r^p} \cdot \frac{1}{r^p} = O(\frac{1}{r^{p+1}}) \quad \text{if} \quad p \leq 1$$

Need solution B to lin equ with $|B| = O(\frac{1}{r})$.

$p = 1$ is interesting

Thm continued:

$$SpecP \cap (-3, 0) = \{-1, -2\}.$$

When $j = 0,\ -1,\ -2,\ -3,$

$$Eigen_j P = H^1[\mathbb{P}^2, (EndE)(j)]$$

Diffeomorphisms χ_t. Solve

$$\star_t(F^0_{A+a} \wedge \chi_t^\star \psi) + \text{monopole term} = 0$$

with gauge fixing (A is instanton).

$$Aux(X) = \star[F^0_A \wedge d(X \lrcorner \psi)]$$

$$\text{if } d\psi = 0.$$

Partial analytic obstruction

$$\frac{H^1[\mathbb{P}^2,\ (EndE)(-1)]}{\{Atiyah\ Classes\}}$$

vanishes iff $E = T^{1,0}\mathbb{P}^2(k)$.

$\{Atiyah\ Classes\} = \{r(Y \lrcorner F_A^0) | Y \in \mathbb{R}^6\}$.

Proposition

Under model data, local universal linearized equation

$$\widetilde{L}B = f$$

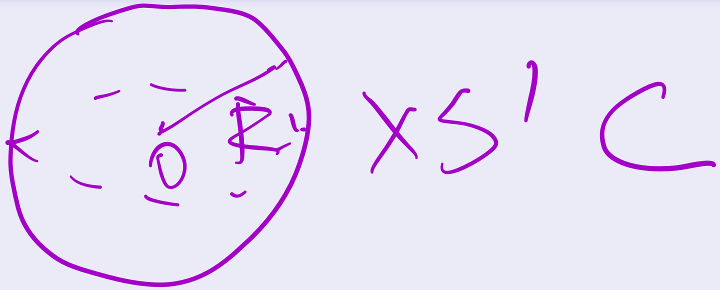
has $O(\frac{1}{r})-$right inverse/ Green's function with $C^0_{O(\frac{1}{r})}-$apriori estimate $\Longrightarrow$

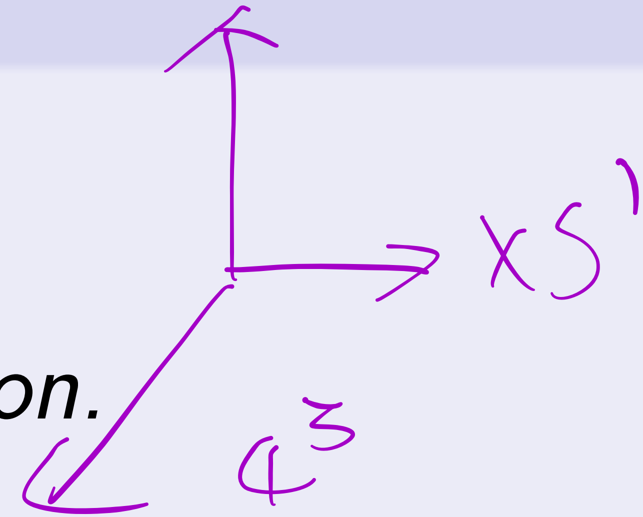
$$A|_E = \textit{twisted Fubini-Study connection} \, |_{T^{1,0}\mathbb{P}^2(k)}$$

$\exists$ global version, no extension assumption anymore.

Proposition

Model data

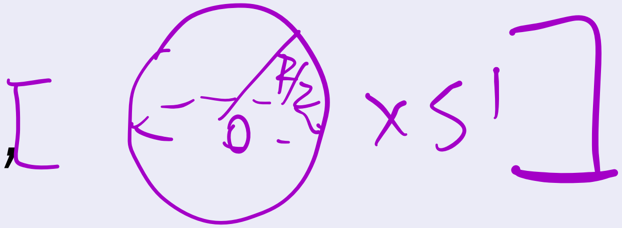




Fubini-Study tan cone connection.

$L\alpha = 0$,

$\langle \alpha, L\xi \rangle \in L^1$,



$\alpha \perp$ *RangeAux* $\Longrightarrow$

$$\alpha \;=\; \textit{section pulled back from } \mathbb{C}^3$$
$$+\textit{non pullback with } \mathbb{I}_\mu \textit{ but not } \mathbb{K}_\mu$$

Corollary

$$\alpha = \frac{1}{r}(\mathit{Eigen}_{-1} \quad components)$$

$$+O(1)$$

$Lb = 0$ on $CY^3 \times \mathbb{S}^1$

$$b = \Sigma_{k \in \frac{2\pi}{\rho_0}\mathbb{Z}} e^{iks} b_k$$

$$\Box_{CY^3} b_k = k b_k.$$

We can adjust length of circle to construct non-pullback harmonic sections.

On spectrum:
Moroianu-Semmelmann (deformation of nearly Kähler structures).
Charbonneau-Harland (deformation of nearly Kähler instantons).
Ikeda–Taniguchi ($Spec\Delta_{Hodge}$ on $\mathbb{P}^n$ and $\mathbb{S}^n$).

Examples of G_2-instantons multiple approaches, deformations/local structure of moduli:
Sá Earp-Walpuski, Sá Earp, Walpuski,
Sá Earp-Menet-Nordström, Platt, Gutwein.
Clarke, Oliveira, Oliveira-Lotay, Ball-Oliveira, Alonso, Waldron, Driscoll, Singhal, Fadel-Nagy-Oliveira etc.

Explicit info on Sasakian $\mathbb{S}^5 \Longrightarrow$

P is 5×5 "matrix":

Block-diagonal Bochner formulas for

$$P^2 + 2P - 3 \ \ (row \ \ 1, 2 = \nabla^\star \nabla)$$

$$P^2 + 4P \ \ (row \ \ 3, 4)$$

$$\mu^2 + 2\mu - 3 = \lambda$$

$AdE-$component (row 1 and 2) of an eigen-section of μ of P must also be $\lambda-$eigensection of $\nabla^\star\nabla$ or 0. But if μ is not a root, it can not be eigen-section. $\Longrightarrow$

it must be 0.

ODEs of Fourier co-efficients

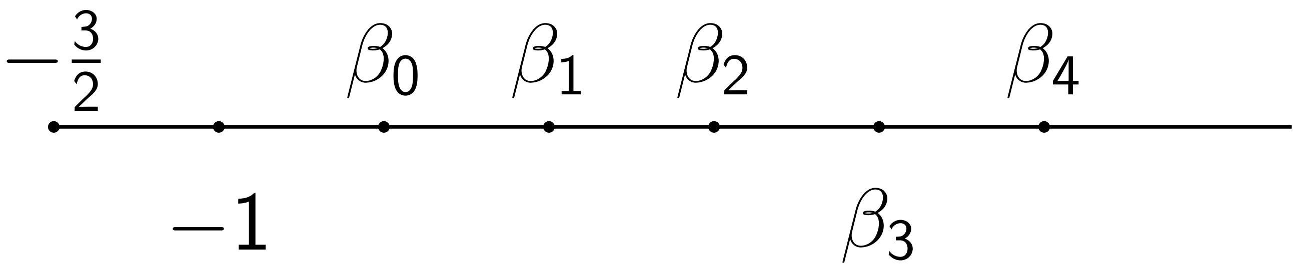
$$x'' + \frac{3x'}{r} - \frac{\beta(\beta+2)x}{r^2} - k^2 x = h.$$

Solutions to homogeneous equ:

$$\frac{K_{\beta+1}(kr)}{r}, \quad \frac{I_{\beta+1}(kr)}{r}$$

In $SpecP \cap (-\frac{3}{2}, \infty)$

only $\beta = -1$ could cause problem.



$\exists\ h_k(r)$ (forcing terms) supported away from 0 s.t. $O(1)-$Fourier co-efficient must be

$$0 \cdot \frac{I_0(kr)}{r} + 0 \cdot \frac{K_0(kr)}{r}$$

$$+ \text{a good term}$$

$$+ \quad \frac{I_0(kr)}{r} \int_0^r K_0(ky) y^2 h_k(y) dy$$

$$I_0(kr) \sim \frac{e^{kr}}{\sqrt{2\pi kr}},$$

But

$$\int_0^r K_0(ky)........$$

decays at most polynomially in k, norms of the forcing terms grow at most polynomially in k.

Weight shift: $|\phi_\beta|_{\mathbb{C}^3} = \frac{1}{r}|\phi_\beta|_{\mathbb{S}^5}$ (1−form nature).

$O(1)$−Fourier coefficient $x_\beta \Longrightarrow$

$$O(\frac{1}{r})\text{−Fourier modes } x_\beta\phi_\beta.$$

Hopf fibration $\mathbb{S}^5 \longrightarrow \mathbb{P}^2$ has Sasakian Quaternion structure.

$$\xi : \text{Reeb vector field } |_{\mathbb{S}^5}.$$

$$\eta : \text{contact } 1\text{−form } |_{\mathbb{S}^5}.$$

$$[\frac{1}{2r^3}(r\frac{\partial}{\partial r} - i\xi)] \lrcorner dz_1 dz_2 dz_3$$

$$= H - iG$$

$$P = \begin{bmatrix} 1 & -L_\xi & 0 & 0 & -(d_0\cdot)\lrcorner H \\ L_\xi & 1 & 0 & 0 & (d_0\cdot)\lrcorner G \\ 0 & 0 & -4 & -L_\xi & d_0^{\star 0} \\ 0 & 0 & L_\xi & -4 & -(d_0\cdot)\lrcorner \frac{d\eta}{2} \\ J_H d_0 & -J_G d_0 & d_0 & J_0 d_0 & -L_\xi J_0 \end{bmatrix}$$

B is $P-$eigen-section, but not of $S_{\nabla^\star \nabla} \Longrightarrow$

$B = b$ is semi-basic $1-$form on $\mathbb{S}^5 \Longrightarrow$

$$d_0 b \lrcorner H = d_0 b \lrcorner G =$$

$$d_0 b \lrcorner \frac{d\eta}{2} = d_0^{\star 0} b = 0$$

$$G,\ H \text{ “}(2,0)\text{”},$$

$$\frac{d\eta}{2} \text{ pullback Fubini-Study “}(1,1)\text{”}$$

$\Longrightarrow$

$$b^{0,1} \text{ is } \overline{\partial}_0 - \text{harmonic}$$

$EndE(l)$: Fourier Series twisted by $O(l)$.

$$b^{0,1} = b^{0,1}(k)(X_0, X_1, X_2)^k$$

$EndE(k)-$valued
$\overline{\partial}-$harmonic
$1-$form on $\mathbb{P}^2$

trivialization
of $O(-k)|_{\mathbb{S}^5}$

Example:
$T\mathbb{P}^2(k)$+representation theory/Peter Weyl formulation
$\Rightarrow$ explicit spectrum.

Row 1: Eigenvalues in $[-4, 1]$.

-4	$-2\sqrt{2}-1$	-2	-1	$2\sqrt{2}-2$	1
12	16	6	6	16	12

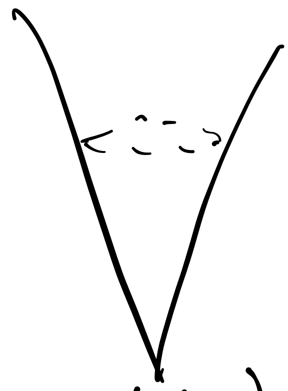
Row 2: Multiplicities.

Corollary

$\mathcal{F} \longrightarrow CY^3$ non-locally free admissible reflexive sheaf. At an admissible Hermitian connection,

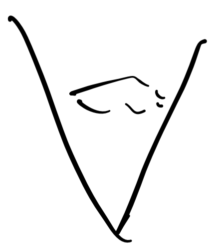
$$
\begin{aligned}
& Index\,\square_{CY^3} \\
= \; & -2\Sigma_j h^1[\mathbb{P}^2, EndE_j(-1)] \\
& -2\Sigma_j h^1[\mathbb{P}^2, EndE_j].
\end{aligned}
$$

Deformation of $HYM|_{F \to CY^3}$



(with Thomas Walpuski)

of spin(7)-instantons with



(with Alex Waldron, and Thomas Walpuski).

More colleague might join.

Clm: For any $l \in \mathbb{Z}_+$, $(a, b) = (2, l-1)$ is the unique solution among $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ to

$$\begin{cases} a^2 + b^2 + ab + 3a + 3b = l^2 + 3l + 6, \\ \\ 3 - a - 2b, b - a - 3 \leq l \\ \leq 2a + b - 3, 3 + b - a. \end{cases}$$

Putnam prob?

$$
\begin{aligned}
& \widetilde{L}_{model} \\
= \; & L_{model} - \\
& \{\frac{\partial X_i}{\partial s} \cdot [(J_{\mathbb{C}^3} e_i) \lrcorner F_A^0] + \frac{\partial X_i}{\partial r} \cdot J_H(e_i \lrcorner F_A^0)\} \\
& X = X_s \frac{\partial}{\partial s} + \Sigma_{i=1}^6 X_i e_i.
\end{aligned}
$$

$X_s, \; X_i$ only depend on $r, \; s$.

supp near $0 \times S^1$

$$\int_{<\epsilon \times S^1} < Aux(X), \text{har sect } b = O(\frac{1}{r^3}) >$$

$$= \ \ldots \int_{\mathbb{S}^1} \langle X, (b \to \mathbb{R}^6) \rangle_{R^6} ds$$

Formula: Let X be C^1 vector field supported in $B_O(R) \times \mathbb{S}^1$, $Lb = 0$, $b = O(\frac{1}{r^3}) \Longrightarrow$

$$\int_{B_O(R)\times\mathbb{S}^1} < \star[F_A \wedge d(X \lrcorner \psi)], b >$$

$$= \; -c_0 \Sigma_j \int_{\mathbb{S}^1} < X(O_j, s),$$

$$\boxminus^{-1} J_H[\lim_{r\to 0}(r^2 P_{Eigen_{-2}} b)] >_{\mathbb{R}^6} ds,$$