Resolutions of (Compact) Spin(7)-Orbifolds

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May 15, 2024

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Spin(7)-Orbifolds

Definition

Let X be an n-dimensional, real orbifold and let X^{sing} denote the union of its singular strata. Let S be a connected component of X^{sing}.

- *X is of singularity type (I) at S, if X at $S \subset X$ is locally modelled on*

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- *X is of singularity type (IIb) at S, if it is neither type (I) nor type (IIa)*

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Let (X, Ω) be a Spin(7)-orbifold. A torsion-free resolution is given by a smooth family of Spin(7)-manifold $(X^t, \widetilde{\Omega}^t)$ and a map

$$\rho^t : X^t \dashrightarrow X$$

that restricts to a diffeomorphism onto $X \backslash X^{sing}$ such that the exceptional set $E^t = (\rho^t)^{-1}(X^{sing})$ is of codimension > 0.

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that restricts to a diffeomorphism onto $X \backslash X^{sing}$ such that the exceptional set $E^t = (\rho^t)^{-1}(X^{sing})$ is of codimension > 0.Furthermore, the family $(X^t, \widetilde{\Omega}^t) \xrightarrow{t \to 0} (X, \Omega)$ in a "Gromov-Hausdorff sense", i.e.

$$\rho^t_* \widetilde{\Omega}^t \xrightarrow[C^\infty_{loc}]{t \to 0} \Omega$$

and

$$\mathrm{vol}_{\widetilde{\Omega}^t}(E^t) \xrightarrow{t \to 0} 0.$$

- let $i : (S, g_S) \hookrightarrow (X, \Omega, g)$ be a connected singular stratum of type (i) of codimension four and whose isotropy group is $\Gamma \subset Sp(1) \subset Spin(7)$. Let further

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- think of NS as a vector bundle $\nu:NS\to S$ with a fiberwise Γ-action (action by $Isot(S)$)
- normal cone bundle of S is given by the quotient orbifold

$$\nu_0:NS/Isot(S)=X_0\to S$$

- Riemannian orbifold structure g induces a splitting

$$TX_0 \cong H_0 \oplus \nu_0^* NS \cong \nu_0^* TS \oplus \nu_0^* NS$$

and a CF-Spin(7)-orbifold structure

$$\Omega_0 = \Omega_0^{4,0} + \Omega_0^{2,2} + \Omega_0^{0,4} \in \Omega^4(X_0)$$

and

$$g_0 = \nu_0^* g_S + g_{0;V} \in \Gamma\left(X_0, \mathrm{Sym}^2 \nu_0^* T^\vee S \oplus \mathrm{Sym}^2 \nu_0^* NS\right).$$

Remark

Notice, that there exists an $\mathbb{R}_{\geq 0}$-action on X_0

$$\Psi_t : X_0 \to X_0$$

such that

$$\Psi_t^* \Omega_0 = \Omega_0^{4,0} + t^2 \cdot \Omega_0^{2,2} + t^4 \cdot \Omega_0^{0,4} \qquad \text{and} \qquad \Psi_t^* g_0 = \nu_0^* g_S + t^2 g_{0;V}$$

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Notice, that on S *there exists a natural reduction of the Spin(7)-frame bundle* $Fr_{Spin(7),X}|_S \to S$ *to a* $N_\Gamma = Norm(\Gamma, Spin(7))$*-principal bundle*

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Moreover, F_S is a torsion-free extension of the SO(4)-frame bundle $Fr_{SO,S}$ of S. Let in the following φ denote Levi-Civita connection on F_S.

- given a tubular neighbourhood

$$j : \mathrm{Tub}_{2\epsilon}(S) \hookrightarrow X$$

we expand the Spin(7)-structure

$$j^*\Omega = \Omega_0 + \Omega_{hot} \qquad |\Omega_{hot}|_{g_0} = \mathcal{O}(r)$$

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Remark

We will later need to C^0-estimate of the higher order terms. In order to get them "small" with respect to the gluing parameter, we need to choose $\epsilon \sim t^\lambda$, for $0 \leq \lambda < 1$.

(Pre-)Resolving Spin(7)-Orbifolds

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- *today only $\mathbb{H}/\Gamma$*

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Proposition ([Kro90])

There exists a vertical $\mathrm{Im}(\mathbb{H})$-valued two form $\underline{\omega}$ on $\mathbb{M} \xrightarrow{\kappa} \Theta_{\mathrm{Im}(\mathbb{H})}$ and codimension three walls $\mathcal{W} \subset \Theta_{\mathrm{Im}(\mathbb{H})}$ such that for all $\zeta \in \Theta_{\mathrm{Im}(\mathbb{H})} \backslash \mathcal{W}$, $(\kappa^{-1}(\zeta), \underline{\omega}) = (M_\zeta, \underline{\omega}_\zeta) \dashrightarrow (\mathbb{H}/\Gamma, \underline{\omega}_0)$ is a hyperkähler ALE space of rate -4.

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- *permutes Weyl chambers*
- κ *is equivariant*
- *universal vertical hyperkähler structure is invariant*

Definition

Let $C \subset Com(im(N_\Gamma), Norm(\Gamma, SO(4)) \ltimes Weyl(\Gamma))$ and $\mathfrak{C} \to S$ a C-principal bundle. We define the twisted vector bundle

$$\mathfrak{H} = (F_S \times_S \mathfrak{C}) \times_{N_\Gamma \times C} \Theta_{\mathrm{Im}(\mathbb{C})} \quad \textit{and} \quad \mathfrak{H}^2_+ = (F_S \times_S \mathfrak{C}) \times_{N_\Gamma \times C} \Theta_{\mathrm{Im}(\mathbb{H})}.$$

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- let $\zeta \in \Omega^2_+(S, \mathfrak{H})$, define the fibration

$$\begin{array}{ccccc} & & \overline{\zeta}^*\mathbb{M} & \xrightarrow{\tilde{\zeta}} & \mathbb{M} \\ & \swarrow^{\psi_\zeta} & \downarrow{\scriptstyle \kappa_\zeta} & & \downarrow{\scriptstyle \kappa} \\ X_\zeta & & F_S \times_S \mathfrak{C} & \xrightarrow{\overline{\zeta}} & \Theta_{\mathrm{Im}(\mathbb{H})} \\ & \searrow_{\nu_\zeta} & \downarrow{\scriptstyle \phi_S} & & \\ & & S & & \end{array}$$

Lemma ([Majc])

Define the four form

$$\hat{\Omega}_\zeta = \hat{\Omega}_\zeta^{4,0,0} + \hat{\Omega}_\zeta^{2,2,0} + \hat{\Omega}_\zeta^{0,4,0} \in \Omega^{4,0}(\overline{\zeta}^*\mathbb{M})^{N_\Gamma \times C}$$

by

$$\begin{aligned}\hat{\Omega}_\zeta^{4,0,0} =& (\kappa_\zeta \circ \phi_S)^* \mathrm{vol}_{g_S}\\ \hat{\Omega}_\zeta^{2,2,0} =& -\frac{1}{3}\left\langle \kappa_\zeta^* \theta_+^S \wedge \tilde{\zeta}^* \underline{\omega}^{0,2,0} \right\rangle\\ \hat{\Omega}_\zeta^{0,4,0} =& \frac{1}{6}\left\langle \tilde{\zeta}^* \underline{\omega}^{0,2,0} \wedge \tilde{\zeta}^* \underline{\omega}^{0,2,0} \right\rangle\end{aligned}$$

The space

$$\pi_\zeta : (X_\zeta, \Omega_\zeta, g_\zeta) \dashrightarrow (X_0, \Omega_0, g_0)$$

is a Spin(7)-resolution.

Lemma ([Majc])

There exists a "scaling map" lift

$$\begin{array}{ccc} (X_\zeta, \Omega^t_\zeta) & \xrightarrow{\Psi_t} & (X_{t^2\cdot\zeta}, \Omega_{t^2\cdot\zeta}) \\ \downarrow{\scriptstyle \pi_\zeta} & & \downarrow{\scriptstyle \pi_{t^2\cdot\zeta}} \\ (X_0, \Omega^t_0) & \xrightarrow{\Psi_t} & (X_0, \Omega_0) \end{array}$$

where $\Psi_t^*\Omega_{t^2\zeta} = \Omega_\zeta^{4,0} + t^2 \cdot \Omega_\zeta^{2,2} + t^4 \cdot \Omega_\zeta^{0,4}$

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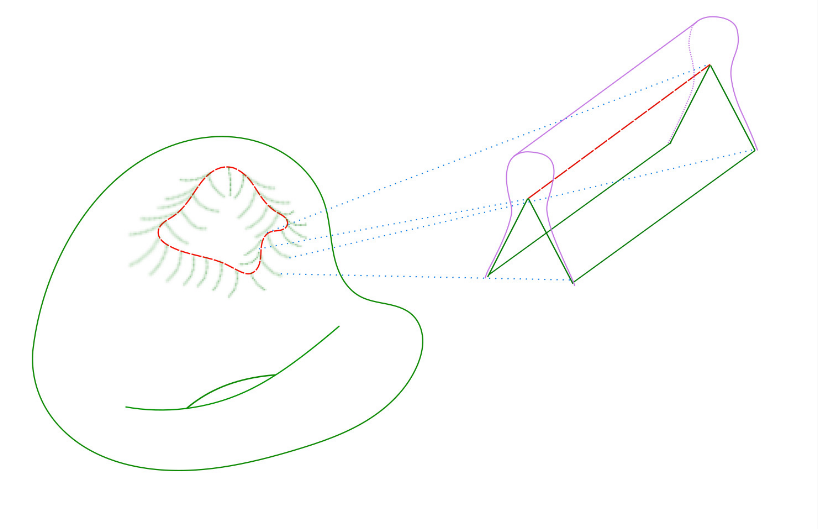
- *if $\zeta \pitchfork 0$ conically singular over $\mathbb{S}^7/\Gamma$*

Theorem ([Majc])

If $\mathrm{d}_\varphi \zeta = 0$ *then*

$$\mathrm{d}\Omega_\zeta = \mathrm{d}^{2,-1}(\Omega_\zeta^{2,2} + \Omega_\zeta^{0,4})$$

In particular, $\mathrm{d}\Omega_{t^2\cdot\zeta} = (\Psi_t)_* \mathrm{d}\Omega_\zeta^t = t \cdot \mathrm{d}\Omega_\zeta$ *and hence* $\Omega_{t^2\cdot\zeta}$ *defines an adiabatic Spin(7)-structure.*



- the composition of maps

$$\Gamma^t_\zeta : B_{2t^{-1}\epsilon}(X_\zeta)\backslash U^t_0 \xrightarrow{\Psi_t} B_{2\epsilon}(X_{t^2\cdot\zeta})\backslash U_0 \overset{\pi_{t^2\cdot\zeta}}{\dashrightarrow} \mathrm{Tub}_{2\epsilon}(S)\backslash S \overset{j}{\hookrightarrow} X$$

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and

$$\begin{aligned} \Omega^t =& \Omega^t_\zeta + \chi^t_2 \cdot (\Omega^t_0 - \Omega^t_\zeta + \Omega^t_{hot}) \\ \mathrm{d}\Omega^t =& (1-\chi^t_2)\mathrm{d}\Omega^t_\zeta + \mathrm{d}\chi^t_2 \wedge (\Omega^t_0 - \Omega^t_\zeta + \Omega^t_{hot}) \end{aligned}$$

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- $\rho^t : (X^t, \Omega^t) \dashrightarrow (X, \Omega)$ converges in a Gromov-Hausdorff sense

Can we find a torsion-free Spin(7)-structure close to Ω^t?

Existence of Torsion-Free Resolutions of Spin(7)-Orbifolds

- the preglued Spin(7)-structure Ω^t has "small torsion"

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- condition of a Spin(7)-structure in a small neighbourhood of Ω^t to be torsion free is given by

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- instead of constructing a $\mathrm{Diff}_0(X^t)$-orbit full of solutions it is more convenient to construct a solution in $\mathfrak{Spin}(7)[X^t] \Rightarrow \mathrm{Diff}_0(X^t)$-gauge slice

$$\begin{aligned} 0 =& \mathrm{d}\Omega^t + \mathrm{d}\pi_{\tau,\Omega^t}(\eta) + \mathrm{d}Q_{\Omega^t}(\eta) \\ 0 =& \pi_{1\oplus 7,\Omega^t}\eta. \end{aligned}$$

- the gauge-fixed equations are equivalent to

$$-\,\mathrm{d}\eta = \mathrm{d}\Omega^t + \mathrm{d}Q_{\Omega^t}(\eta) \tag{1}$$

$$-\,\mathrm{d}^*\eta = *\,\mathrm{d}\Omega^t + *\mathrm{d}Q_{\Omega^t}(\eta). \tag{2}$$

or equivalently

$$-(\mathrm{d}+\mathrm{d}^{*^t})\eta = (\mathrm{d}+\mathrm{d}^{*^t})\pi_{-,\Omega^t}\left\{Q_{\Omega^t}(\eta)\right\} + (\mathrm{d}-\mathrm{d}^{*^t})\left(\Omega^t + \pi_{+,\Omega^t}\left\{Q_{\Omega^t}(\eta)\right\}\right).$$

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- let $R_{\mathrm{d}+\mathrm{d}^{*^t}}$ denote a right inverse to $(\mathrm{d}+\mathrm{d}^{*^t})$

$$\eta = -(\mathrm{d}+\mathrm{d}^{*^t})\pi_{-,\Omega^t}\left\{Q_{\Omega^t}(R_{\mathrm{d}+\mathrm{d}^{*^t}}\eta)\right\} - (\mathrm{d}-\mathrm{d}^*)\left(\Omega^t + \pi_{+,\Omega^t}\left\{Q_{\Omega^t}(R_{\mathrm{d}+\mathrm{d}^{*^t}}\eta)\right\}\right). \qquad (3)$$

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Remark

This behaviour of D^t is share by Dirac operators on orbifold resolutions (more general set-up in upcoming paper [Majb]).

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- instead consider the operator $D^t_\zeta = \Psi^*_t \circ D_{t^2 \cdot \zeta} \circ \Psi^*_{-t} = D^t_{\zeta;H} + t^{-1} \cdot D_{\zeta;V}$ where $D_{\zeta;V}$ is the vertical Hodge-de Rham-operator

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- D^t_ζ is the Dirac operator of an adiabatic family of Dirac bundles $(\wedge^\bullet T^\vee X_\zeta, \mathrm{cl}^{g^t_\zeta}, g^t_\zeta, \nabla^{g^t_\zeta})$ (see [Goe14])

- we are interested in the operator $D_{t^2\cdot\zeta}$ on $X_{t^2\cdot\zeta}$
- problems: no Fredholm theory and no insides into the adiabatic limit $t \to 0$
- instead consider the operator $D^t_\zeta = \Psi^*_t \circ D_{t^2\cdot\zeta} \circ \Psi^*_{-t} = D^t_{\zeta;H} + t^{-1} \cdot D_{\zeta;V}$ where $D_{\zeta;V}$ is the vertical Hodge-de Rham-operator
- D^t_ζ is the Dirac operator of an adiabatic family of Dirac bundles $(\wedge^\bullet T^\vee X_\zeta, \mathrm{cl}^{g^t_\zeta}, g^t_\zeta, \nabla^{g^t_\zeta})$ (see [Goe14])
- "**harmonic forms on** (X_ζ, g^t_ζ) **concentrate along vertically harmonic forms**" [Bis86, BC89, BL91, BL95, Goe14]

- the push forward along ν_ζ allows us to view the operator D^t_ζ as an operator

$$\begin{array}{ccc} \Omega^\bullet_\beta(X_\zeta) & \xrightarrow{D^t_\zeta} & \Omega^\bullet_\beta(X_\zeta) \\ \| & & \| \\ \Gamma(S, (\nu_\zeta)_{*_\beta} T^\vee X_\zeta) & \xrightarrow{D^t_\zeta} & \Gamma(S, (\nu_\zeta)_{*_\beta} T^\vee X_\zeta) \end{array}$$

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- to stay consistent with the literature (see [Wal17, Pla20], etc.) we will work with (weighted) Hölder norms on X^t
- using Hölder-completions of the fibres of the push forward

$$t^{-1}\cdot D_{\zeta;V} : (\nu_\zeta)_{*^{k+1,\alpha}_\beta} T^\vee X_\zeta \to (\nu_\zeta)_{*^{k,\alpha}_{\beta-1}} T^\vee X_\zeta$$

and split

$$(\nu_\zeta)_{*^{k+1,\alpha}_\beta} \wedge^\bullet T^\vee X_\zeta = \mathcal{I}^{k,\alpha}_{AC;\beta}(X_\zeta/S) \oplus \mathcal{K}_{AC}(X_\zeta/S)$$

- we decompose the Dirac operator

$$D_\zeta^t = \begin{pmatrix} D^t_{\zeta;\mathcal{I}\mathcal{I}} & D^t_{\zeta;\mathcal{K}\mathcal{I}} \\ D^t_{\zeta;\mathcal{I}\mathcal{K}} & D^t_{\zeta;\mathcal{K}\mathcal{K}} \end{pmatrix} = \begin{pmatrix} D^t_{\zeta;H;\mathcal{I}\mathcal{I}} + t^{-1} \cdot D_{\zeta;V;\mathcal{I}\mathcal{I}} & D^t_{\zeta;H;\mathcal{K}\mathcal{I}} \\ D^t_{\zeta;H;\mathcal{I}\mathcal{K}} & D^t_{\zeta;H;\mathcal{K}\mathcal{K}} \end{pmatrix}$$

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- "**harmonic forms on (X_ζ, g^t_ζ) concentrate along vertically harmonic forms**" $\Rightarrow$ understand $D^t_{\zeta;\mathcal{K}\mathcal{K}}$

Proposition ([Majb])

The vertical kernel bundles are isomorphic to

$$\mathcal{K}_{AC}(X_\zeta/S) \cong \wedge^\bullet T^\vee S \otimes \mathcal{H}^\bullet(X_\zeta/S)$$

where $\mathcal{H}^\bullet(X_\zeta/S)$ consist of vertically harmonic (anti-self dual) two forms that decay with rate -4. Moreover, this is again a Hermitian $Cl(TS, g_S)$-module, whose Hermitian structure is given by the fiberwise L^2_t-norm.

Definition

The effective Dirac operator of D_ζ^t is the induced operator

$$D^t_{\zeta;\mathcal{K}\mathcal{K}} = D_{\mathcal{K}} + t \cdot \mathrm{cl}^{g_\zeta}(F_{H_\zeta})_{\mathcal{K}\mathcal{K}}$$

and the adiabatic residue of

$$\mathfrak{D}_\zeta = \lim_{t\to 0} D^t_{\zeta;\mathcal{K}\mathcal{K}} = \mathfrak{D}_{GM}$$

is given by the Gauß-Manin-Hodge-de Rham operator. The adiabatic kernel is defined by

$$\mathfrak{Ker}(D^t_\zeta) = \ker(\mathfrak{D}_{GM})$$

Definition

We say D_ζ^t *is isentropic (i.e. adiabatic and reversible) if* $\ker(D_\zeta^t) \cong \mathfrak{Ker}(D_\zeta^t)$

Definition

We say D^t_ζ is isentropic (i.e. adiabatic and reversible) if $\ker(D^t_\zeta) \cong \mathfrak{Ker}(D^t_\zeta)$

Remark

There exist topological obstructions on the isentropicity of D^t_ζ that can be computed using spectral sequences.

Definition

Let define the projection

$$\tilde{\pi}^t : \Omega^\bullet(X^t) \to \Omega^\bullet\left(S, \mathcal{H}^\bullet(X_\zeta/S)\right) : \tilde{\iota}^t$$

and a section of $\tilde{\pi}^t$. We further define the maps

$$\tilde{\pi}_{\mathcal{K}} = \tilde{\iota}^t \tilde{\pi}^t : \Omega^{\bullet;l,\alpha}_{\beta;t}(X^t) \to \Omega^{\bullet;l,\alpha}_{\beta;t}(X^t).$$

and $\tilde{\pi}_{\mathcal{I}} = 1 - \tilde{\pi}_{\mathcal{K}}$.

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and $\tilde{\pi}_{\mathcal{I}} = 1 - \tilde{\pi}_{\mathcal{K}}$. Let $\kappa > 0$ and α, β and λ be fixed. We define the D^t-adiabatic norms

$$||\gamma||_{\mathfrak{X}^t} = ||\tilde{\pi}_{\mathcal{I}}\gamma||_{C^{1,\alpha}_{\beta;t}} + t^{-\kappa} \left|\left|\tilde{\pi}^t\gamma\right|\right|_{C^{1,\alpha}_t} \quad (4)$$

$$||\gamma||_{\mathfrak{Y}^t} = ||\tilde{\pi}_{\mathcal{I}}\gamma||_{C^{0,\alpha}_{\beta-1;t}} + t^{-\kappa} \left|\left|\tilde{\pi}^t\gamma\right|\right|_{C^{0,\alpha}_t} . \quad (5)$$

Theorem ([Majb])

Assume that D^t is isentropic. We can choose α, β, λ and κ in a reasonable way. Then there exists a right-inverse R_{D^t} to D^t satisfying

$$||R_{D^t}\eta||_{\mathfrak{X}^t} \lesssim ||\eta||_{\mathfrak{Y}^t}$$

Theorem ([Majc])

Assume that D^t is isentropic. We can choose α, β, λ and κ such that there exists a torsion-free Spin(7)-structure $\widetilde{\Omega}^t$ such that

$$\left|\left|\Omega^t - \widetilde{\Omega}^t\right|\right|_{\mathfrak{X}^t} \lesssim t^{\vartheta}$$

whereby ϑ depends on all the choices.

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$\mathrm{codim}(S)$	α	β	λ	κ	l_j	ϑ_ζ	ϑ
4	0.01	-1.0	0.4	1.0	1		0.8
-	-	-0.67	0.428	-	2		1.14
-	-	-	0.01	-	∞	∞	3.96

Table 1: Good choice of parameters for $\mathfrak{X}^t$-norms.

$\mathrm{codim}(S)$	α	β	λ	κ	l_j	ϑ_ζ	ϑ
4	0.01	-0.5	0.444	0.5	1		0.656
-	-	-	-	-	2		1.1
-	-	-	0.4	-	∞	∞	2.6

Table 2: Good choice of parameters for $C^{1,\alpha}_{\beta;t}$-norms.

Corollary ([Majc])

Assume that D^t is isentropic. We can choose α, β, λ and κ such that there exists a torsion-free G_2-structure $\widetilde{\varphi}^t$ such that

$$||\varphi^t - \widetilde{\varphi}^t||_{\mathfrak{X}^t} \lesssim t^{\vartheta}$$

whereby ϑ depends on all the choices.

Examples

Theorem ([Majc])

Let $\widetilde{X^t} \dashrightarrow \mathbb{T}^8/\Gamma$ be one of the Joyce manifolds constructed in [Joy96] and $\widetilde{\Omega}^t$ its torsion-free Spin(7)-structure. Let Ω^t be the Spin(7)-structure on X^t constructed from the pregluing process. Then

$$\left|\left|\widetilde{\Omega}^t - \Omega^t\right|\right|_{\mathfrak{X}^t} \lesssim t^{\sim 4}.$$

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Theorem ([Majc])

Let $\widetilde{X^t} \dashrightarrow \mathbb{CP}^{[a_0,\dots,a_4]}$ be one of the manifolds constructed in [Joy99] and $\widetilde{\Omega}^t$ its torsion-free Spin(7)-structure. Let Ω^t be the Spin(7)-structure on X^t constructed from the pregluing process. Then

$$\left|\left|\widetilde{\Omega}^t - \Omega^t\right|\right|_{\mathfrak{X}^t} \lesssim t^{\sim 4}$$

Definition

Let $(\hat{X}, \hat{\omega}, \hat{\theta}, \hat{g})$ be a Calabi-Yau four-fold given by a complete intersection. Assume that the real locus S of $\hat{X}$ is of real codimension four and is smooth.

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Proposition ([Majc])

Let $\mathfrak{z}_2 \to S$ be a two-fold, unramnified cover of S. If there exists a harmonic, self-dual, $\mathfrak{z}_2$-twisted, non vanishing two form ζ, then the quotient $\hat{X}/\mathbb{Z}_2$ admits a resolution

$$\pi_\zeta : X^t \dashrightarrow \hat{X}/\mathbb{Z}_2.$$

Moreover, if D^t is isentropic the resolution carries a family of torsion-free Spin(7)-structure $\widetilde{\Omega}^t$ resolving the natural one on $(\hat{X}/\mathbb{Z}_2, \Omega, g_\Omega)$ induced by the Calabi-Yau structure.

- pick the same K3 surface as in [JK21], that admits a branched double cover

$$K3 \to \mathbb{CP}^2$$

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by swapping the sheets of the branched covering and consequently $\mathrm{Fix}(\alpha_{K3}) = C$

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- the $\mathbb{Z}_2$ action on $\mathbb{CP}^2$ by complex conjugation can be lifted to an action

$$\begin{array}{ccc} K3 & \xrightarrow{\beta_{K3}} & K3 \\ \downarrow & & \downarrow \\ \mathbb{CP}^2 & \xrightarrow{\bar{\cdot}} & \mathbb{CP}^2 \end{array}$$

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- non-tivial argument one can show that $\mathrm{Fix}(\beta_{K3}) = \mathbb{S}^2$ and $\mathrm{Fix}(\alpha_{K3}\beta_{K3}) = \emptyset$

- set

$$\mathbb{S}^1 = [0,4]/\sim$$

an define the $\mathbb{Z}_2^3$-action

$$\begin{aligned}
\alpha_{\mathbb{T}^4} &: (x_1, x_2, x_3, x_4) \mapsto (-x_1, 2 - x_2, x_3, x_4) \\
\beta_{\mathbb{T}^4} &: (x_1, x_2, x_3, x_4) \mapsto (2 - x_1, x_2, -x_3, x_4) \\
\gamma_{\mathbb{T}^4} &: (x_1, x_2, x_3, x_4) \mapsto (-x_1, -x_2, 2 - x_3, 2 - x_4)
\end{aligned}$$

on the four torus $\mathbb{T}^4$, with

$$\mathrm{Fix}(\alpha_{\mathbb{T}^4}) = \left\{ \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \mathbb{S}^1, \mathbb{S}^1 \right\}$$

$$\mathrm{Fix}(\beta_{\mathbb{T}^4}) = \left\{ \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \mathbb{S}^1, \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \mathbb{S}^1 \right\}$$

$$\mathrm{Fix}(\gamma_{\mathbb{T}^4}) = \left\{ \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\}.$$

- lift the action of $\mathbb{Z}_2^3$ to $\mathbb{T}^4 \times K3$ via $\alpha = \alpha_{\mathbb{T}^4} \times \alpha_{K3}$, $\beta = \beta_{\mathbb{T}^4} \times \beta_{K3}$ and $\gamma = \gamma_{\mathbb{T}^4} \times \mathrm{id}_{K3}$, whose fix point sets are given by

$$\mathrm{Fix}(\alpha) = \left\{ \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \mathbb{S}^1, \mathbb{S}^1 \right\} \times C$$

$$\mathrm{Fix}(\beta) = \left\{ \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \mathbb{S}^1, \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \mathbb{S}^1 \right\} \times \mathbb{S}^2$$

$$\mathrm{Fix}(\gamma) = \left\{ \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\} \times K3.$$

and

$$\alpha\beta, \alpha\gamma, \beta\gamma, \alpha\beta\gamma$$

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$$\alpha\beta, \alpha\gamma, \beta\gamma, \alpha\beta\gamma$$

  act freely on $\mathbb{T}^4 \times K3$
- this $\mathbb{Z}_2^3$ action preserves the torsion-free Spin(7)-structure

$$\hat{\Omega} = \mathrm{vol}_{\mathbb{T}^4} + \mathrm{vol}_{K3} - \mathrm{tr}_+ \left(\underline{\omega}_{\mathbb{T}^4} \wedge \underline{\omega}_{K3} \right)$$

  on $\mathbb{T}^4 \times K3$ and hence the torsion-free structure $\hat{\Omega}$ descends to a torsion-free Spin(7)-structure Ω on the orbifold

$$(\mathbb{T}^4 \times K3)/\mathbb{Z}_2^3.$$

- singular strata in $(\mathbb{T}^4 \times K3)/\mathbb{Z}_2^3$ are non-intersecting and given by two copies of $\mathbb{T}^2 \times C$, two copies of $\mathbb{T}^2 \times \mathbb{S}^2$ and four copies of $K3$
- pick the resolution data on $\mathbb{T}^2 \times C$ to be determined by $\zeta = \mathrm{vol}_{\mathbb{T}^2} + \mathrm{vol}_C$, the resolution data on $\mathbb{T}^2 \times \mathbb{S}^2$ by $\zeta = \mathrm{vol}_{\mathbb{T}^2} + \mathrm{vol}_{\mathbb{S}^2}$ and on the $K3$ we can freely pick any combination of the hyperkähler triple as ζ

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Lemma ([Majc])

There exists a torsion-free Spin(7)-structure $\widetilde{\Omega}^t$ on X^t satisfying

$$\left|\left|\tilde{\Omega}^t - \Omega^t\right|\right|_{C^{1,\alpha}_{\beta;t}} \lesssim t^{\sim 2.6}.$$

Moreover, as $b^1(X^t) = b^2_7(X^t) = 0$ the resolved Spin(7)-manifold has full holonomy.

- $\mathrm{Sym}^2(K3) = (K3 \times K3)/\mathbb{Z}_2$ is a Spin(7)-orbifold whose torsion-free Spin(7)-structure is induced by the product structure

$$\hat{\Omega} = \mathrm{vol}_1 + \mathrm{vol}_2 - \mathrm{tr}(\underline{\omega}_1 \wedge \underline{\omega}_2).$$

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- singular strata of $\mathrm{Sym}^2(K3)$ is the diagonal $K3$ and its normal bundle can be identified with its tangent bundle, i.e.

$$0 \longrightarrow TK3 \longrightarrow T\mathrm{Sym}^2(K3) \longrightarrow NK3 \longrightarrow 0$$

splits by $NK3 \cong TK3 \ni v \mapsto (\pm v \oplus \mp v)$

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  splits by $NK3 \cong TK3 \ni v \mapsto (\pm v \oplus \mp v)$
- there exists a two sphere of non-vanishing, harmonic, self-dual two forms corresponding to resolutions

$$X_{\omega_I} \dashrightarrow NK3/\mathbb{Z}_2.$$

  The spaces X_{ω_I} are diffeomorphic to $\mathrm{Bl}_I(\Delta : K3 \hookrightarrow \mathrm{Sym}^2(K3))$ and the resolved space is diffeomorphic to

$$\widetilde{\mathrm{Sym}^2(K3)}^t \cong \mathrm{Hilb}^2(K3).$$

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Proposition ([Majc])

There exists a family of hyperkähler metrics $\widetilde{g}^t$ *on* $\mathrm{Hilb}^2(K3)$ *satisfying*

$$||\widetilde{g}^t - g^t||_{C^{1,0.01}_{-0.5;t}} \lesssim t^{1.1}$$

and $\lambda = 0.444$ *and* $\kappa = 1/2$.

Summary

- correspondence between resolutions of X_0 and harmonic $\zeta \in \Omega^2_+(S, \mathfrak{H})$

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- improved existence result for Spin(7)-orbifold resolutions
- new examples of compact Spin(7)-manifolds and approximations of hyperkähler metrics on $Hilb^2(K3)$

Further Research

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- generalise construction to all types of orbifold singularities using Mazzeo's Edge calculus
- computing η-invariants of G_2-manifolds (extending work of [For])

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