Moduli Spaces of G_2 Holonomy Metrics

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The Model Case

Definition

On $\mathbb{R}^7 = \mathbb{R}^3 \oplus \mathbb{H}$:

$$\varphi_{\mathrm{std}} = \mathrm{d}\, x^1 \wedge \mathrm{d}\, x^2 \wedge \mathrm{d}\, x^3 - \sum_{\alpha=1}^{3} \mathrm{d}\, x^\alpha \wedge \omega_\alpha$$

- $\mathrm{G}_2 = \{A \in \mathrm{GL}(7) \,:\, A^* \varphi_{\mathrm{std}} = \varphi_{\mathrm{std}}\} \leq \mathrm{SO}(7)$
- $\Lambda^{3,+}\mathbb{R}^7 = \mathrm{GL}^+(7)/\mathrm{G}_2 \subseteq \Lambda^3\mathbb{R}^7$ open subset of positive forms.

Fact:

- $(\iota_v \varphi_{\mathrm{std}}) \wedge (\iota_w \varphi) \wedge \varphi = 6\langle v, w\rangle_{\mathrm{std}} \mathrm{vol}_{\mathbb{R}^7}$
  - $\varphi_{\mathrm{std}} \rightsquigarrow \langle -, - \rangle_{\mathrm{std}}$ and $\varphi_{\mathrm{std}} \rightsquigarrow \mathrm{vol}_{\mathbb{R}^7}$

G_2 Structures on Manifolds

A G_2-structure on M is a choice of $\varphi \in \mathcal{G}_2(M) := \Gamma(\Lambda^{3,+} T^\vee M)$.

- $\varphi \rightsquigarrow g_\varphi$ Riemannian metric and $\varphi \rightsquigarrow \mathrm{vol}_{g_\varphi}$ orientation

$T(\varphi) = \nabla^{g_\varphi} \varphi$ torsion tensor

- $T(\varphi) = 0 \Longleftrightarrow \mathrm{d}\,\varphi$ and $\mathrm{d}^{*\varphi}\,\varphi = 0$ (torsion-free)
- $T(\varphi) = 0$ implies $\mathrm{Hol}(g_\varphi) \leq \mathrm{G}_2$
- For M closed:
    - $\mathrm{Hol}(g_\varphi) = \mathrm{G}_2$ if and only if $T(\varphi) = 0$ and $\pi_1(M)$ finite
    - $|\,\mathrm{Iso}(g_\varphi)\,| < \infty$ if $\mathrm{Hol}(g_\varphi) = \mathrm{G}_2$

Moduli Space of G_2-Manifolds

Theorem (Joyce '96)

The G_2 moduli space

$$\mathfrak{X}(M) := \mathcal{G}_2^{tf}(M)/\mathrm{Diff}(M)_0$$

is a smooth manifold of dimension $b_3(M)$.

Remark:

- completely local result, $\mathfrak{X}(M) = \emptyset$ possible
- scarce knowledge about global topological properties:
  - not compact if non empty
  - Crowley-Goette-Nordström '16: May be disconnected
  - no knowledge about higher homotopy groups so far

Main Result

Theorem (Goette-H. '23)

Joyce's first example M^7_{Joyce} of a closed G_2 manifold satisfies

$$\pi_2(\mathfrak{X}(M^7_{Joyce})) \neq 0.$$

Proof Strategy

Given $M \hookrightarrow (E, \varphi) \to S^k$ fibre bundle with

- structure group $\mathrm{Diff}(M)_0$
- $\varphi = \{\varphi_b\}_{b \in S^k}$ fibrewise torsion-free G_2 structure

$$f_{E,\varphi} \colon S^k \to \mathfrak{X}(M), \qquad b \mapsto [F^* \varphi_b]$$

Comparison Theorem (Goette-H. '23)

For all $k \geq 2$, we have

$$[f_{E,\varphi}] \neq 0 \in \pi_k(\mathfrak{X}(M)) \qquad \text{if } E \ncong M \times S^k$$

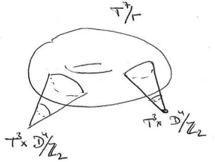
If $k \geq 3$, then all elements of $\pi_k(\mathfrak{X}(M))$ arise by this construction.

The Flat Orbifold

Setup:

$(T^7, \varphi_{\mathrm{std}}) \curvearrowleft \Gamma \cong \mathbb{Z}_2^3$ such that $g^* \varphi_{\mathrm{std}} = \varphi_{\mathrm{std}}$

$\left(T^7/\Gamma, \varphi_{\mathrm{std}}\right)$ flat orbifold



Singular set: 12 copies of T^3

locally around: $T^3 \subseteq U \cong T^3 \times \mathbb{H}/\mathbb{Z}_2$

Resolve singularities with Eguchi-Hanson spaces

The Blow Up Model

Definition

The Eguchi-Hanson metric on $\mathbb{H}/\mathbb{Z}_2 \setminus \{0\}$ given by

$$g_{EH,a} = \left(1 + (a/r)^4\right)^{-1/2} \left(\mathrm{d}\, r^2 + r^2\sigma_{\mathrm{i}}^2\right) + \left(1 + (a/r)^4\right)^{1/2} \left(r^2\sigma_{\mathrm{j}}^2 + r^2\sigma_{\mathrm{k}}^2\right)$$

extends to a hyper Kähler ALE metric on

$$EH = T^{\vee}\mathbb{C}P^1 = (S^3 \times \mathbb{C})/\sim \qquad (p, \lambda) \sim (\mathrm{e}^{\mathrm{i}\theta}, \mathrm{e}^{-2\mathrm{i}\theta}\lambda)$$

$\rightsquigarrow$ torsion-free G_2 structure on $T^3 \times EH$ via

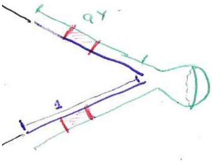
$$\varphi^{EH,a} := \mathrm{d}\, t^{\mathrm{i}} \wedge \mathrm{d}\, t^{\mathrm{j}} \wedge \mathrm{d}\, t^{\mathrm{k}} - \sum_{\alpha \in \{\mathrm{i,j,k}\}} \mathrm{d}\, t^{\alpha} \wedge \omega_{\alpha}^{EH,a}$$

$$\varphi^{EH,a} = \varphi_{\mathrm{std}} \quad \text{at infinity}$$

Resolving Singularities

$$M^7_{Joyce} = T^7/\Gamma \setminus \bigcup_{\alpha=1}^{12} T^3 \times D^4/\mathbb{Z}_2 \ \cup \ \bigcup_{\alpha=1}^{12} T^3 \times EH_{\leq a^{-1}}$$

$$\varphi \text{ "="} \ \varphi_{\text{std}} \cup \varphi^{EH,a}$$



$$\mathrm{d}\,\varphi = 0 \qquad \text{and} \qquad ||\mathrm{d}^{\varphi}\,\varphi|| \ll 1$$

Theorem (Joyce '96)

If $a \ll 1$, then φ can be perturbed to a torsion-free G_2 structure $\tilde{\varphi}$.

Topological Properties of M^7_{Joyce}

- M^7_{Joyce} is simply connected
  - **Consequence:** $\mathrm{Hol}(g_{\tilde{\varphi}}) = \mathrm{G}_2$

- $b_2(M) = 12,\ b_3(M) = 7 + 12 \cdot 3 = 43$
  - **Consequence:** $\mathfrak{X}(M^7_{Joyce}) \neq \emptyset$ is a 43-dimensional manifold

Proposition

$$p_1(M^7_{Joyce}) \in \mathrm{Hom}(H_4(M;\mathbb{Z}), 3\mathbb{Z}) \subseteq H^4(M;\mathbb{R})$$

Consequence: $\int_{N^4} f^* p_1(M^7_{Joyce}) \in 3\mathbb{Z}$ for all $f : N^4 \to M^7_{Joyce}$

Families of Eguchi Hanson Spaces

Definition

The Eguchi-Hanson fibre metric on $\mathbb{H}/\mathbb{Z}_2 \setminus \{0\} \times \mathbb{C}P^1 \to \mathbb{C}P^1$ given by

$$g_{\mathcal{EH},a,[q]} := \left(1+(a/r)^4\right)^{-1/2} \left(\mathrm{d}\, r^2 + r^2 \sigma^2_{q^{-1}\mathrm{i}q}\right) + \left(1+(a/r)^4\right)^{1/2} \left(r^2 \sigma^2_{q^{-1}\mathrm{j}q} + r^2 \sigma^2_{q^{-1}\mathrm{k}q}\right)$$

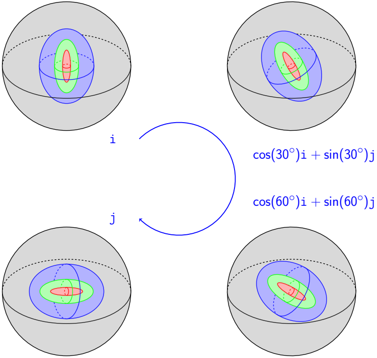
extends to an ~~hyper Kähler~~ ALE fibre-metric on

$$\mathcal{EH} = (S^3 \times \mathbb{C}P^1 \times \mathbb{C})/\sim \qquad (p,[q],\lambda) \sim (p,[q],\mathrm{e}^{q^{-1}\mathrm{i}q\theta},\mathrm{e}^{-2\mathrm{i}\theta}\lambda)$$

$\rightsquigarrow$ fibrewise torsion-free G_2 structure on $T^3 \times \mathcal{EH} \to \mathbb{C}P^1$ via

$$\varphi^{\mathcal{EH},a}_{[q]} := \mathrm{d}\, t^{\mathrm{i}} \wedge \mathrm{d}\, t^{\mathrm{j}} \wedge \mathrm{d}\, t^{\mathrm{k}} - \sum_{\alpha \in \{\mathrm{i},\mathrm{j},\mathrm{k}\}} \mathrm{d}\, t^{q^{-1}\alpha q} \wedge \omega^{\mathcal{EH},a}_{q^{-1}\alpha q}$$

Eguchi-Hanson Family - Visualised



$$g_{\mathcal{EH},a,[q]} = \left(1+(a/r)^4\right)^{-1/2}\left(\mathrm{d}\,r^2 + r^2\sigma^2_{q^{-1}\mathrm{i}q}\right) + \left(1+(a/r)^4\right)^{1/2}\left(r^2\sigma^2_{q^{-1}\mathrm{j}q} + r^2\sigma^2_{q^{-1}\mathrm{k}q}\right)$$

Topological Properties of Eguchi-Hanson Bundle

Proposition

$$EH \longrightarrow \mathcal{EH} \xrightarrow{\mathrm{pr}_2} \mathbb{C}P^1$$

$$\mathbb{C} \hookrightarrow \mathcal{EH} \xrightarrow{[p,[q],\lambda]\mapsto([qp],[q])} \mathbb{C}P^1 \times \mathbb{C}P^1, \qquad \iota : \mathbb{C}P^1 \times \mathbb{C}P^1 \dashrightarrow \mathcal{EH}$$

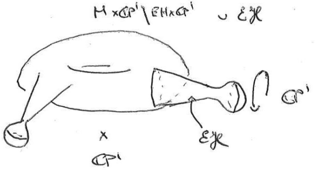
Properties:

1. $\mathcal{EH} \setminus \iota(\mathbb{C}P^1 \times \mathbb{C}P^1) \cong \mathbb{H}/\mathbb{Z}_2 \setminus \{0\} \times \mathbb{C}P^1$
2. $\varphi^{\mathcal{EH},a}_{[q]} = \varphi^{EH,a} = \varphi_{\mathrm{std}}$ at infinity
3. $p_1(T^{\mathrm{vert}}\mathcal{EH}) = c_1(\mathcal{EH})^2 = -8xy \in H^4(\mathcal{EH}) = H^4(\mathbb{C}P^1 \times \mathbb{C}P^1) \cong \mathbb{Z}$
   - **Consequence:** $\langle p_1(T^{\mathrm{vert}}\mathcal{EH}), [\mathbb{C}P^1 \times \mathbb{C}P^1]\rangle = -8$

Family of Joyce Manifolds

$$E_{M^7_{Joyce}} = \left(M^7_{Joyce} \setminus T^3 \times EH_{\leq a^{-1}}\right) \times \mathbb{C}P^1 \cup T^3 \times \mathcal{EH}_{\leq a^{-1}}$$

$$\varphi^{E_M}_{[q]} \text{ "=" } \varphi^M \cup \varphi^{\mathcal{EH},a}_{[q]}$$



$$\mathrm{d}\,\varphi^{E_M}_{[q]} = 0 \qquad \text{and} \qquad ||\mathrm{d}^{\varphi^{E_M}_{[q]}}\,\varphi^{E_M}_{[q]}|| \ll 1$$

Theorem (Joyce '96) in families

If $a \ll 1$, then the fibre G_2 structure $\varphi^{E_M} = \{\varphi^{E_M}_{[q]}\}_{\{q \in \mathbb{C}P^1\}}$ can be perturbed to a fibrewise torsion-free G_2 structure $\tilde{\varphi}^{E_M}$.

The Topology of $E_{M^7_{Joyce}}$

- $M^7_{Joyce} \hookrightarrow (E_{M^7_{Joyce}}, \tilde{\varphi}^{E_{M^7_{Joyce}}}) \xrightarrow{\mathrm{pr}_2} \mathbb{C}P^1$ fibre bundle

- $\mathbb{C}P^1 \times \mathbb{C}P^1 \xrightarrow{\iota} \mathcal{EH}_{\leq a^{-1}} \hookrightarrow E^7_{M_{Joyce}}$ canonical embedding

Proposition

$$E_{M^7_{Joyce}} \ncong M^7_{Joyce} \times \mathbb{C}P^1$$

Consequence of Homotopy Theorem:

$$\mathrm{const} \not\simeq f_{E_M, \tilde{\varphi}^{E_M}} : S^2 = \mathbb{C}P^1 \to \mathfrak{X}(M^7_{Joyce}), \qquad [q] \mapsto [F^* \tilde{\varphi}^{EM}_{[q]}]$$

Proof of Non-Triviality

Proposition

$$E_{M^7_{Joyce}} \not\cong M^7_{Joyce} \times \mathbb{C}P^1$$

Proof.

- Evaluation at $\mathbb{C}P^1 \times \mathbb{C}P^1 \xrightarrow{\iota} \mathcal{EH}_{\leq a^{-1}} \subseteq E_M$ yields

$$\begin{aligned}\langle p_1(T^{\mathrm{vert}} E_M); \iota_*[\mathbb{C}P^1 \times \mathbb{C}P^1]\rangle &= \langle p_1(T^{\mathrm{vert}} \mathcal{EH}); \iota_*[\mathbb{C}P^1 \times \mathbb{C}P^1]\rangle \\ &= -8\end{aligned}$$

- Assume $\mathfrak{T}\colon E_M \xrightarrow{\cong} M \times \mathbb{C}P^1$, then $T^{\mathrm{vert}} E_M = \mathfrak{T}^* \mathrm{pr}_1^* TM$ and

$$\begin{aligned}\langle p_1(T^{\mathrm{vert}} E_M); \iota_*[\mathbb{C}P^1 \times \mathbb{C}P^1]\rangle &= \langle \mathfrak{T}^* \mathrm{pr}_1^* p_1(TM); \iota_*[\mathbb{C}P^1 \times \mathbb{C}P^1]\rangle \\ &= \langle p_1(M); (\mathrm{pr}_1 \circ \mathfrak{T} \circ \iota)_*[\mathbb{C}P^1 \times \mathbb{C}P^1]\rangle \\ &= 3k\end{aligned}$$

□

The Comparison Theorem

Comparison Theorem (Goette-H. '23) (geometrical version)

$(E, \varphi) \to S^k$ fibre bundle, φ fibrewise torsion-free G_2 structure. If $E \ncong M \times S^k$ with $k \geq 2$, then

$$[f_{E,\varphi}] \neq 0 \in \pi_k(\mathfrak{X}(M)) \neq 0, \qquad f_{E,\varphi} \colon b \mapsto [F^*\varphi_b].$$

If $k \geq 3$, then all elements of $\pi_k(\mathfrak{X}(M))$ arise by this construction

Comparison Theorem (Goette-H. '23) (homotopy-theoretical version)

$$\pi_k\big(\mathcal{G}_2^{tf}(M)/\!\!/\mathrm{Diff}(M)_0\big) \xrightarrow{\pi_k(p)} \pi_k\big(\mathcal{G}_2^{tf}(M)/\mathrm{Diff}(M)_0\big) = \pi_k(\mathfrak{X}(M))$$

- injective if $k \geq 2$
- bijective if $k \geq 3$

The Homotopy Quotient

Setup: $G \curvearrowright X$ proper right action.

$G \to EG \simeq \{*\} \to BG$ universal G-principal bundle

- homotopy functor: $[B, BG] = \{G \hookrightarrow P \to B\}/\sim_{\mathrm{Iso}}$

$X /\!\!/ G := (EG \times X)/G = EG \times_G X$ homotopy quotient

- $X \to X /\!\!/ G \to BG$ universal X-bundle with structure group G
- homotopy functor: $[B, X /\!\!/ G] = \bigsqcup_{[P]} \big(\pi_0 \Gamma(P \times_G X)\big)/_{\sim_{\mathrm{Iso}}}$
- $X /\!\!/ G \xrightarrow{\mathrm{pr}_2} X/G$ is weak homotopy equivalence if $G \curvearrowright X$ is free

$H \leq G$ closed subgroup, then fibre bundle

$$G/H \to X /\!\!/ H \to X /\!\!/ G$$

Applications to G_2 Geometry

Setup: $G = \mathrm{Diff}(M)_0 \curvearrowright \mathcal{G}_2^{tf}(M) = X$ via pull back.

$\mathcal{G}_2^{tf}(M)/\!\!/\mathrm{Diff}(M)_0$ homotopy moduli space

- $(M_{univ}, \varphi_{univ}) \to \mathcal{G}_2^{tf}(M)/\!\!/\mathrm{Diff}(M)_0$ universal family
- classifies $(E, \varphi) \to B$ up to homotopy and isomorphism

$\mathcal{G}_2^{tf}(M)/\!\!/\mathrm{Diff}(M)_0 \xrightarrow{\mathrm{pr}_2} \mathcal{G}_2^{tf}(M)/\mathrm{Diff}(M)_0$ comparison map

- map of interest, but not weak homotopy equivalence

$H := \mathrm{Diff}_{x_0}(M) = \{\varphi \in \mathrm{Diff}(M)_0 \ : \ D_{x_0}\varphi = \mathrm{id}\}$ obs. diffeo group

- $\mathrm{Diff}_{x_0}(M) \curvearrowright \mathcal{G}_2^{tf}(M)$ free

  $\rightsquigarrow$ $\mathcal{G}_2^{tf}(M)/\!\!/\mathrm{Diff}_{x_0}(M) \xrightarrow{\simeq} \mathcal{G}_2^{tf}(M)/\mathrm{Diff}_{x_0}(M)$
- $G/H = \mathrm{Fr}^+(M) \to \mathcal{G}_2^{tf}(M)/\!\!/\mathrm{Diff}_{x_0}(M) \to \mathcal{G}_2^{tf}(M)/\!\!/\mathrm{Diff}(M)_0$

Moduli Space versus Homotopy Moduli Space

$\mathcal{G}_2^{tf}(M)_\varphi$ path component of φ, $\Gamma = \mathrm{Stab}_{\mathrm{Diff}(M)_0}(\varphi)$.

$$\begin{array}{ccccc} \mathrm{Fr}^+(M) & \longrightarrow & \mathcal{G}_2^{tf}(M)_\varphi /\!\!/ D_{x_0} & \longrightarrow & \mathcal{G}_2^{tf}(M)_\varphi /\!\!/ D_0 \\ & & \downarrow & & \downarrow \\ & & \mathcal{G}_2^{tf}(M)_\varphi / D_{x_0} & \longrightarrow & \mathcal{G}_2^{tf}(M)_\varphi / D_0 \end{array}$$

Fibration Theorem

The observer moduli space is a fibre bundle over the moduli space:

$$\Gamma \setminus \mathrm{Fr}^+(M) \longrightarrow \mathcal{G}_2^{tf}(M)_\varphi / D_{x_0} \longrightarrow \mathcal{G}_2^{tf}(M)_\varphi / D_0$$

Moduli Space versus Homotopy Moduli Space

$\mathcal{G}_2^{tf}(M)_\varphi$ path component of φ, $\qquad \Gamma = \mathrm{Stab}_{\mathrm{Diff}(M)_0}(\varphi)$

$$\begin{array}{ccccc}
\mathrm{Fr}^+(M) & \longrightarrow & \mathcal{G}_2^{tf}(M)_\varphi /\!\!/ D_{x_0} & \longrightarrow & \mathcal{G}_2^{tf}(M)_\varphi /\!\!/ D_0 \\
\Gamma : \downarrow 1 & & \downarrow & & \downarrow \\
\Gamma \setminus \mathrm{Fr}^+(M) & \longrightarrow & \mathcal{G}_2^{tf}(M)_\varphi / D_{x_0} & \longrightarrow & \mathcal{G}_2^{tf}(M)_\varphi / D_0
\end{array}$$

Fibration Theorem

The observer moduli space is a fibre bundle over the moduli space:

$$\Gamma \setminus \mathrm{Fr}^+(M) \longrightarrow \mathcal{G}_2^{tf}(M)_\varphi / D_{x_0} \longrightarrow \mathcal{G}_2^{tf}(M)_\varphi / D_0$$

Thank You For Your Attention