G_2-instantons over generalised Kummer constructions via finite group actions

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G_2-Geometry

7-manifold Y

- G_2-structure: $\phi \in \Omega^3(Y)$ such that $(T_y Y, \phi_y) \cong (\mathbb{R}^7, \phi_{\text{model}})$ for $\forall y \in Y$.
  - $\text{Stab}_{\text{Gl}(\mathbb{R}^7)}(\phi_{\text{model}}) = G_2 < SO(\mathbb{R}^7)$
  - $\rightarrow$ metric g_ϕ and orientation $\rightarrow \psi := *_\phi \phi \in \Omega^4(Y)$.

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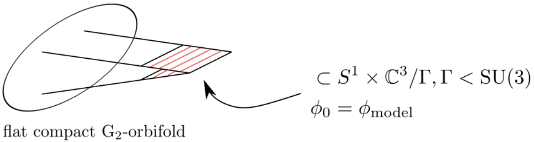
  - For $\psi_{\text{model}} = *\phi_{\text{model}}$: $\text{Stab}_{\text{Gl}(\mathbb{R}^7)}(\psi_{\text{model}}) = \mathbb{Z}_2 \times G_2 < O(\mathbb{R}^7)$
- G_2-structure is torsion-free $\Leftrightarrow$ $d\phi = 0$ and $d\psi = 0$.
  - G_2-manifold: (Y, ϕ).

Joyce's generalised Kummer construction

Produces G_2-manifolds as desingularisations of compact, flat G_2-orbifolds.

1. (Y_0, ϕ_0) compact, flat G_2-orbifold
   Assume: $\forall$ conn. comp. $S \subset \mathrm{Sing}(Y_0)$, ε-nbhd $B_\varepsilon(S)$ is isometric to nbhd in $S^1 \times \mathbb{C}^3/\Gamma$ for $\Gamma < \mathrm{SU}(3)$ acting freely on $\mathbb{C}^3 \setminus \{0\}$



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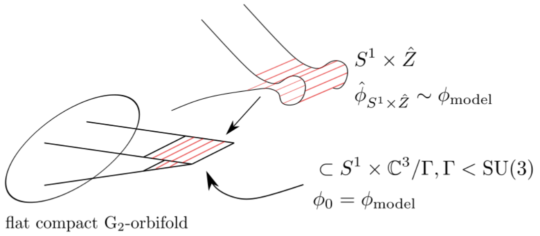

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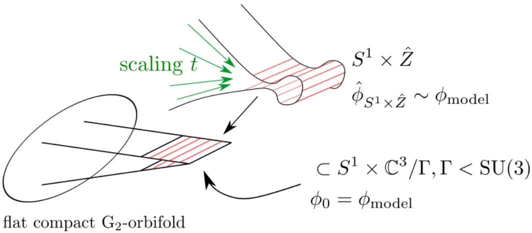
4. interpolation: G_2-structure $\tilde{\phi}_t \to$ deformation: torsion-free ϕ_t for $t \ll 1$



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Example (Joyce '96)

- G_2-orbifold $(\mathbb{R}^7/\Gamma, \phi_0)$ where Γ is generated by

$$\alpha(x_1,\ldots,x_7) = (x_2, x_3, x_7, -x_6, -x_4, x_1, x_5)$$
$$\beta(x_1,\ldots,x_7) = (\tfrac{1}{2} - x_1, \tfrac{1}{2} - x_2, -x_3, -x_4, \tfrac{1}{2} + x_5, \tfrac{1}{2} + x_6, x_7)$$
$$\tau_1(x_1,\ldots,x_7) = (x_1 + 1, x_2, x_3, x_4, x_5, x_6, x_7).$$

- $\mathrm{Sing}(\mathbb{R}^7/\Gamma) = S^1$ with model $S^1 \times \mathbb{C}^3/\mathbb{Z}_7$
  action generated by $\mathrm{diag}(e^{2\pi i/7}, e^{4\pi i/7}, e^{8\pi i/7}) \in SU(3)$

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Interlude: solutions to PDE's via Newton-Iteration

- Goal: solve (non-linear, elliptic) PDE $F(v) = 0$.
- Have: almost solution v_0 (i.e. $F(v_0)$ small)
- Hope: find small v s.t.

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- If L_{v_0} has right-inverse R, then (1) is satisfied by $v = Rw$ where

$$w = -F(v_0) - \mathcal{N}_{v_0}(Rw).$$

- often in gluing constructions: find such w via Banach's Fixedpoint Theorem

Interlude: solutions to PDE's via Newton-Iteration

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- Assume: F equivariant

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Observation

If v_0 is H-invariant, R is equivariant, and fixpoint w is unique, then w and $v = Rw$ are H-invariant. (Because $h \cdot w$ also solves fixpoint-problem for every $h \in H$.)

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- Involution $\sigma := -\mathrm{Id} \in \mathrm{Norm}_{O(\mathbb{R}^7)}(\Gamma)$ descends to $\mathbb{R}^7/\Gamma$ and preserves ψ_0

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- Fact: Obtain lift $\hat{\sigma}$ to $\hat{Y}$ such that $\hat{\sigma}^*\psi_t = \psi_t$ (and $\hat{\sigma}^*\phi_t = -\phi_t$)

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- Applications:
  - Joyce '96: (Co-) Associative submanifolds as fixpoint sets of G_2-(anti)-involutions.
  - Dwivedi–Platt–Walpuski '23: Associative submanifolds by deforming Morse–Bott families.
    $\rightarrow$ Use group action to ensure that deformed family contains associatives
  - G. '24: (obstructed) equivariant G_2-instantons via gluing.
    $\rightarrow$ Use group action to overcome possible obstructions

G_2-instantons

(Y, ϕ) compact G_2-manifold, $\pi\colon\ P \to Y$ principal G-bundle

Definition

A connection A on $\pi\colon\ P \to Y$ is called G_2-instanton, if $F_A \wedge \psi = 0$.

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Fix connection A_0. Then $A_0 + a$ for $a \in \Omega^1(Y, \mathfrak{g}_P)$ is a G_2-instanton (in Uhlenbeck gauge) iff there exists a $\xi \in \Omega^7(Y, \mathfrak{g}_P)$ such that

$$\left.\begin{aligned} 0 &= F_{A_0+a} \wedge \psi + d^*_{A_0+a}\xi \\ 0 &= d^*_{A_0} a \end{aligned}\right\} = F_{A_0} \wedge \psi + L_{A_0}(a,\xi) + Q_{A_0}(a,\xi)$$

$$=: \Upsilon_{A_0}(a,\xi)$$

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- (Z, ω, Ω) Calabi–Yau 3-fold
- $\rightarrow$ G_2-manifold: $S^1 \times Z$ with $\psi := \frac{1}{2}\omega \wedge \omega + \mathrm{d}s \wedge \mathrm{Re}\,\Omega$

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- $\to$ $\mathrm{pr}_Z^* A$ is G_2-instanton on $\mathrm{pr}_Z^*\pi\colon\ \mathrm{pr}_Z^* P \to S^1 \times Z$ iff
  $F_A \wedge \omega \wedge \omega = 0$ and $F_A \wedge \mathrm{Re}\,\Omega = 0$
- $\Leftrightarrow$ A is Hermitian Yang–Mills (assume G is real)

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- Walpuski '13: G_2-instantons over generalised Kummer constructions
- Platt '22: G_2-instantons over orbifold resolutions

Group actions

Assumption

Assume:

- H finite group
- H-action $\tilde{\lambda}\colon\ H \to \mathrm{Isom}(P)$, i.e.

$$\begin{array}{ccc} P & \xrightarrow{\tilde{\lambda}(h)} & P \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\lambda(h)} & Y \end{array}$$

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- $\Rightarrow$ H acts on all $\Omega^k(Y, \mathfrak{g}_P)$
- Observation: If A_0 is H-invariant, then

$$\Upsilon_{A_0}\colon\ \Omega^1(Y,\mathfrak{g}_P) \oplus \Omega^7(Y,\mathfrak{g}_P) \to \Omega^0(Y,\mathfrak{g}_P) \oplus \Omega^6(Y,\mathfrak{g}_P)$$

is equivariant. (Sanity check: $\Upsilon_{A_0}(0) = F_{A_0} \wedge \psi$ invariant)

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Pretend that V, W are Banach-spaces

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Schematic Theorem

Assume A_0 is H-invariant and

1. There exists a bounded right-inverse $R_{A_0}\colon\ W^H \to V^H$ to $L_{A_0}|_{V^H}$.
2. $\|F_{A_0}\wedge\psi\|$ is 'sufficiently small' (depending on op-norm of R_{A_0} and bounds on Q_{A_0}).

Then there exists a H-invariant $a \in \Omega^1(Y,\mathfrak{g}_P)$ such that $A_0 + a$ is a G_2-instanton.

$\to$ use this to construct instantons over generalised Kummer construction

G_2-instantons over Kummer constructions

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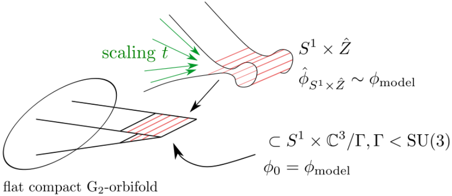

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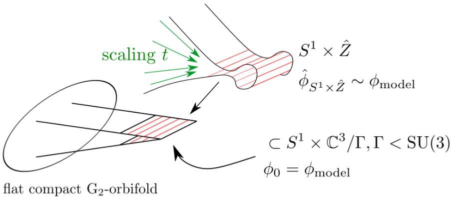

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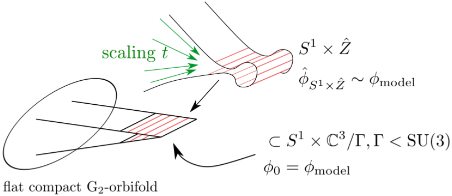
- ϕ_t 'close' to $\tilde{\phi}_t$ 'close' to ϕ_0 on $Y_0 \setminus \text{Sing}$ and $\phi_{S^1 \times \hat{Z}, t}$ on $S^1 \times \hat{Z}$



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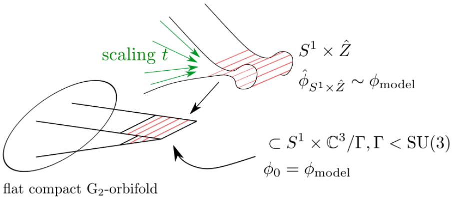
- idea: almost instanton on $(\hat{Y}, \phi_t)$ by interpolating G_2-instanton on Y_0 and G_2-instanton on $S^1 \times \hat{Z}$ matching at infinity



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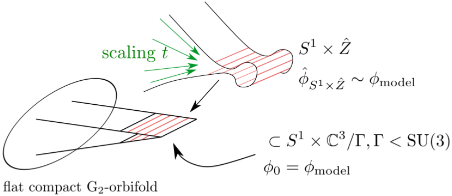
- easier: almost instanton by interpolating flat connection on Y_0 and asymptotically flat Hermitian Yang–Mills connection on $S^1 \times \hat{Z}$



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- (Y_0, ϕ_0) flat compact G_2-orbifold such that all singularities are modelled on $S^1 \times \mathbb{C}^3/\Gamma$ ($\Gamma < SU(3)$ acts freely outside 0)
- (P_0, A_0) flat principal G-bundle over Y_0
- $\rightarrow$ over (nbhd. in) $S^1 \times \mathbb{C}^3/\Gamma$:

$$(P_0, A_0) \cong (\mathbb{R} \times (P_\infty, A_\infty))/\mathbb{Z}$$

with (P_∞, A_∞) flat bundle over $\mathbb{C}^3/\Gamma$.

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Gluing data

- flat principal G-bundle over Y_0
- framed principal G-bundle $\hat{P}_{\hat{Z}}$ over $\hat{Z}$ (isomorphic to P_∞ outside the except. divisor)
- Hermitian Yang–Mills connection $\hat{A}_{\hat{Z}}$ on $\hat{P}_{\hat{Z}}$ asymptotic to A_∞ (with rate -5)
- lift of the $\mathbb{Z}$-action to $\hat{P}_{\hat{Z}}$ that preserves $\hat{A}_{\hat{Z}}$

Theorem (G. '24)

Assume there are

G_2-instantons over Kummer constructions

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- Gluing data:
  - a flat bundle (P_0, A_0) over Y_0
  - bundles $\hat{P}_{\hat{Z}}$ over $\hat{Z}$ with HYM connection $\hat{A}_{\hat{Z}}$ asymptotic to A_0 at infinity (with rate -5) + lift of monodromy around S^1

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- matching H-actions on P_0 and $\hat{P}_{\hat{Z}}$ that preserve ψ_0, A_0, $\hat{\psi}_{S^1 \times \hat{Z}}$, and $\hat{A}_{\hat{Z}}$

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If

- $\operatorname{coker}(L_{A_0})^H = 0$ and
- The HYM connections $\hat{A}_{\hat{Z}}$ do not have $H_{\hat{Z}}$-invariant obstructions where $H_{\hat{Z}} < H$ stabilises $S^1 \subset \operatorname{Sing}(Y_0)$ resolved by $\hat{Z}$,

then $\tilde{A}_t$ can be deformed to a G_2-instanton for $t \ll 1$.

How to apply the previous theorem:

set-up: G_2-orbifold $(\mathbb{R}^7/\Gamma, \phi_0)$, action by $H < \mathrm{Norm}_{O(\mathbb{R}^7)}(\Gamma)$

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1. flat connection:
   - flat principal G-bundles over $\mathbb{R}^7/\Gamma \Leftrightarrow \mathrm{Hom}(\Gamma, G)$
   - flat principal G-bundles (P_0, A_0) over $\mathbb{R}^7/\Gamma$ with H-action $\Leftrightarrow \rho \in \mathrm{Hom}(H \ltimes \Gamma, G)$
   - $\mathrm{coker}(L_{A_0})^H = 0 \Leftrightarrow (\mathfrak{g} \oplus \Lambda^6\mathbb{R}^7 \otimes \mathfrak{g})^{H \ltimes \Gamma, \rho} = 0$

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2. Hermitian Yang–Mills connections over $\hat{Z}$ (crep. resolution of $\mathbb{C}^3/\Gamma$)
   - Degeratu–Walpuski '15: For every flat $U(n)$-bundle (P_∞, A_∞) over $\mathbb{C}^3/\Gamma$ there exists a framed $U(n)$-bundle $\hat{P}_{\hat{Z}}$ over $\hat{Z}$ together with an unobstructed HYM connection $\hat{A}_{\hat{Z}}$ asymptotic to A_∞ (with rate -5).
   - Check if S^1-monodromy lifts to $(\hat{P}_{\hat{Z}}, \hat{A}_{\hat{Z}})$

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set-up: G_2-orbifold $(\mathbb{R}^7/\Gamma, \phi_0)$, action by $H < \mathrm{Norm}_{O(\mathbb{R}^7)}(\Gamma)$

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   - Check if S^1-monodromy lifts to $(\hat{P}_{\hat{Z}}, \hat{A}_{\hat{Z}})$
3. Caution: Don't want $U(1)$-instanton or flat connection over $\hat{Y}$

G_2-instantons over Kummer constructions

Example (Joyce '96)

- G_2-orbifold $(\mathbb{R}^7/\Gamma, \phi_0)$ where Γ is as before
  $\rightarrow$ Kummer construction: $(\hat{Y}, \phi_t)$
- Recall: Involution $\sigma := -\mathrm{Id} \in \mathrm{O}(\mathbb{R}^7)$ lifts to $\hat{Y}$ such that $\hat{\sigma}^*\psi_t = \psi_t$ (and $\hat{\sigma}^*\phi_t = -\phi_t$)
  $\rightarrow$ $\mathbb{Z}_2$-action on $\hat{Y}$.

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  $\to \mathbb{Z}_2$-action on $\hat{Y}$.

Proposition (G. '24)

In above example:

- There exists a SO(14)-bundle over $\hat{Y}$ and a S^1-family of pairwise not gauge equivalent 'almost instantons' $(\tilde{A}_\theta)_{\theta \in S^1}$.
- All $\tilde{A}_\theta$ are $\mathbb{Z}_2$-invariant.
- If $\theta \in S^1 \setminus \{\pm 1\}$, then $\tilde{A}_\theta$ satisfies condition of previous theorem ($\to$ can be deformed to G_2-instantons)

$\Rightarrow$ For each pairwise distinct $\theta_1, \ldots, \theta_n \in S^1 \setminus \{\pm 1\}$ there are (for $t \ll 1$) $\mathbb{Z}_2$-invariant (non-flat) G_2-instantons $A_{\theta_1}, \ldots, A_{\theta_n}$ which are pairwise not gauge-equivalent and infinitesimally irreducible.

Hope:

In prior example: deform a 1-parameter family $(\tilde{A}_\theta)_{\theta \in K}$ for $K \subset S^1 \setminus \{\pm 1\}$ a compact submanifold with boundary to obtain a corresponding family of G_2-instantons.

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Hope:

In prior example: deform a 1-parameter family $(\tilde{A}_\theta)_{\theta \in K}$ for $K \subset S^1 \setminus \{\pm 1\}$ a compact submanifold with boundary to obtain a corresponding family of G_2-instantons.

- Such a family would be obstructed but $\mathbb{Z}_2$-unobstructed
- $\Rightarrow$ obtain family of unobstructed 'instantons' on $\hat{Y}/\mathbb{Z}_2$ (non-orientable but has ψ)

Thank you for your attention!