Stability of Cayley fibrations

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Spin(7)-geometry

Definition

Standard **Cayley form** on $\mathbb{R}^8 = \mathbb{R}^4 \oplus \mathbb{R}^4$:

$$\Phi_0 = \mathrm{dvol}_{\mathbb{R}^4 \oplus 0} + \omega_1 \wedge \tau_1 + \omega_2 \wedge \tau_2 - \omega_3 \wedge \tau_3 + \mathrm{dvol}_{0 \oplus \mathbb{R}^4} \in \Lambda^4(\mathbb{R}^8).$$

Here ω_i, τ_i are positive bases of the anti-self-dual two-forms on both copies of $\mathbb{R}^4$. For $v_1, v_2, v_3, v_4 \in \mathbb{R}^8$ have **calibration property**:

$$\Phi_0(v_1, v_2, v_3, v_4) \leqslant \mathrm{vol}(v_1, v_2, v_3, v_4).$$

A manifold M^8 with 4-form Φ is a $\mathrm{Spin}(7)$-**manifold** if (T_pM, Φ_p) is pointwise isomorphic to $(\mathbb{R}^8, \Phi_0)$. A Riemannian metric g_Φ is induced, as $\mathrm{Stab}(\Phi_0) = \mathrm{Spin}(7) \subset \mathrm{SO}(8)$ via a Spin representation.
If $d\Phi = 0$, then $\mathrm{Hol}(g_\Phi) \subset \mathrm{Spin}(7)$.

Definition

A submanifold $N^4 \subset (M, \Phi)$ is **Cayley** if $\Phi|_N = \mathrm{dvol}_N$.

Existence of fibrations via gluing

We hope $\mathrm{Spin}(7)$ manifolds to be foliated by Cayley submanifolds, motivated by the SYZ conjecture for special Lagrangians in Calabi-Yau manifolds.

Goal: Construct a fibration on a compact torsion-free $\mathrm{Spin}(7)$-manifold via gluing of complex fibrations (on top of gluing a $\mathrm{Spin}(7)$-manifold).

Strategy:

1. Construct exact fibration on (M, Φ) with $\|\mathrm{d}\Phi\|$ small via gluing from simpler pieces.
2. Show that fibration property stable under small change of Φ. This requires knowledge of the **deformation theory** of the Cayley fibres and **desingularisation theory** of the singular fibres.
3. Use Existence theorem for torsion-free $\mathrm{Spin}(7)$-structures near small-torsion structures and stability of fibration to conclude.

Singularities of fibrations on compact manifolds

It is expected that if (M, Φ) is a compact $\mathrm{Spin}(7)$ manifold and $f : M^8 \longrightarrow B^4$ a smooth fibre bundle with Cayley fibres, then $\mathrm{Hol}(g_\Phi) \neq \mathrm{Spin}(7)$ (has been proven for coassociative fibrations in G_2-manifolds). In other words, *interesting fibrations are expected to have singularities*.

Analogy: A holomorphic fibration $K3 \longrightarrow \mathbb{C}P^1$ needs to have nodal fibres, since otherwise $24 = \chi(K3) = \chi(C) \times \chi(\mathbb{C}P^1) = 2(2-2g)$ with $g \geqslant 0$.

In the $\mathrm{Spin}(7)$ case, the simplest singularities are conical and the simplest plausible one is the following complex singularity:

$$f : \mathbb{C}^4 \longrightarrow \mathbb{C}^2; \quad (x, y, z, w) \longmapsto (x^2 + y^2 + z^2, w).$$

The singular fibres $f^{-1}(0, a)$ are **conically singular**. We have a good Fredholm theory and can handle them analytically.

Deformation theory

Definition

A submanifold $N^4 \subset (M, \Phi)$ is α-**almost Cayley** if $\Phi|_N \geqslant \alpha \, \mathrm{dvol}_N$ with $0 < \alpha < 1$.

For (M, Φ) a compact $\mathrm{Spin}(7)$-manifold, there is a global α_{min} such that if $N \subset M$ is α-almost Cayley with $\alpha > \alpha_{\mathrm{min}}$ then there is a well-defined bundle E_N and a nonlinear elliptic differential operator

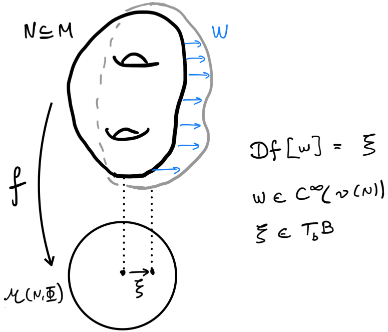
$$F : L^p_{k+1}(\nu_\epsilon(N)) \longrightarrow L^p_k(E_N), \quad v \longmapsto \pi_E(\star_N \exp_v^*(\tau|_{N_v})).$$

for $p > 4, k \geqslant 1$ such that $N_v = \exp(v)$ is Cayley exactly when $F(v) = 0$. The map F is smooth and automatically Fredholm at its zeros. Thus if $F(v) = 0$ and $\mathrm{Coker}\,\mathrm{D}F(v) = 0$ (unobstructed case) for some v, then $\mathcal{M}(N, \Phi)$, the moduli space of Cayley submanifolds isotopic to N is a smooth, finite dimensional manifold near N, modelled on $F^{-1}(0)$. F depends smoothly on N and Φ.

Fibrations near non-singular fibres

We assume a Cayley $N \subset M$ is unobstructed with $\operatorname{ind} \not{D} = 4$, and that there is a basis $w_1, \ldots, w_4$ of $\ker \not{D}$ which is pointwise independent. In other words there is a map:

$$\det = \det(w_1, \ldots, w_4) : N \longrightarrow \mathbb{R}_{>0}, \quad w_1 \wedge \cdots \wedge w_4 = \det \cdot \mathrm{dvol}_{\nu(N)}$$



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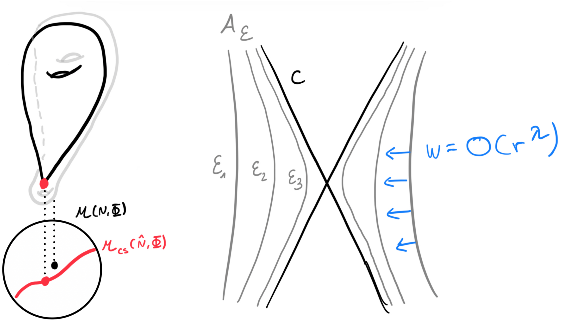
By compactness of N we have $\det > K > 0$ uniformly on an open set $U \subset \mathcal{M}(N, \Phi)$ containing N. If now $\{(N_t, \Phi_t)\}_{t \in (-\epsilon, \epsilon)}$ vary smoothly so that N_t is Cayley for Φ_t, then one can find $w_{i,t}$ as above that vary smoothly as well, by the compact deformation theory (because F varies smoothly in N and Φ and is elliptic). Hence, for $0 < t < t_{\min}$ we still have:

$$\begin{aligned}\det(\tilde{w}_{1,t}, \ldots, \tilde{w}_{4,t}) &= \det(w_1 + t\partial_t w_1, \ldots, w_4 + t\partial_t w_4) + O(t^2) \\ &> K/2 > 0,\end{aligned}$$

as $|\partial_t w_i| \leqslant C$ on a neighbourhood of N. From this we can later prove that the fibration property is maintained.

Trouble near singular fibres

We can prove stability of fibrations outside a neighbourhood of the singularities, as this region of M is compact. Near the singularities we cannot rely on compactness arguments and we need to show by hand that $\partial_t w_i$ is small in a suitable sense. Hence we need to study the **deformation theory** of singular Cayleys and their almost singular neighbours (**desingularisation theory**).



Deformation theory of conical Cayleys I

Take $A \subset \mathbb{R}^8$ an **asymptotically conical** Cayley of rate $\lambda < 1$.

Definition

Fix a weight $\delta \in \mathbb{R}$ and bundle of tensors $F \longrightarrow A$. For $s \in C^\infty_c(F)$ define the norm

$$\|s\|_{L^p_{k,\delta}} = \left(\sum_{i=0}^{k} \int_A |r^{-\delta+i} \nabla^i s|^p r^{-4} \, \mathrm{dvol} \right)^{\frac{1}{p}}.$$

The **weighted Sobolev space** $L^p_{k,\delta}(F)$ is the completion of $C^\infty_c(F)$ with regards to $\| \cdot \|_{L^p_{k,\delta}}$.

We can then define the deformation operator as follows, where $\lambda < 1$ is the rate of A:

$$F_{\mathrm{AC}} : L^p_{k+1,\lambda}(\nu_\epsilon(A)) \longrightarrow L^p_{k,\lambda-1}(E_A).$$

Then any v such that $F_{\mathrm{AC}}(v) = 0$ corresponds to an AC_λ Cayley and the deformation theory can be developed analogously to the compact case.

Deformation theory of conical Cayleys II

Proposition (E.'23)

F_{AC} is smooth and $\not{D}_{\mathrm{AC}} = \mathrm{D}F_{\mathrm{AC}}(0)$ Fredholm for $\lambda \in \mathbb{R} \setminus \mathcal{D}_{crit}(C)$. As λ passes a critical rate, $\operatorname{ind} \not{D}_{\mathrm{AC}}$ jumps by a fixed amount. If λ not critical and $\operatorname{Coker} \not{D}_{\mathrm{AC}} = 0$, then

$$\mathcal{M}^{\lambda}_{\mathrm{AC}}(A) = \{\tilde{A} : \tilde{A} \text{ is an } \mathrm{AC}_\lambda \text{ Cayley submanifold of } (\mathbb{R}^8, \Phi_0) \text{ isotopic to } A \text{ and asymptotic to the same cone}\}$$

is a smooth finite dimensional manifold, locally.

Can also define $\overline{\mathcal{M}}^{\lambda}_{\mathrm{AC}}(A) = \mathcal{M}^{\lambda}_{\mathrm{AC}}(A) \cup \{C\}$, including the cone.

Example

Let $-1 < \lambda < 0$. Consider the complex surface $A_\epsilon = \{x^2 + y^2 + z^2 = \epsilon, w = 0\} \subset \mathbb{C}^4$. This is an unobstructed Cayley and has a real 2-dimensional moduli space (from varying $\epsilon \in \mathbb{C}$). If instead $0 < \lambda < 1$ the moduli space becomes $2 + 8 = 10$-dimensional

Desingularisation theory I

Similarly there is a moduli space of CS_μ Cayleys in (M, Φ), denoted by $\mathcal{M}^\mu_{\mathrm{CS}}(N, \Phi)$ $(1 < \mu < 2)$, as well as a family moduli space $\mathcal{M}^\mu_{\mathrm{CS}}(N, \mathcal{S})$, where $\mathcal{S}$ is a family of $\mathrm{Spin}(7)$-structures. Moreover, one can include rates up to $\mu = 1$ and $\mu = 0$ manually, denote these moduli spaces by $\mathcal{M}^1_{\mathrm{CS}}$ and $\mathcal{M}^0_{\mathrm{CS}}$ respectively. Assume that $\mathcal{M}^0_{\mathrm{CS}}(N, \Phi)$ and $\mathcal{M}^\lambda_{\mathrm{AC}}(A)$ contain only unobstructed Cayley with the same cone C, that $\mathcal{D}_{crit}(C) \cap [\lambda, 0] = \{0\}$ and that the linear deformations of the cone are unobstructed.

Theorem (E.'23)

There is a stratified smooth gluing map:

$$\Gamma : \mathcal{M}^0_{\mathrm{CS}}(N) \times (U \subset \overline{\mathcal{M}}^\lambda_{\mathrm{AC}}(A)) \times \mathcal{S} \longrightarrow \mathcal{M}(N \sharp A, \mathcal{S}).$$

Here U is a small open neighbourhood of the cone C in $\overline{\mathcal{M}}^\lambda_{\mathrm{AC}}(A)$. It is a local diffeomorphism.

Desingularisation theory II

Question: Suppose that $N \subset (M, \Phi)$ is a CS Cayley with cone C and A is an AC Cayley with the same cone. Can you desingularise N by gluing in A?

We first define almost Cayleys $N_t = N \sharp tA$ which are approximations of desingularisations of N. Write $F_t(v) = F_t(0) + \not{D}_t v + Q_t(v)$, where $\not{D}_t v = \mathrm{D}F_t(0)[v]$. Generally, $\dim \operatorname{Ker} \not{D}_t > 0$ so solutions come in families. Hence we solve modulo **pseudo-kernel** $\kappa_t \subset C^\infty(\nu(N_t))$ to find unique solution. Since we want to perturb a family of almost Cayleys $\{N_t\}_{t \in (0,\epsilon)}$ with uniformly small perturbations we need to chose our function spaces $L^p_{k,\delta,t}$ $(0 < \delta < \mu)$ so that:

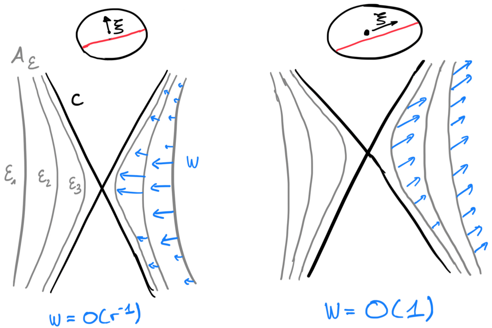
- $\|F_t(0)\|_{L^p_{k,\delta-1,t}} \leqslant t^\alpha C_F$ with $\alpha > 0$.
- $\|u\|_{L^p_{k+1,\delta,t}} \leqslant C_D \|\not{D}_t u\|_{L^p_{k,\delta-1,t}}$ for uniform $C_D > 0$ and $u \perp \kappa_t$.
- $\|Q_t(u)\|_{L^p_{k,\delta-1,t}} \leqslant C_Q \|u\|^2_{L^p_{k+1,\delta,t}}$ for uniform $C_Q > 0$.

These estimates can be obtained by combining the CS and AC deformation theory. For sufficiently small neck size $t > 0$ we can find v_t with $F_t(v_t) = 0$ by Banach fixed point theorem, with $\|v_t\|_{L^p_{k+1,\delta,t}} \leqslant t^\beta$, where $\beta > 0$.

Determinant map near singular fibres

Goal: replicate the stability argument for non-singular fibres in a neighbourhood of the singular fibres.

Want maps $\det : \Gamma(N, A, \Phi) \to \mathbb{R}$ for $(N, A) \in \mathcal{M}_{\mathrm{CS}}(N, \Phi) \times \overline{\mathcal{M}}^{\lambda}_{\mathrm{AC}}(A)$ with $0 < C < \det < C^{-1}$ uniformly. If $\{\Phi_s\}$ is a smooth family then $\partial_s \det$ should be uniformly bounded, even as the neck size shrinks to 0.



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$$\det(w_1, w_2, r \cdot w_3, r \cdot w_4),$$

where w_1, w_2 are the translations, corresponding to varying $w = \eta, \eta \in \mathbb{C}$, and w_3, w_4 are the rescalings (of rate r^{-1}, r being the distance to the singular point), corresponding to varying $x^2 + y^2 + z^2 = \epsilon, \epsilon \in \mathbb{C}$.

Gluing of infinitesimal deformation vector fields

The first order deformations w_i can be obtained by solving the linearised Cayley equation:

$$\not{D} w = 0.$$

We can solve this on $\Gamma(N, tA, \Phi)$ starting from known glued approximation. By using appropriate Banach spaces $L^p_{k,\delta,t}$, we can make the error small.

Proposition

Given solutions $w_{\mathrm{CS}}, w_{\mathrm{AC}}$ to $\not{D} w = 0$ on N and A with the same asymptotic behaviour $r^\delta \sigma$ for $\delta = -1, 0$ (+technical conditions), then there is a solution to the Cayley equation on $\Gamma(N, tA, \Phi)$ of the form:

$$w_{\mathrm{CS}} + t^\delta w_{\mathrm{AC}} + \tilde{w}_t,$$

where $\|\tilde{w}_t\|_{L^p_{k+1,\delta+\epsilon,t}} \to 0$ as $t \to 0$. Varying Φ varies $\tilde{w}_t$ uniformly continuously in $L^p_{k+1,\delta+\epsilon,t}$. $\Rightarrow |\partial_\Phi w|_{C^0_{\delta+\epsilon}} \leqslant C$.

Stability of fibration

We can cover the base of our initial fibration $f : M \to B$ with neighbourhoods where we can apply the local stability results.

Theorem (Main theorem (E.'24))

If all the fibres in a fibration modelled on the quadratic cone are unobstructed and the initial fibration has $C_1 > \det > C_2 > 0$, then the fibration property is stable under deformation of the $\mathrm{Spin}(7)$ structure.

Proof.

Consider the evaluation map, dependent on Φ:

$$\mathrm{ev}_\Phi : \mathrm{Univ}(\overline{\mathcal{M}}(N, \Phi)) \longrightarrow M$$

Desingularisation theory $\Rightarrow$ stratified smooth map, degree 1 in the sense of pseudo-cycles. Initially an orientation-preserving diffeomorphism on smooth stratum. Since $\det > 0$ it stays orientation preserving of degree one, thus any point must have exactly one pre-image. $\square$

A nice quartic in $\mathbb{C}P^4$

$$P = x_0^4 + x_1^4 + x_2^4 + x_3^4 + x_4^4 + x_3^3(x_0 + 10x_1 + 100x_2).$$

Consider the Fano threefold $X = \{P = 0\} \subset \mathbb{C}P^4$. The anticanonical bundle is $\omega_X^\star = \mathcal{O}(1)|_X$, and the anticanonical divisors are exactly the hyperplane sections of X.Take the sections:

$$\begin{aligned} S_0 =& \{x_3 = 0\} \cap X \simeq \{x_0^4 + x_1^4 + x_2^4 + x_4^4 = 0\} \subset \mathbb{C}P^3 \\ S_\infty =& \{x_4 = 0\} \cap X \simeq \{x_0^4 + x_1^4 + x_2^4 + x_3^4 + x_3^3(x_0 + 10x_1 + 100x_2)\} \subset \mathbb{C}P^3. \end{aligned}$$

Both are smooth K3 surfaces. They intersect transversely in a curve $C \simeq \{x_0^4 + x_1^4 + x_2^4 = 0\} \subset \mathbb{C}P^2$. We obtain a building block Z by blowing up C and the pencil defined by S_0, S_∞ lifts to a holomorphic map $f : Z \to \mathbb{C}P^1$ which extends

$$f : Z \setminus E \longrightarrow \mathbb{C}P^1, \quad [x_0 : x_1 : x_2 : x_3 : x_4] \longmapsto [x_3 : x_4].$$

There are exactly $3^3 \cdot 4$ singular fibres of f, each with a unique singular point modelled on $\{x^2 + y^2 + z^2 = 0\}$.

Twisted connected sum from building block

Having conically singular fibres with at most one singular point each is a Zariski open condition on X in the deformation type of the quartic and the space of choices for S_0, S_∞. **Claim:** This is sufficient to allow for matching of two building blocks related by a hyperkähler rotation. From two such blocks we can build a G_2 manifold of small torsion together with a coassociative fibration. We can then deform this manifold to a torsion free **twisted connected sum** manifold M. Applying the main theorem to $M \times S^1$, we see that the Cayley fibration persists if the neck of M is sufficiently long. As the $\mathrm{Spin}(7)$-structure on $M \times S^1$ is of product type, the fibration splits into a coassociative fibration times S^1.