Geometric flows of G_2 and Spin(7)-structures

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based on arXiv:2404.00870
and
arXiv:2311.05516 (joint work with P. Gianniotis & S. Karigiannis).

Overview

- We study general flows of G_2 and Spin(7)-structures. General here means no conditions on the G_2-structures φ (closed, co-closed, isometric etc.) and the Spin(7)-structures Φ.
- We find all the linearly independent second order differential invariants of a G_2/Spin(7)-structure, which can occur in a variation of G_2/Spin(7)-structures.
- We write the most general flow of G_2/Spin(7)-structures and find sufficient conditions for short-time existence and uniqueness (amenable to a DeTurck type trick).

Flows of G_2-structures.

Throughout this part, we will be working on **7-dimensional manifolds.**

A G_2-structure on M^7 is the reduction of the structure group of the frame bundle $\mathrm{Fr}(M)$ from $\mathrm{GL}(7, \mathbb{R})$ to the Lie group $G_2 \leq \mathrm{SO}(7)$.

Definition

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Let $X, Y \in \Gamma(TM)$, then

$$(X \lrcorner \varphi) \wedge (Y \lrcorner \varphi) \wedge \varphi = 6 g_\varphi(X, Y) \operatorname{vol}_\varphi .$$

M^7 admits G_2-structures $\iff$ it is orientable and spinnable.

The space of nondegenerate (or positive) 3-forms Ω^3_+ is an open subbundle of Ω^3.

Introduction contd.

G_2 structure $\leftrightsquigarrow$ "non-degenerate" 3-form $\varphi \rightsquigarrow g_\varphi$ and orientation nonlinearly.

Thus, we have a Hodge star operator $*_\varphi$ and dual 4-form $*_\varphi \varphi = \psi$.

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Definition

Let (M^7, φ) be a manifold with a G_2 structure φ and let ∇ be the Levi-Civita connection of g_φ. We call (M, φ) a G_2 manifold if $\nabla\varphi = 0$. $\nabla\varphi$ is interpreted as the torsion T of the G_2 structure.

G_2 manifolds, i.e., those having torsion-free G_2 structure φ are always Ricci-flat and have special holonomy contained in the Lie group $G_2 \subset \mathrm{SO}(7)$.

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In order to find second order differential invariants of φ, we need to understand the decomposition of differential forms into irreducible G_2-representations.

Introduction contd.

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In particular,

$$\Omega^2 = \Omega^2_7 \oplus \Omega^2_{14}, \quad \Omega^3 = \Omega^3_1 \oplus \Omega^3_7 \oplus \Omega^3_{27}.$$

$$\begin{aligned}
\Omega^2_7 = \{X \lrcorner \varphi \mid X \in \Gamma(TM)\} &= \{\beta \in \Omega^2 \mid *(\varphi \wedge \beta) = 2\beta\}, \\
\Omega^2_{14} = \{\beta \in \Omega^2 \mid \beta \wedge \psi = 0\} &= \{\beta \in \Omega^2 \mid *(\varphi \wedge \beta) = -\beta\}
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For $\sigma \in \Omega^k$ and $A = A_{ij} dx^i \otimes dx^j \in \mathcal{T}^2$, we define

$$(A \diamond \sigma)_{i_1 i_2 \cdots i_k} = A_{i_1}{}^p \sigma_{p i_2 \cdots i_k} + A_{i_2}{}^p \sigma_{i_1 p i_3 \cdots i_k} + \cdots + A_{i_k}{}^p \sigma_{i_1 i_2 \cdots i_{k-1} p},$$

so, e.g., $g \diamond \sigma = k\sigma$ and in particular $(A \diamond \varphi)_{ijk} = A_i{}^p \varphi_{pjk} + A_j{}^p \varphi_{ipk} + A_k{}^p \varphi_{ijp}$.

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Since $\mathcal{T}^2 \cong \Omega^0 \oplus S^2_0 \oplus \Omega^2_7 \oplus \Omega^2_{14}$, it can be proved that

$A \in \ker(\cdot \diamond \varphi) \iff A \in \Omega^2_{14}\ (\cong \mathfrak{g}_2)$

$A \mapsto A \diamond \varphi$ is an isomorphism between $S^2 \oplus \Omega^2_7$ and Ω^3.

Introduction contd.

Thus, we can describe the 3-forms as

$$\Omega^3_1 = \{f\varphi \mid f \in \Omega^0\}, \quad \Omega^3_7 = \{A \diamond \varphi \mid A \in \Omega^2_7\} = \{X \lrcorner \psi \mid X \in \Gamma(TM)\},$$
$$\Omega^3_{27} = \{A \diamond \varphi \mid A \in S^2_0\}.$$

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For $A \in \mathcal{T}^2$, we set

$$(\mathcal{V}A)_k = A_{ij}\varphi^{ij}{}_k.$$

Only the Ω^2_7 part of A contributes to $\mathcal{V}A$, and we call it the vector part of A. In fact,

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Thus on (M, φ), *any* 3-form can be equivalently described by a pair (h, X) with h a symmetric 2-tensor and $X \in \Gamma(TM)$. We will write

$$\gamma_{ijk} = (h \diamond \varphi)_{ijk} + (X \lrcorner \psi)_{ijk} = h^p_i \varphi_{pjk} + h^p_j \varphi_{ipk} + h^p_k \varphi_{imp} + X_p \psi^p{}_{ijk}.$$

for a 3-form γ.

- The torsion T is a 2-tensor and is explicitly given as

$$T_{pq} = \frac{1}{24} \nabla_p \varphi_{ijk} \psi_q{}^{ijk} \qquad (\psi = *_\varphi \varphi)$$

Since $T \in \mathcal{T}^2$, it can be further decomposed as $T = T_1 + T_7 + T_{14} + T_{27}$.

- T satisfies a "Bianchi"-type identity **(Karigiannis '09)**

$$\nabla_i T_{jk} - \nabla_j T_{ik} = T_{ia} T_{jb} \varphi_a{}^{bk} + \frac{1}{2} R_{ijab} \varphi_k{}^{ab}.$$

We expect this as $\phi^*(T_\varphi) = T_{\phi^*\varphi}$ for any diffeo. ϕ (and also by commuting two derivatives on φ). We crucially use this for some of our results.

Theorem (Fernàndez–Gray)

(M, φ) is torsion-free, i.e., $T = 0$ ($\iff \nabla \varphi = 0$) if and only if $d\varphi = d\psi = 0$.

Flows of G_2-structures

Suppose (M^7, φ) is a compact manifold with a G_2 structure.

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A geometric flow is an evolution equation which improves a geometric structure, starting from a given one.

Given a G_2 structure (not necessarily torsion-free), it is natural to attempt to improve it in some sense to a "better" G_2 structure (for instance, torsion-free) by using a geometric flow. (Compare: Ricci flow of metrics; mean curvature flow of immersions.)

- General study of flows of G_2 structures - Karigiannis, '09
  D.–Gianniotis–Karigiannis, '23
  Weiss–Witt, '12

- Laplacian flow of closed G_2 structures - Bryant, '05, Bryant–Xu, '11
  ($\frac{\partial\varphi}{\partial t} = \Delta_\varphi \varphi, \quad d\varphi = 0$) Lotay–Wei, '15

- Laplacian co-flow of co-closed G_2 structures - Karigiannis–McKay–Tsui, '12
  ($\frac{\partial\psi}{\partial t} = \Delta_\varphi \psi, \quad d\psi = 0$)

- **Modified** Laplacian co-flow of co-closed G_2 structures - Grigorian,'13
  ($\frac{\partial\psi}{\partial t} = \Delta_\varphi \psi + d((\underbrace{A}_{constant} - \operatorname{tr} T)\varphi), \quad d\psi = 0$)

- Isometric Flow of G_2-structures- D.–Gianniotis–Karigiannis '19, Grigorian, '19,
  Loubeau–Sá Earp, '19
  ($\frac{\partial\varphi}{\partial t} = \operatorname{div} T \lrcorner \psi, \quad \text{on} \quad \{\varphi \mid g_\varphi = g_{\varphi_0}\}$)

Flows of G_2-structures

Recall that on (M, φ), any 3-form can be described by a pair $(h, X),\ h \in S^2(TM),\ X \in \Gamma(TM)$. Thus, any flow of G_2-structures can be written as

$$\frac{\partial \varphi(t)}{\partial t} = (h(t) \diamond_t \varphi(t)) + X(t) \lrcorner \psi(t). \qquad \text{(GF)}$$

Facts: Along (GF), $\partial_t g(t) = 2h(t),\ \partial_t g(t)^{-1} = -2h(t),\ \partial_t \operatorname{vol}_t = \operatorname{tr} h(t) \operatorname{vol}_t$.

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One possible approach to write the most general (and reasonable) flow of G_2-structures is to classify all linearly independent second order differential invariants of a G_2-structure (upto lower order terms) which are invariant under diffeomorphisms and then take a linear combination of those which can be made into a 3-form.

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We use representation theoretic aspects of the Lie group G_2 to decompose the Riemann curvature tensor Rm and ∇T.

Definition On (M, φ) $\exists$ another Ricci-type tensor F given explicitly as

$$F_{jk} = R_{abcd} \varphi^{ab}{}_j \varphi^{cd}{}_k \underbrace{=}_{symm. of\ \mathrm{Rm}} R_{cdab} \varphi^{ab}{}_j \varphi^{cd}{}_k = F_{kj}.$$

$\operatorname{tr}(F) = -2R$, $R =$scalar curvature. F has another geometric interpretation.

2nd order differential invariants of φ from Rm

The curvature decomposition is

$$\mathrm{S}^2(\Lambda^2) = \mathrm{S}^2(\mathbf{7} \oplus \mathbf{14}) = \mathrm{S}^2(\mathbf{7}) \oplus (\mathbf{7} \otimes \mathbf{14}) \oplus \mathrm{S}^2(\mathbf{14})$$

which can be further decomposed into irreducible G_2-representations as

$$(\mathbf{1} \oplus \mathbf{27}) \oplus (\mathbf{64} \oplus \mathbf{7} \oplus \mathbf{27}) \oplus (\mathbf{77} \oplus \mathbf{1} \oplus \mathbf{27}).$$

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However, it must be orthogonal to $\Lambda^4 = \mathbf{1} \oplus \mathbf{7} \oplus \mathbf{27}$ by the first Bianchi identity. This cuts down the curvature to an element of

$$\underbrace{\mathbf{1} \oplus \mathbf{27}}_{\text{Ricci}} \oplus \underbrace{\mathbf{27} \oplus \mathbf{64} \oplus \mathbf{77}}_{\text{Weyl}}.$$

That is, the Bianchi identity says that the **7** part is zero, that the two **1**'s are multiples of each other, and that the three **27**'s reduce to just two independent **27**'s. Only the **1** and the two **27** components can be made into a 3-form.

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Upshot: The only second order invariants from Rm which could appear for a flow of G_2-structures are: Rg, Ric_0 and W_{27}.

Since

$$W_{27} = F + \frac{2}{7}Rg - \frac{4}{5}\,\mathrm{Ric}_0,$$

we'll use Rg, Ric_0 and F.

2nd order differential invariants of φ from ∇T

- In a similar way we can decompose $\nabla T \in \Gamma(T^*M \otimes \mathcal{T}^2)$ into irreducible G_2-representations and look for those 2*nd* order differential invariants which can be made into a 3-form. There are many of these.

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• However, not all invariants obtained from Rm and ∇T are independent because these quantities are related by the G_2-Bianchi identity.

The G_2-Bianchi identity is

$$G_{qij} = \nabla_i T_{jq} - \nabla_j T_{iq} - T_{ia} T_{jb} \varphi^{ab}{}_q - \frac{1}{2} R_{ijab} \varphi^{ab}{}_q = 0.$$

G_{qij} are the components of a tensor $G \in \Gamma(T^*M \otimes \Lambda^2(T^*M))$, because G_{qij} is skew in $i, j \rightsquigarrow$ decomposed into two components $G^7 + G^{14}$, where $G^k \in \Gamma(T^*M \otimes \Lambda^2_k(T^*M))$ for $k = 7, 14$. Using the decompositions

$$\mathbf{7} \otimes \mathbf{7} = \mathbf{1} \oplus \mathbf{27} \oplus \mathbf{7} \oplus \mathbf{14} \quad \text{and} \quad \mathbf{7} \otimes \mathbf{14} = \mathbf{64} \oplus \mathbf{27} \oplus \mathbf{7},$$

we can therefore decompose $G = 0$ into seven independent relations.

Doing all this, we prove the following lemma.

All 2nd order differential invariants of φ

Lemma (D.–Gianniotis–Karigiannis, '23)

Up to lower order terms, there are 6 independent 2nd order differential invariants of φ which can be made into a 3-form. The choices are

$$h = Ric_0, Rg, F, \mathcal{L}_{T_7} g \qquad \textit{and} \qquad X = \operatorname{div} T, \operatorname{div} T^t.$$

Note that T_7 is the 7-component of the torsion and $(\operatorname{div} T)_k = \nabla^i T_{ik}$, $(\operatorname{div} T^t)_k = \nabla^i T_k{}^i$ are vector fields on M.

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These h's and X's appear in the first variation of the L^2-norm of the torsion components, i.e, in $\frac{d}{dt}\big|_{t=0} \int_M |T_i|^2 \text{vol}$, $i = 1, 7, 14, 27$. The formulas are:

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$$\frac{d}{dt}\int_M |T_1|^2 \operatorname{Vol} = \int_M h^{ip}((\operatorname{tr} T)^2 g_{ip} - 2 \operatorname{tr} T T_{ip}) \operatorname{Vol} - 2\int_M X^p(\operatorname{tr} T (T_7)_p + (\operatorname{div} T^t)_p) \operatorname{Vol}$$

$$\frac{d}{dt}\int_M |T_7|^2 \,\mathrm{Vol} = \int_M 6h^{ip}\Big[(\mathcal{L}_{T_7}g)_{ip} + Rg_{ip} + \mathrm{tr}(T^2)g_{ip} - (\mathrm{tr}\,T)^2 g_{ip} + T_{la}T_{jq}\psi^{lajq}g_{ip} - |T_7|^2 g_{ip} - 4(T_{\mathrm{skew}})_{is}T_p^s - 2T_{mn}T_{is}\psi^{mns}{}_p\Big]\mathrm{Vol}$$

$$+ \int_M 6X^q\Big[2(\mathrm{div}\,T)_q + 2(\mathrm{div}\,T^t)_q + 2\nabla_p T_{mn}\psi^{pmn}{}_q + 4(T_{\mathrm{skew}})_{pq}{T_7}^p + 2(T^2)_{pn}\varphi^{pn}{}_q\Big]\mathrm{Vol}$$

$$\frac{d}{dt}\int_M |T_{14}|^2 \,\mathrm{Vol} = \int_M h^{ip}\Big[R_{ip} - \frac{11}{2}(\mathcal{L}_{T_7}g)_{ip} - \frac{1}{4}F_{ip} - 6Rg_{ip} + 2(T^2)_{pi} - (T\circ T^t)_{ip} - \mathrm{tr}\,T\,T_{pi} - \frac{1}{2}T_{ms}T_{nt}\varphi^{mn}{}_i\varphi^{st}{}_p - 2T_{km}(T_{\mathrm{skew}})_{pq}\psi^{kmq}{}_i$$
$$+ 12T_{mn}T_{is}\psi^{mns}{}_p + 24(T_{\mathrm{skew}})_{is}{T_p}^s + \frac{1}{2}|T|^2 g_{ip} - \frac{13}{2}\mathrm{tr}(T^2)g_{ip} + 6(\mathrm{tr}\,T)^2 g_{ip} + 6|T_7|^2 g_{ip} - 6T_{la}T_{jq}\psi^{lpjq}g_{ip}\Big]\mathrm{Vol}$$
$$- X^q\int_M \Big[13(\mathrm{div}\,T)_q + 13(\mathrm{div}\,T^t)_q - 24(T_{\mathrm{skew}})_{pq}(T_7)^p - 13(T^2)_{pl}\varphi^{pl}{}_q - 12\nabla_p T_{mn}\psi^{pmn}{}_q\Big]\mathrm{Vol}$$

$$
\begin{aligned}
\frac{d}{dt}\int_M |T_{27}|^2 \,\mathrm{Vol} = \int_M h^{ip}\Big[& R_{ip} + \frac{1}{2}(\mathcal{L}_{T_7} g)_{ip} + \frac{1}{4}F_{ip} + \frac{1}{2}T_{ms}T_{nt}\varphi^{mn}_{\ \ \ i}\varphi^{st}_{\ \ \ p} - \frac{5}{7}\operatorname{tr} T\, T_{pi} \\
& - 2T_{km}(T_{\mathrm{sym}})_{pq}\psi^{kmq}_{\ \ \ \ \ i} - (T \circ T^t)_{ip} + \frac{1}{2}|T|^2 g_{ip} + \frac{1}{2}\operatorname{tr}(T^2) g_{ip} \\
& - \frac{1}{7}(\operatorname{tr} T)^2 g_{ip}\Big] \,\mathrm{Vol} \\
& - \int_M X^q \Big[(\operatorname{div} T)_q - \frac{9}{7}(\operatorname{div} T^t)_q + T^2_{pl}\varphi^{pl}_{\ \ \ q} + \frac{2}{7}\operatorname{tr} T (T_7)_q \\
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& - \frac{1}{7}(\mathrm{tr}\,T)^2 g_{ip}\Big]\,\mathrm{Vol} \\
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& + \frac{2}{7}(T_7)^p T_{qp}\Big]\,\mathrm{Vol}\,.
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Thus, we are led to define the following family of flows of G_2-structures.

[Flows of G_2-structures]

Let (M^7, φ_0) be a compact manifold. The general flow of G_2-structures is the initial value problem

$$\frac{\partial \varphi}{\partial t} = (-\operatorname{Ric} + a\mathcal{L}_{T_7} g + \beta F) \diamond \varphi + (b_1 \operatorname{div} T + b_2 \operatorname{div} T^t) \lrcorner \psi + \text{l.o.t.} \qquad \text{(GGF)}$$
$$\varphi(0) = \varphi_0$$

with $a, \beta, b_1, b_2 \in \mathbb{R}$.

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Special cases

- $a = \beta = b_1 = b_2 = 0$ and no l.o.t. gives the usual **Ricci flow of G_2-structures**, i.e., a flow of G_2-structures which induce the Ricci flow of metrics.

Special cases contd.

$\frac{\partial\varphi}{\partial t} = (-\operatorname{Ric} + a\mathcal{L}_{T_7}g + \beta F)\diamond\varphi + (b_1 \operatorname{div} T + b_2 \operatorname{div} T^t)\lrcorner\psi + \text{l.o.t.}$

- $a = \beta = b_2 = 0$, $b_1 = 1$ and no Ric and l.o.t term gives the **isometric/harmonic flow of** G_2**-structures** $\partial_t\varphi = \operatorname{div} T\lrcorner\psi \rightsquigarrow$ **negative gradient flow of** $\varphi \mapsto \int_M |T|^2 \operatorname{Vol}$ **restricted to** $[[\varphi_0]]_{\text{iso}}$**.** Analytic properties well-understood and we have a monotonicity formula, entropy functional, ε-regularity theorem.

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- $a = \beta = b_2 = 0$ and no l.o.t. gives the **Ricci flow coupled with the isometric flow of** G_2**-structures.** We prove short-time existence and uniqueness of solutions and one can get *a priori* estimates using Gao Chen's arguments.

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- $a = -\frac{1}{2}$, $\beta = 0$, $b_1 = 1$, $b_2 = 0$ so we have $\partial_t \varphi = (-\operatorname{Ric} - \frac{1}{2}\mathcal{L}_{T_7} g) \diamond \varphi + \operatorname{div} T \lrcorner \psi + \underbrace{\text{l.o.t.}}_{\textit{explicit}} \rightsquigarrow$ **negative gradient flow of** $\varphi \mapsto \int_M |T|^2 \operatorname{Vol}$ **on** *all* **G_2-structures.** Studied by Weiss–Witt (2012). We have short-time existence and uniqueness of solutions.

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What can we say about the short-time existence and uniqueness of solutions of (GGF) in general?

Main Theorem

Theorem (D.–Gianniotis–Karigiannis, '23)

Let (M, φ_0) *be a compact 7-manifold with a* G_2*-structure* φ_0*. Then there exists a unique* $\varphi(t)$*,* $t \in [0, \varepsilon)$*, such that*

$$\frac{\partial \varphi(t)}{\partial t} = (-\,Ric + a\mathcal{L}_{T_7} g + \beta F) \diamond \varphi + (b_1 \operatorname{div} T + b_2 \operatorname{div} T^t) \lrcorner \psi$$

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provided that $0 \leq b_1 - a - 1 < 4$*,* $b_1 + b_2 \geq 1$ *and* $|\beta| < \dfrac{c}{4}$*, where* $c = 1 - \frac{1}{4}(b_1 - a - 1)$*.*

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Idea of the proof:

• Let the RHS of (GGF) be P_φ. Calculate the principal symbols of the operators involved: $\mathrm{Ric}, \mathcal{L}_{T_7} g, F, \operatorname{div} T, \operatorname{div} T^t$. It turns out that $\dim \ker \; (\sigma(DP_\varphi)(h, X)) \geq 7$ because of diffeomorphism invariance of the tensors involved.

Proof contd.

- We prove that dim ker $(\sigma(DP_\varphi)(h, X)) = 7$ and hence the failure of parabolicity of (GGF) is only due to diffeomorphism invariance of the tensors involved. Remark: We needed to introduce a new operator to show this; we have $B_1 : S^2(T_x^* M) \to T_x^* M$ with $B_1(h)_k = \xi_a h^a{}_k - \frac{1}{2}\xi_k \operatorname{tr} h$ which is the usual *Bianchi operator* in the Ricci-flow. We introduce $B_2 : T_x^* M \to T_x^* M$ by $B_2(X)_k = \xi_a X_b \varphi^{ab}{}_k$ and use both of these.

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• Because of above, we can use the DeTurck's trick: look at the modified operator $P_\varphi + \mathcal{L}_W\varphi$ with $W \in \Gamma(TM)$ given by

$$W^k = g^{ij}\left(\Gamma_{ij}^k - \bar{\Gamma}_{ij}^k\right) - 2a(T_7)^k$$

where $\bar{\Gamma}$ are the Christoffel symbols w.r.t. a fixed background G_2-structure, e.g., φ_0.

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• The symbol of $P_\varphi + \mathcal{L}_W \varphi$ is a multiple of Id and hence we can prove short-time existence and uniqueness using DeTurck's trick.

Flows of Spin(7)-structures.

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• A Spin(7)-structure on M^8 is given by $\Phi \in \Omega^4(M)$ and the subgroup of $\mathrm{GL}(8, \mathbb{R})$ preserving Φ is the Lie group Spin(7) which is the double cover of SO(7). The existence of such a structure is again a topological condition.

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• Unlike the G_2-case, the space of 4-forms Φ on M which determine a Spin(7)-structure is a subbundle $\mathcal{A}$ called the bundle of admissible 4-forms. This is not a vector subbundle and is not even an open subbundle. In fact, for $p \in M$, $\mathcal{A}_p$ has codimension 27 in $\Lambda^4(T_p^* M)$.

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$$\Omega^2 = \Omega^2_7 \oplus \Omega^2_{21}, \quad \Omega^3 = \Omega^3_8 \oplus \Omega^3_{48}, \quad \Omega^4 = \underbrace{\Omega^4_1 \oplus \Omega^4_{35}}_{\cong \diamond S^2(M)} \oplus \underbrace{\Omega^4_7}_{\cong \diamond \Omega^2_7} \oplus \Omega^4_{27}.$$

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• The torsion tensor $T \in \Omega^1_8 \otimes \Omega^2_7$ and is given in terms of $\nabla\Phi$ as

$$T_{m;ab} = \frac{1}{96}(\nabla_m \Phi_{ajkl} \Phi_b{}^{jkl})$$

and in fact, T can be viewed as a 3-form with $T = T_8 + T_{48}$ with T_8 being identified as a vector field.

On (M, φ), any 4-form in Ω^4_{1+7+35} can be described by a pair (h, X), $h \in S^2(TM)$, $X \in \Omega^2_7$. From (Karigiannis, '07), any flow of Spin(7)-structures can be written as

$$\frac{\partial \Phi(t)}{\partial t} = (h(t) + X(t)) \diamond_t \Phi(t).$$

Facts: Along the above flow,
$\partial_t g(t) = 2h(t),\ \partial_t g(t)^{-1} = -2h(t),\ \partial_t \operatorname{vol}_t = \operatorname{tr} h(t) \operatorname{vol}_t .$

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Using similar strategies as in the G_2-case, we show that the only linearly independent second order differential invariants of Φ that can occur for a flow of Spin(7)-structures are: Ric_0, Rg, $\mathcal{L}_{T_8} g$ for h and $(\operatorname{div} T)_{ab} = \nabla^m T_{m;ab}$ for X.

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Since the above h and X all occur in the negative gradient flow of

$$E(\Phi)=\int_M |T_\Phi|^2\operatorname{vol}_\Phi \qquad \text{(EF)}$$

we look at this flow as this naturally gives us the lower order terms as well. Also studied by Ammann–Weiss–Witt, '16 using spinorial approach (and more generally by Collins–Phong.)

Let (M^8, Φ_0) be a compact manifold. The negative gradient flow of (EF) is the initial value problem

$$\begin{aligned}\frac{\partial \Phi}{\partial t} &= \Big(-\operatorname{Ric} + 2(\mathcal{L}_{T_8} g) + (T * T) - |T|^2 g + 2\operatorname{div} T\Big) \diamond \Phi, \\ \Phi(0) &= \Phi_0.\end{aligned} \tag{GSF}$$

with $(T * T)_{ij} = 8 T_{b;i}{}^{l} T_{j;l}{}^{b} - 8 T_{j;i}{}^{l} T_{b;l}{}^{b} + 2 T_{i;lb} T_{j;}{}^{lb}$.

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- The **isometric flow of Spin(7)-structures** $\partial_t \Phi(t) = \operatorname{div} T \diamond \Phi$ studied by D.–Loubeau–Sa' Earp, '21 is a special case of (GSF).

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Theorem (D., '24)

Let (M^8, Φ_0) be a compact 8-manifold with a Spin(7)-structure Φ_0. Then there exists a unique $\Phi(t)$, $t \in [0, \varepsilon)$ which is a solution to (GSF).

Solitons

A soliton for (GSF) is a triple (Φ, Y, λ) with $Y \in \Gamma(TM)$ and $\lambda \in \mathbb{R}$ such that

$$\left(-\operatorname{Ric} + 2(\mathcal{L}_{T_8} g) + T * T - |T|^2 g + 2\operatorname{div} T\right) \diamond \Phi = \lambda\Phi + \mathcal{L}_Y \Phi$$

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Lemma (D., '24)

Let (Φ, Y, λ) be a soliton of the (GSF).

1. *There are no compact expanding solitons.*
2. *The only compact steady solitons are given by torsion-free Spin(7)-structures.*

• Find the flow with the "nicest" evolution of the torsion/norm of torsion. This will involve the lower order terms as well. We have a "heat-type" equation for T or $|T|^2$ along the negative gradient flow of $\varphi \mapsto \int_M |T|^2 \mathrm{Vol}$ functional ($\beta = b_2 = 0, a = -\frac{1}{2}, b_1 = 1$ case).

• Examine the flows which might give some "preserved" conditions for the torsion or closedness of certain tensors along the flow (e.g., strong G_2T-structures as in A. Fino's talk).

• A monotone quantity, just like the case of the isometric flow and if possible, an entropy functional, "smallness" of which guarantees long time existence.

• Examples of solutions and solitons.

• Dynamical stability of stationary points (e.g. torsion-free G_2/Spin(7)-structures) along the flows considered here.

Thank you for your attention.