On the Donaldson-Scaduto conjecture

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based on arXiv:2401.15432

joint with Yang Li (MIT)

SCSHGAP meeting, 16 May, 2024

Introduction

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- Local Model, away from singularities,

$$M^7 \approx K3 \times \mathbb{R}^3.$$

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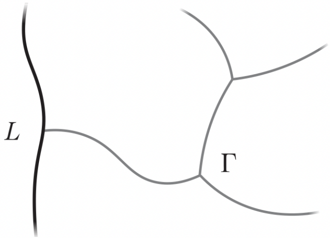

Figure 1: Local diagram of gradient cycle Γ which has one univalent vertex terminating at the link L.

Conjecture

- In search of the building block pair of pants!

Donaldson-Scaduto conjecture

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Conjecture 1 *Let $\alpha_1, \alpha_2, \alpha_3$ be -2 classes on the K3 manifold X with $\alpha_1 + \alpha_2 + \alpha_3 = 0$. Let $\mathbf{R}^3 = H \subset H^2(X)$ be a maximal positive subspace corresponding to a hyperkähler structure and v_i be the projection of α_i to H. Assume that the (α_i, H) are irreducible. Then there is an associative submanifold $\Pi \subset X \times \mathbf{R}^3$ with three ends asymptotic to $\Sigma_i \times \mathbf{R}^+ v_i$ where Σ_i is the complex curve representing α_i, for the complex structure defined by v_i, and Π is unique up to the translations of $\mathbf{R}^3$.*

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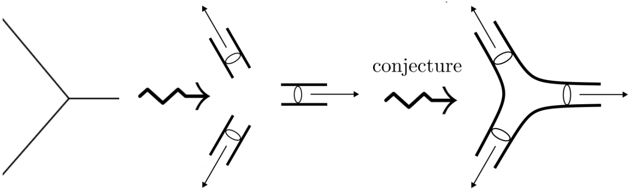
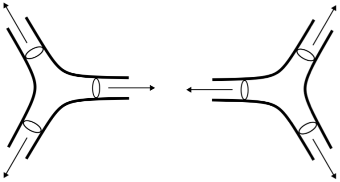
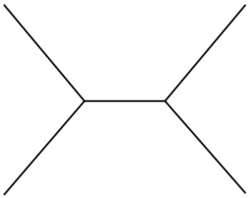


Donaldson-Scaduto conjecture.

Gluing

- A plumbing conjecture:





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- Idea: the non-compact manifold X_{A_2} can be emebedded in a Kummer $K3$ surface + deformation theory $\to$ global Donaldson-Scaduto conjecture for an open subset of moduli space of $K3$ surfaces.

- **Local Donaldson-Scaduto conjecture.** There exists a $U(1)$-invariant associative submanifold $L \subset X_{A_2} \times \mathbb{R}^3$ homeomorphic to a three-holed 3-sphere, with three ends asymptotic to the associative cylinders.

Gluing

- Features of the conjecture:

  (1) Compactness with respect to the deformation of the hyperkähler structure of X.

  (2) Relevance to the Joyce conjecture.

Local Donaldson-Scaduto conjecture

- **Theorem (E- Yang Li)** Local Donaldson-Scaduto conjecture holds.

II. The model Calabi-Yau 3-fold

Model

- The conjectured associative can be interpreted as a special Lagrangian

$$L \subset X \times \mathbb{R}^2 \subset X \times \mathbb{R}^3.$$

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- Kodaira: Any compact hyperkähler 4-manifold is either a K3 surface or a torus $\mathbb{T}^4$.

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- There is a 2-sphere family of complex structures on X.

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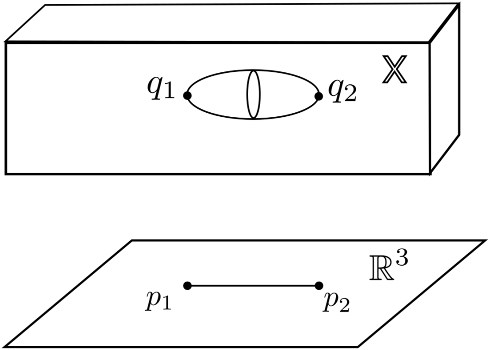

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- Gibbons-Hawking Ansatz: non-compact hyperkähler 4-manifolds.
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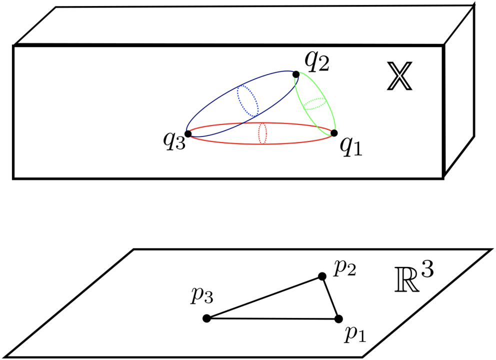

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  $V(u_1, u_2, u_3) = A + \sum_{i=1}^{n} \frac{1}{2|u - p_i|}, \quad u = (u_1, u_2, u_3) \in \mathbb{R}^3, \quad A = \text{constant} \geq 0.$

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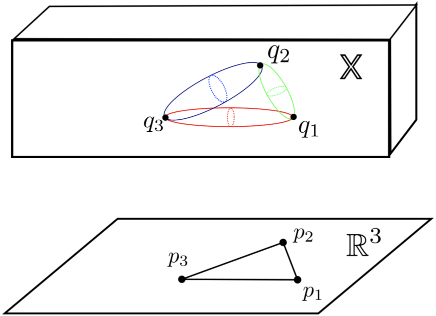
- The metric is given by $g_X = V^{-1}\theta^2 + V \sum_{i=1}^3 du_i^2$.

Gibbons-Hawking

- A_2 type ALE hyperkähler manifold X_{A_2}: let $n = 3$ and $V = \sum_{i=1}^{3} \frac{1}{2|u-p_i|}$.
- Three 2-sphere $\Sigma_i := \pi^{-1}[p_i, p_{i+1}] \subset X$ is holomorphic.



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- Special interests:

$$Z = \mathrm{K3} \times \mathbb{R}^2, \qquad Z = X_{A_2} \times \mathbb{R}^2.$$

III. Donaldson-Scaduto conjecture

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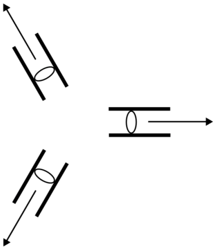
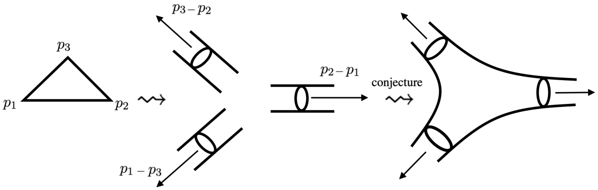


Donaldson-Scaduto conjectures

Theorem (E - Yang Li, Local Donaldson-Scaduto)

There exists special Lagrangian $P \subset X_{A_2} \times \mathbb{R}^2$ homeomorphic to a three-holed 3-sphere, with three ends asymptotic to the half-cylinders L_1, L_2, L_3.



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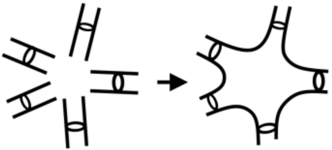
Theorem (Generalized local Donaldson-Scaduto conjecture, E-Li)

There exists an $(n-3)$-dimensional family of special Lagrangians $L \subset X \times \mathbb{R}^2$ homeomorphic to a n-holed 3-sphere, with n ends asymptotic to the translations of half-cylinders $L_1, \ldots, L_n$.



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- The dimensionally reduced Lagrangian is

$$L_{\text{red}} := L/U(1) \subset Z_{\text{red}} := u_3^{-1}(0)/U(1).$$

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- The SLag condition reduces to a holomorphicity condition:

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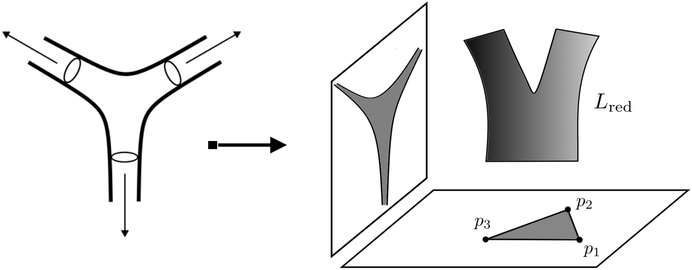

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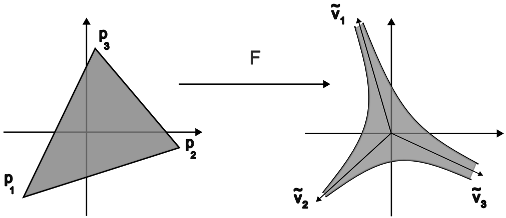

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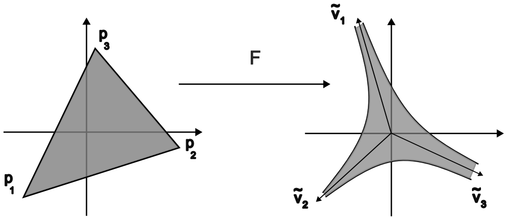

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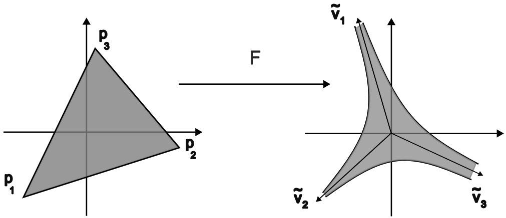

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- The second condition (Lag) implies a degenerate Monge–Ampère equation:

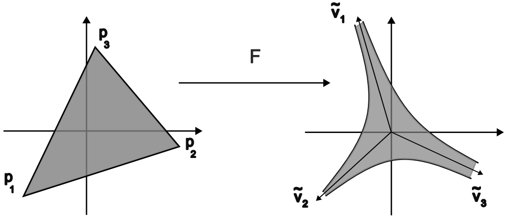
$$\det D^2\varphi = V = A + \sum_{i=1}^{n} \frac{1}{2|u - p_i|}.$$

A 'good' PDE

- Find $\varphi : U \subset \mathbb{R}^2 \to \mathbb{R}$ such that

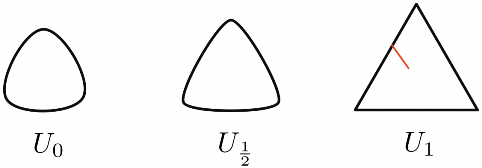
$$\det D^2\varphi = V = A + \sum_{i=1}^{n} \frac{1}{2|u - p_i|},$$

with a suitable Dirichlet boundary condition.



Solve the PDE

- Solving the Dirichlet problem: an approximation method and a compactness argument.



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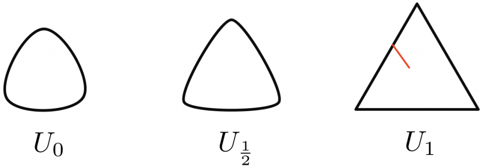

- Uniform bound:

$$\bar{\phi}_t(u) - C\text{dist}(u, \partial U)^{1/2} \leq \varphi_t(u) \leq \bar{\phi}_t(u) + C\text{dist}(u, \partial U).$$

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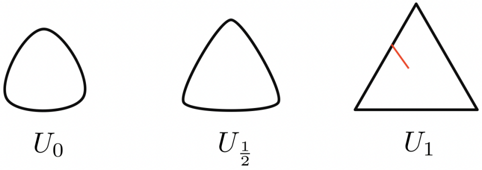

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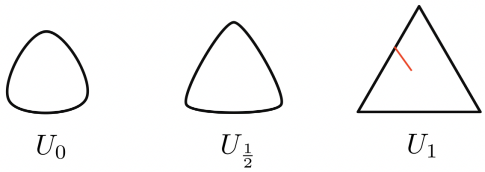

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    1. Caffarelli: The singular set must propagate along some line segment to the boundary.
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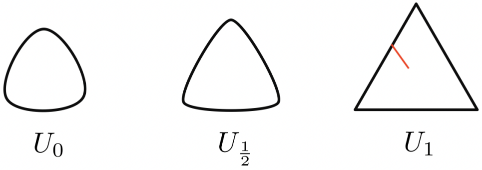
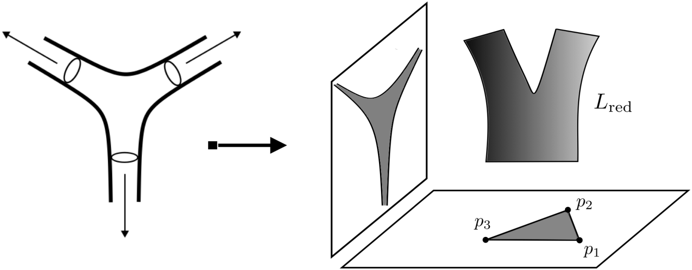


- The interior smoothness of the solution is based on two facts:
  1. Caffarelli: The singular set must propagate along some line segment to the boundary.
  2. Mooney's partial regularity: The singular set has codimension one Hausdorff measure zero.
- This proves the existence of $\varphi \Rightarrow$ the dimensionally reduced conjectured SLag.

Smoothness

- $L^\circ = \pi^{-1}(\mathrm{Graph}(F)_U)$.



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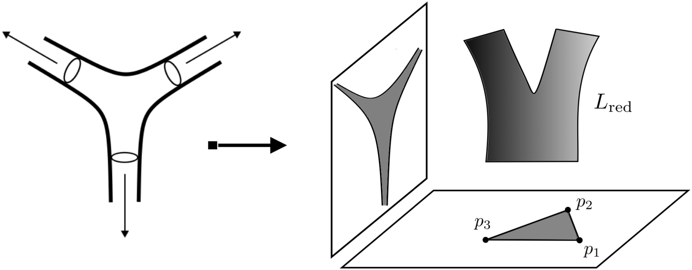

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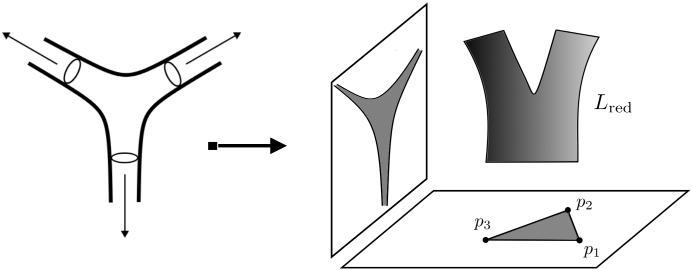

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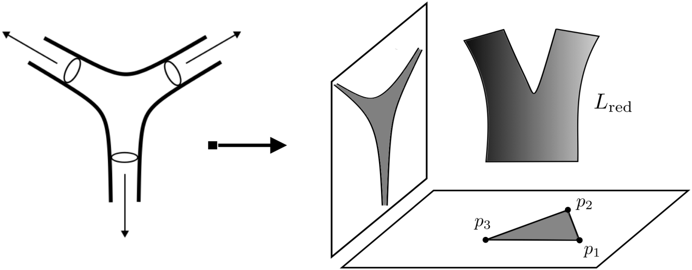

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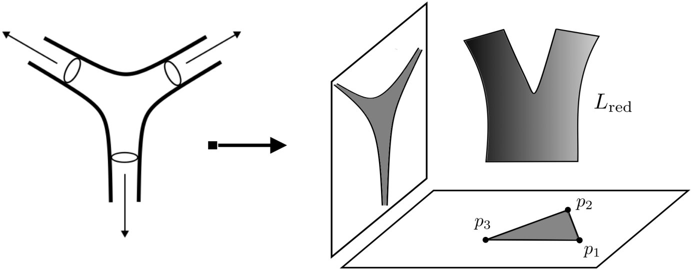
- The only possibility $\pi_{(u_1,u_2)}(x) \in \{p_1, \ldots, p_n\}$.



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- Proposition: A point $x \in \text{supp}(L)$ is a smooth point if and only if every tangent cone $N \subset \mathbb{C}^3$ at x is a 3-plane with multiplicity one.
- There is a classification of $U(1)$-invariant special Lagrangian cones in $\mathbb{C}^3$ due to Joyce/Haskins.

Smoothness

- Proposition (Joyce/Haskins): Let N be a $U(1)$-invariant SLag cone in $\mathbb{C}^3$. Then, exactly one of the following holds:

  1. N is a $\mathbb{T}^2$-cone.
  2. N is the singular union of two flat 3-planes.
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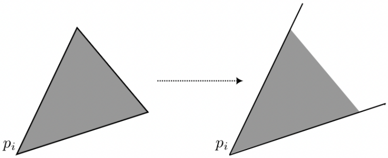
  4. N is a flat 3-plane with multiplicity $m \in \mathbb{Z}$.
- Using properties of the Monge–Ampère equation, and the geometry of the problem, we rule out every case but a flat 3-plane with $m = 1$.

- The projection of $\mathbb{T}^2$-cone is surjective.



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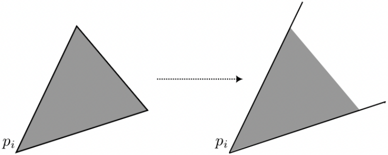
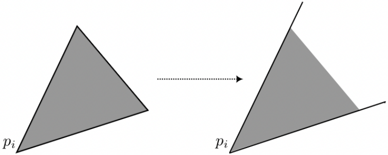


Tangent cone analysis

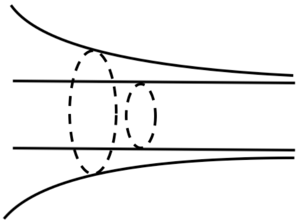
- The projection of $\mathbb{T}^2$-cone is surjective.
- The projection of the union of two planes includes a line.
- Ruling out the Jacobi elliptic case and flat 3-plane with $|m| \geq 1$ follows from a variation of Joyce graphicallity argument + some GMT ingredients.



Asymptotic analysis

- Asymptotically cylindrical:
    1. Using Legendre transform + quasi-Elliptic property: C^0 convergence.
    2. Allard's regularity: $C^{1,\alpha}$-decay, and then C^k-decay.
    3. Exponential decay: three-annulus lemma or iteration method

$$F(R) = \int_{\{y_2 \leq -R\} \cap L} |\nabla \varphi|^2 \Longrightarrow -CF' \geq F.$$



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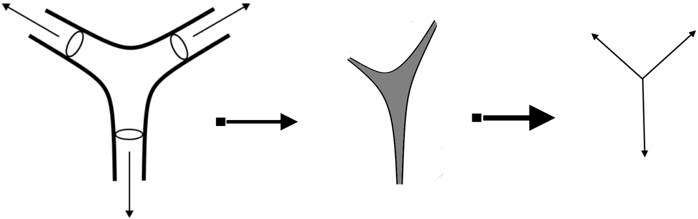
- This completes the proof.

Adiabatic limit

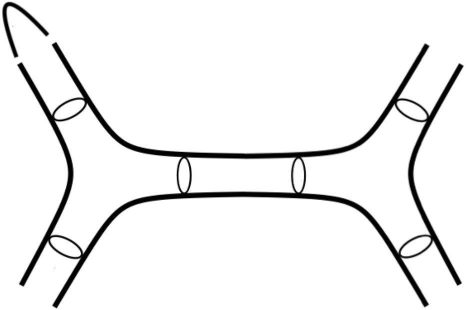
- Remark: The adiabatic limit of the pair of pants special Lagrangian is a trivalent graph.



V. Epilogue

Epilogue

- Deformation + gluing + capping-off:



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   Joint with Yang Li (in preparation): a Compactness theorem for the monopole Fueter sections.

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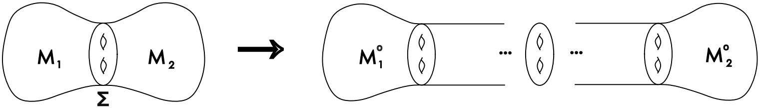

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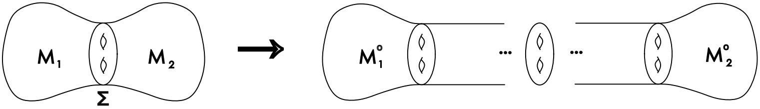

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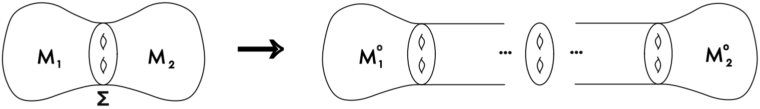

Atiyah-Floer type questions + categorified Donaldson-Segal conjecture!

Thank you for your attention!