Tower Counting for the Weak Gravity Conjecture

- Weak Gravity Conjecture in M-theory:
  w/ Cesar Cota, Alessandro Mininno, Max Wiesner 2212.09758
- Emergent String Conjecture:
  w/ Seung-Joo Lee and Wolfgang Lerche 1910.01135

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Motivation

Swampland Conjectures make profound predictions for geometry.

Key Example: see talks by Langlais, Grimm

Swampland Distance Conjecture and asymptotics of moduli spaces

This talk:

1) Interpretation and tests of Swampland Distance Conjecture

- Propose universal interpretation of asymptotically massless states
  $\Longrightarrow$ Emergent String Conjecture
- Test in classical Kähler moduli space of CY 3-folds (5d M-theory)

2) Tower Weak Gravity Conjecture

- Predicts existence of certain states, i.e. non-vanishing 'invariants'
- String dualities plus results from state counting imply the WGC.

Motivation

Swampland Conjectures make profound predictions for geometry.

Key Example: see talks by Langlais, Grimm

Swampland Distance Conjecture and asymptotics of moduli spaces

This talk:

1) Interpretation and tests of Swampland Distance Conjecture

2) Tower Weak Gravity Conjecture

Specifically: Cota,Mininno,TW,Wiesner'22

- Show Tower WGC in M-theory on CY3 with a weak coupling limit
  $\Longrightarrow$ **Asymptotic Tower WGC**
- Mathematical Input:
  1. Kähler geometry of CY3 (see part 1)
  2. Counting of BPS *and of non-BPS states* via DT-invariants on CY3

Part I: Emergent String Conjecture

Motivating questions:

- What is the interpretation of the asymptotically massless states?
- What type of theory does one approach at infinite distance in moduli space?

Emergent String Conjecture

Proposal: [Lee,Lerche,TW'19]

If a quantum gravity theory admits an infinite distance limit, then

- *either it reduces to a weakly coupled string theory*
  $\Rightarrow$ *Leading tower of states:*
  string excitations + Kaluza-Klein (KK) states at same scale
- *or it decompactifies*
  $\Rightarrow$ *Leading tower of states: Kaluza-Klein excitations*

Note: The KK states may come in disguise, e.g. as wrapped branes

Confirmed in highly-non-trival (non-perturbative) setups:

Existence and uniqueness of emergent critical string $\Longleftrightarrow$ (Quantum) geometry of string compactification

[Lee,Lerche,TW'19],[Baume,Marchesano,Wiesner'19]

[Klaewer,Lee,TW,Wiesner'20]

String Emergence - Overview

Provide evidence in Kähler moduli space of CY 3-folds X_3 probed by M-theory (classical moduli space)

1) Geometric analysis: [Lee,Lerche,TW,'19]

Classification of infinite distance limits in classical Kähler moduli space of CY3

Up to scaling of overall volume, an infinite distance limit is of the form

$$\begin{array}{rcc} \pi: \quad \mathcal{F} & \rightarrow & X_3 \\ & & \downarrow \\ & & B \end{array} \qquad \mathcal{V}_B \sim \lambda \rightarrow \infty \qquad \mathcal{V}_{\mathcal{F}} \sim \frac{1}{\lambda} \qquad \lambda \rightarrow \infty$$

1. CY3 is **T^2-fibration**
2. CY3 is **K3-fibration**
3. CY3 is **T^4-fibration**

In presence of several fibrations: a **unique fiber** vanishes at **fastest** rate

String Emergence - Overview

2) M-theory at infinite distance (finite volume) [Lee,Lerche,TW'19]

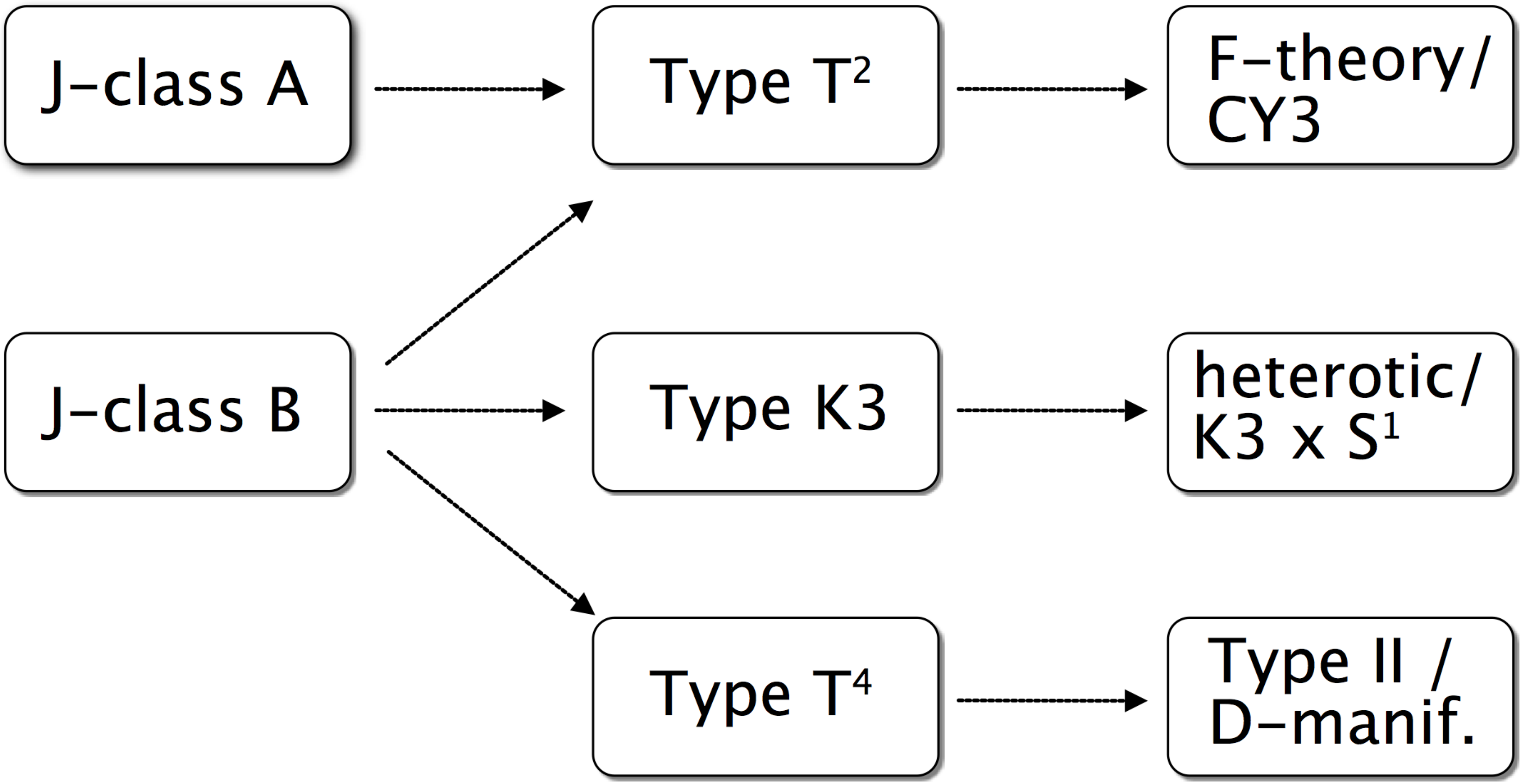
Limit of Type T^2	F-theory limit (decompactification to 6d)
Limit of Type K3	Emergence of heterotic string in 5d
Limit of Type T^4	Emergence of Type II string in 5d

3) Asymptotic tower WGC for 5d M-theory [Cota,Mininno,TW,Wiesner'22]

follows from this, making use of excitations of emergent fundamental string

- Physics: Reinterpretation partially as non-BPS string excitations
  Solves puzzle of missing BPS states raised in [Alim,Heidenreich,Rudelius'21]
- Mathematical machinery: BPS state counting

String Emergence - Overview



Classical infinite distance limits

CY 3-fold X_3 classical Kähler form $J' = \sum_{i\in\mathcal{I}} v'^i J_i\,,\qquad v'^i \geq 0$

$$\mathcal{V}'_{X_3} = \frac{1}{3!}\int_{X_3} J'^3$$

Geometric infinite distance limit:

- (some) $v'^i \to \infty \qquad \Rightarrow \mathcal{V}'_{X_3} \sim \mu \to \infty$ or finite
- rescale $J = \mu^{-1/3} J' =: \sum v^i J_i \qquad \Rightarrow \mathcal{V}_{X_3} = \mu^{-1}\,\mathcal{V}'_{X_3} \qquad \mathcal{V}_{X_3}$ finite

If all v^i finite $\quad\Longrightarrow\quad$ no further inf. distance limit
All vanishing cycles contractible
no towers of weakly coupled states
(except from overall expansion)

If some $v^i \to \infty$
others to zero $\quad\Longrightarrow\quad$ residual finite volume infinite distance limit
non-contractible cycle shrinks

Classical infinite distance limits

$$J = \sum_i v^i J_i\,, \qquad \mathcal{V}_Y = \frac{1}{3!}\int_Y J^3$$

Classify finite volume limits via refinement of analysis in

[Lee,Lerche,TW'18/'19]

$$v^i \sim \lambda \to \infty \quad \forall i \in \mathcal{I}_\lambda\,, \qquad v^j \prec \lambda \quad \forall j \in \mathcal{I} \setminus \mathcal{I}_\lambda$$

Finite volume requires: $J_i^3 = 0 \quad \forall\, i \in \mathcal{I}_\lambda$ [Lee,Lerche,TW'19]

***J*-class A**: $J_i^2 \neq 0$ for some $i \in \mathcal{I}_\lambda$ ***J*-class B**: $J_i^2 = 0\ \forall\, i \in \mathcal{I}_\lambda$

independent classification: [Corvilain,Grimm,Valenzuela'18]

via mirror symmetry to [Grimm,Palti,Valenzuela'18]

Key to understand the physics: [Lee,Lerche,TW '04/19 and '10/19]

By Oguiso's theorem each such limit implies a **fibration structure**

Classical infinite distance limits

Oguiso's theorem:

If there exists a nef divisor D with $D^3 = 0$ on Calabi-Yau 3-fold X_3:

$D^2 \neq 0$ (and D effective or $D \cdot c_2 > 0$)	$\Longrightarrow$	X_3 is genus-one fibration with fiber D^2
$D^2 = 0$		Y is surface fibration with
$D \cdot c_2(X_3) > 0$	$\Longrightarrow$	fiber $\mathcal{F} = D$: K3
$D \cdot c_2(X_3) = 0$		fiber $\mathcal{F} = D : T^4$

Apply to infinite distance limits:

***J*-class A:**

$J_i^2 \neq 0$ for some $i \in \mathcal{I}_\lambda$ $\Longrightarrow$ exists T^2-fibration

***J*-class B:**

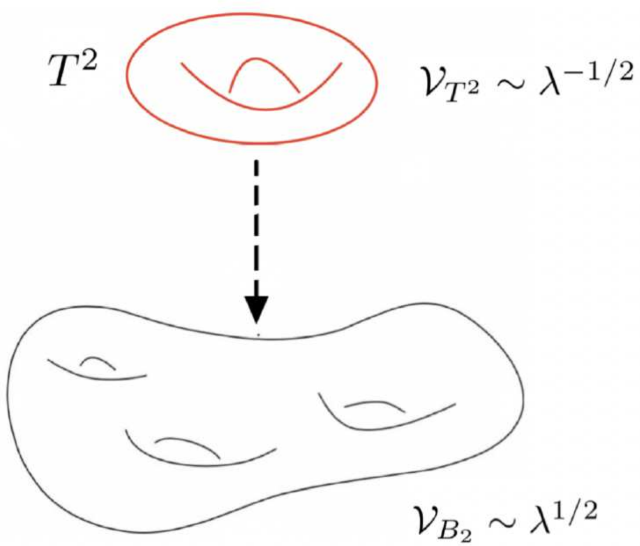
$J_i^2 = 0 \ \forall \, i \in \mathcal{I}_\lambda$ $\Longrightarrow$ exists $K3/T^4$-fibration

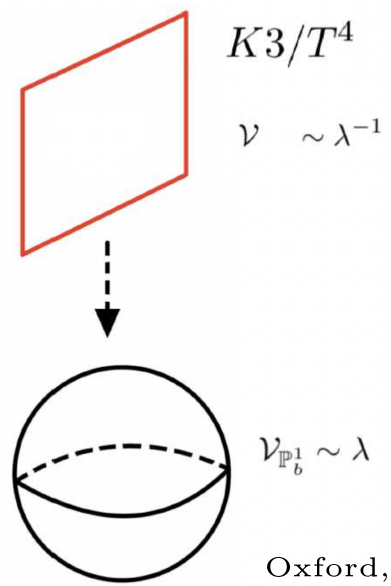
Crucial: several fibrations can coexist and we must look closer

Classical infinite distance limits

- There always exists a **unique fiber** whose volume scales to zero at the **fastest rate**.
- No ambiguity in identification of fastest shrinking curve possible

$$\text{Limit of} \left\{ \begin{array}{l} \text{Type}\, T^2 \\ \text{Type}\, \mathrm{K3} \\ \text{Type}\, T^4 \end{array} \right\} \Longleftrightarrow \textit{fastest shrinking fiber} \text{ of } X_3 \text{ is} \left\{ \begin{array}{c} T^2 \\ \mathrm{K3} \\ T^4 \end{array} \right\}$$





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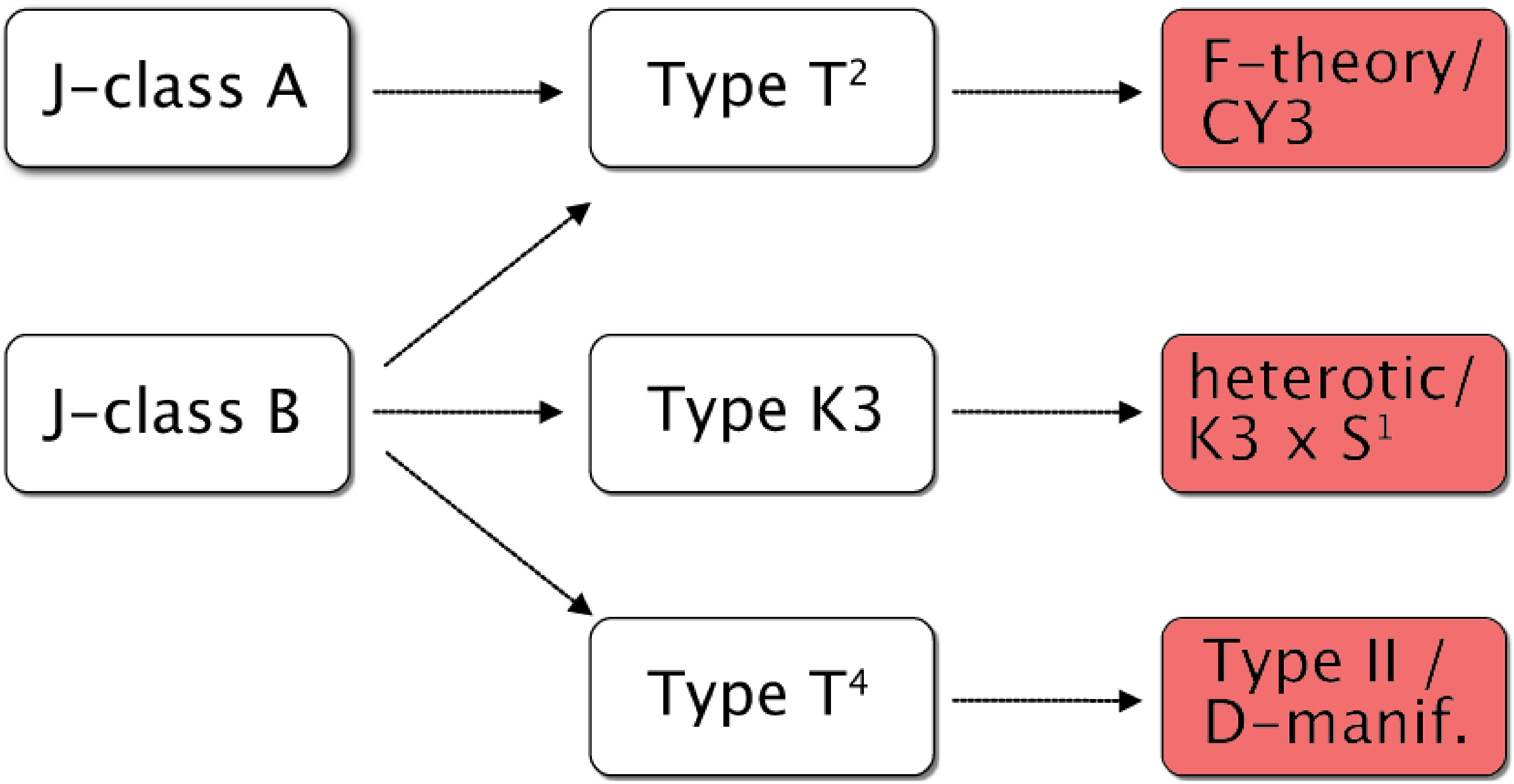
Key result: [Lee,Lerche,TW'19]

Every classical finite volume limit uniquely falls into one of these classes.

Example: Suppose X_3 admits two K3-fibrations with fiber $\mathcal{F}_1$ and $\mathcal{F}_2$ such that $\mathcal{V}_{\mathcal{F}_1} \sim \mathcal{V}_{\mathcal{F}_2} \sim \lambda^{-1} \to 0$:

- Then X_3 admits a T^2-fibration and $\mathcal{V}_{T^2} \sim \lambda^{-1/2-\delta} \prec \sqrt{\mathcal{V}_{\mathcal{F}_i}}$ for $\delta > 0$,
- and if X_3 admits several such T^2-fibrations, then there exists precisely one whose fiber shrinks at fastest rate.


J-class A
Type T2
F-theory/ CY3
J-class B
Type K3
heterotic/ K3 x S1
Type T4
Type II / D-manif.

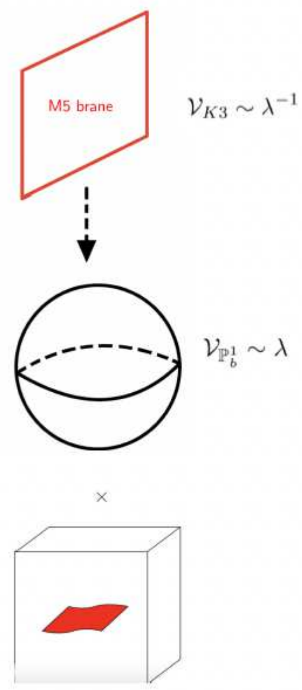
Limits of Type K3

$$\mathcal{V}_{K3} \sim \lambda^{-1}\,, \qquad \mathcal{V}_{\mathbb{P}^1_b} \sim \lambda\,, \qquad \lambda \to \infty$$

M5 brane on K3: (MSW) string: $\frac{T}{M_{\rm Pl}^2} \sim \mathcal{V}_{K3} \to 0$

M5 brane on K3:

- (MSW) string in $\mathbb{R}^{1,4}$
- tension: $\frac{T}{M_{\rm Pl}^2} \sim \mathcal{V}_{K3} \to 0$



Interpretation:

emergent critical heterotic string

Duality:

M-theory on $X_3 \times \mathbb{R}^{1,4}$ $\Longleftrightarrow$ **Heterotic on** $\widehat{K}3_{\rm het} \times S^1_A \times \mathbb{R}^{1,4}$

Limits of Type K3

$$\mathcal{V}_{K3} \sim \lambda^{-1}\,, \qquad \mathcal{V}_{\mathbb{P}^1_b} \sim \lambda\,, \qquad \lambda \to \infty$$

M5 brane on K3: (MSW) string: $\frac{T}{M_{\rm Pl}^2} \sim \mathcal{V}_{K3} \to 0$

M-theory on X_3

M5 on K3-fiber

$\Longrightarrow$ tower of **non-BPS** excitations

M2 on $C \subset K3$

$C \cdot_{\rm K3} C \geq 0$

$\Longrightarrow$ tower of **BPS** excitations

$\Longleftrightarrow$

Heterotic on $\widehat{K}3 \times S^1_A$

fundamental het. string

$\Longrightarrow$ 5d heterotic **string excitations**

winding and KK modes of het string on S^1_A

$\Longrightarrow$ **effective/dual KK tower**

Gopakumar-Vafa invariants for M2-brane on $C \subset K3$: [Harvey,Moore'99], ...

$$N_C = c(\frac{C^2}{2}) \qquad f(q) = \sum_{n=-1}^{\infty} c(n) q^n \quad \textbf{modular form}$$

Infinite tower on $nC \leftrightarrow C \cdot C \geq 0$ = non-contractible curves inside K3

Limits of Type T^4

$$\mathcal{V}_{T^4} \sim \lambda^{-1}\,, \qquad \mathcal{V}_{\mathbb{P}^1_b} \sim \lambda\,, \qquad \lambda \to \infty$$

Emergent Type II string probes (non-geometric) D-manifold

M-theory on Abelian surface fibration X_3 $\longleftrightarrow$ Type IIB theory on $Z \times S^1_A$

Part II: (Asymptotic) Tower Weak Gravity Conjecture

Weak Gravity Conjecture

(Tower) WGC   initiated in [Arkani-Hamed,Motl,Nicolis,Vafa'06]

Consider a gauge theory coupled to quantum gravity, (for simplicity) with abelian gauge factors $U(1)$ and charge lattice $\Lambda_{\mathbf{Q}}$. Then every ray in the lattice $\Lambda_{\mathbf{Q}}$ must support a tower of super-extremal states.

- super-extremal (=self-repulsive for us):

$$F_{\text{Coulomb}} \geq F_{\text{Grav.}} + F_{\text{Yukawa}}$$

  - F_{Yukawa} in presence of massless scalars - first pointed out in [Palti,'17]
  - in general self-repulsiveness not equivalent to super-extremal [Heidenreich,Reece,Rudelius,'19], but in asymptotic weak coupling limit they are [Lee,Lerche,TW'18]

- Tower: [Heidenreich,Reece,Rudelius'15-16] [Montero,Shiu,Soler'16] [Andriolo et al.'16]

$\exists$ super-extremal particle of charge $n\mathbf{Q}$ for $n \in \mathcal{I}$ an *infinite* set

WGC in 5d M-theory

M-theory on CY3 X_3: 5d N=1 theory (8 supercharges)

- Basis of U(1) gauge groups from expanding

$$C_3 = A^\alpha \wedge J_\alpha \,, \qquad \alpha = 1, \ldots, h^{1,1}(X_3)$$

- J_α: Basis of Kähler cone generators $J = v^\alpha J_\alpha$
- Gauge kinetic terms

$$S_{5d} = \frac{M_{\mathrm{Pl}}^3}{2} \int_{\mathbb{R}^{1,4}} R \star 1 - \frac{1}{2g_5^2} \int_{\mathbb{R}^{1,4}} f_{\alpha\beta} F^\alpha \wedge \star F^\beta + \ldots \qquad F^\alpha = dA^\alpha$$

Gauge kinetic matrix $f_{\alpha\beta} \Longleftrightarrow$ Kähler moduli (modulo overall volume)

$$f_{\alpha\beta} = \frac{1}{\mathcal{V}^{1/3}} \int_{X_3} J_\alpha \wedge \star J_\beta = \left(\hat{\mathcal{V}}_\alpha \hat{\mathcal{V}}_\beta - \hat{\mathcal{V}}_{\alpha\beta} \right)$$

$$\mathcal{V} = \frac{1}{6} \int_{X_3} J^3 \,, \qquad \mathcal{V}_\alpha = \frac{1}{2\mathcal{V}} \int_{X_3} J_\alpha \wedge J^2 \,, \qquad \mathcal{V}_{\alpha\beta} = \frac{1}{\mathcal{V}} \int_{X_3} J_\alpha \wedge J_\beta \wedge J \,, \qquad \hat{v}^\alpha = \frac{v^\alpha}{\mathcal{V}}$$

WGC in 5d M-theory

Self-repulsiveness condition for states of

- charges Q_α under $U(1)_\alpha$
- Kähler moduli dependent mass $M_k(v^\alpha)$

$$F_{\text{Coulomb}} \overset{!}{\geq} F_{\text{grav}} + F_{\text{Yukawa}}$$

$$\frac{(M_{\text{Pl}} g_5^2)(Q_\alpha f^{\alpha\beta} Q_\beta)}{M_k^2/M_{\text{Pl}}^2} \overset{!}{\geq} \left.\frac{d-3}{d-2}\right|_{d=5} + \frac{1}{2}\frac{M_{\text{Pl}}^4}{M_k^4}\left(f^{\alpha\beta} - \frac{1}{3}\hat{v}^\alpha \hat{v}^\beta\right)\partial_\alpha\left(\frac{M_k^2}{M_{\text{Pl}}^2}\right)\partial_\beta\left(\frac{M_k^2}{M_{\text{Pl}}^2}\right)$$

basis of U(1) gauge groups from expanding

$$C_3 = A^\alpha \wedge J_\alpha\,, \qquad \alpha = 1, \ldots, h^{1,1}(X_3)$$

J_α: Basis of Kähler cone generators $J = v^\alpha J_\alpha$

$$S_{5d} = \frac{M_{\text{Pl}}^3}{2}\int_{\mathbb{R}^{1,4}} R \star 1 - \frac{1}{2g_5^2}\int_{\mathbb{R}^{1,4}} f_{\alpha\beta} F^\alpha \wedge \star F^\beta + \ldots$$

WGC: BPS Towers

First source for super-extremal tower:
particles in 5d from M2-branes on curves on X_3

Fact: Every BPS state is automatically super-extremal

$\Longrightarrow$ Existence of tower of BPS states along ray in charge lattice sufficient for tower WGC in that direction

Conjecture: [Alim,Heidenreich,Rudelius'21]
Every curve class $C \in \mathrm{Mov}_1(X_3)$ supports a tower of BPS states, i.e.

$$N_{g=0}(nC) \neq 0 \qquad \forall C \in \mathrm{Mov}_1(X_3)$$

Recall: Movable curve cone $\mathrm{Mov}_1(X_3)$ is dual to cone of effective divisors $\mathrm{Eff}^1(X_3)$

✓ confirmed in many examples in [Alim,Heidenreich,Rudelius'21]
✓ extremality $=$ BPS condition for such $C \in \mathrm{Mov}_1(X_3)$

Challenge for tower WGC: What if there are no BPS towers?

Example: Conifold

Beyond BPS Towers

Main result: [Cota,Mininno,TW,Wiesner'22]

Whenever there is no BPS tower, then

- either there is no weak coupling limit for the U(1)s
- or there does exist a super-extremal non-BPS tower.

$\Longrightarrow$ Establishes *asymptotic* tower WGC in 5d M-theory

Strategy:

1. Characterise all weak coupling limits $\Longleftrightarrow$ Kähler geometry
2. Identify towers of super-extremal BPS or non-BPS states for U(1)s with a weak coupling limit
   $\Longleftrightarrow$ DT invariants/Noether-Lefschetz theory

Weak Coupling Limits

$$S_{5d} = \frac{M_{\text{Pl}}^3}{2} \int_{\mathbb{R}^{1,4}} R \star 1 - \frac{1}{2g_5^2} \int_{\mathbb{R}^{1,4}} f_{\alpha\beta} F^\alpha \wedge \star F^\beta + \ldots$$

Gauge kinetic matrix $f_{\alpha\beta} \Longleftrightarrow$ Kähler moduli (modulo overall volume)

$$f_{\alpha\beta} = \frac{1}{\mathcal{V}^{1/3}} \int_{X_3} J_\alpha \wedge \star J_\beta = \left(\hat{\mathcal{V}}_\alpha \hat{\mathcal{V}}_\beta - \hat{\mathcal{V}}_{\alpha\beta} \right)$$

Weak coupling limits:

entries of $f_{\alpha\beta} \to \infty \leftrightarrow$ infinite distance limits (at fixed overall volume $\mathcal{V}$)

More precise formulation:

$U(1)_C = c_\alpha U(1)^\alpha$ basis $U(1)^\alpha \leftrightarrow A^\alpha$ in $C_3 = A^\alpha J_\alpha$

$$\Lambda^2_{\text{WGC}} \left(U(1)_C \right) = g^2_{\text{YM,C}} M^3_{\text{Pl}} = g_5^2 \left(c_\alpha f^{\alpha\beta} c_\beta \right) M^3_{\text{Pl}}$$

$$\frac{\Lambda^2_{\text{WGC}} \left(U(1)_C \right)}{\Lambda^2_{\text{QG}}} \to 0$$

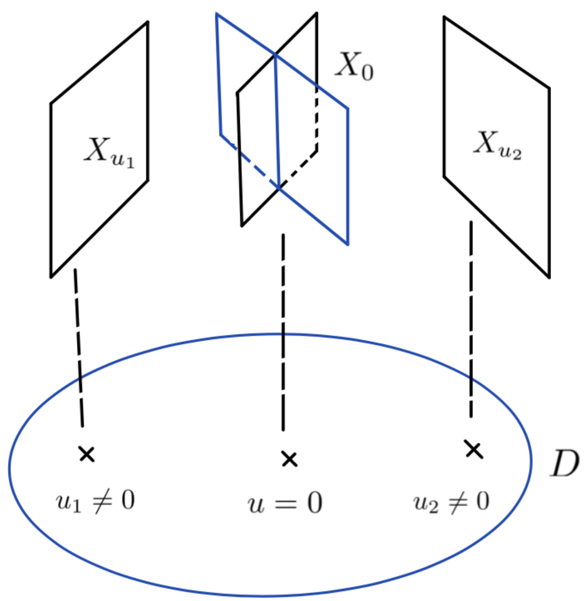
Λ^2_{QG} : species scale

Weak Coupling Limits

Characterisation of weak coupling limits: [Cota,Mininno,TW,Wiesner'22]

In M-theory compactified on a Calabi–Yau X_3, the only $U(1)$s which admit a weak coupling limit are obtained as $U(1)_C = c_\alpha U(1)^\alpha$ for $C = c_\alpha \omega^\alpha \in H_2(X_3)$ a curve class with

1. $C = T^2$ - a generic torus fiber of X_3
2. $C \subset S$ for S a generic K3 or T^4 fiber of X_3 or a degenerate such fiber occuring at finite distance in the fiber moduli space.



Elliptic tower counting

Suppose X_3 admits fibration $\pi : T^2 \to B_2$ with generic fiber $T^2 \equiv \mathcal{E}$.

1. Unless X_3 also admits a K3 or T^4 surface (see next case), the *only* U(1) which can undergo a weak coupling limit is

$$U(1)_{\mathcal{E}} = c_\alpha U(1)^\alpha \qquad\qquad \mathcal{E} = c_\alpha \omega^\alpha$$

2. $\exists$ tower of BPS states charged under $U(1)_{\mathcal{E}}$:
   M2-branes wrapped n-times on T^2 fiber

$$N^0_{n\mathcal{E}} = -\chi(X_3)$$

$\implies$ super-extremal BPS tower in agreement with asymptotic tower WGC

Interpretation: KK tower for decompactification 5d to 6d

K3 tower counting

Suppose X_3 admits fibration $\rho : K3 \to \mathbb{P}^1$

The *only*[a] U(1) which can undergo a weak coupling limit is

$$U(1)_C = c_\alpha U(1)^\alpha \qquad C = c_\alpha \omega^\alpha$$

for C a curve in a generic K3 fiber or in a special K3 fiber at finite distance in moduli space.

Such curves lie in a lattice

$$\Lambda^*_{\mathbb{R}} = \Lambda^*_+ \oplus \Lambda^*_-$$

$\Lambda^*_{\mathbb{R}}$ lattice of charges with respect to such U(1) of signature $(1, r)$, $r \leq 19$

- $\mathbf{Q}^2 \geq 0$: BPS tower exists
  ✓such curves are movable inside K3 fiber and hence in movable cone
  ✓in agreement with BPS index counting via modular forms - see before
- $\mathbf{Q}^2 < 0$: No BPS tower exists
  ✓curves are rigid inside K3 fiber and hence not in movable cone

[a] Unless other fibrations exist of course.

K3 tower counting

Claim: Tower of non-BPS states takes over [Cota,Mininno,TW,Wiesner'22]

- Special set of states from excitations of MSW-type heterotic string obtained by wrapping M5-brane on K3 fiber turns out to be super-extremal.
- Existence of these states established via relation of elliptic genus and 4d BPS invariants in Type IIA on CY3[a]

  [cf. talk by Pioline]

  $\Longleftrightarrow$ Uses related results such as

  [Bouchard,Creutzig,Diaconescu,Doran,Quigley,Sheshmani16] [Pandharipande,Thomas'16]

[a] Analysis most explicitly in absence of multi-components fibers, but expect results to carry over more generally.

K3 tower counting

M-theory on $X_3 \times S^1_M$
bound state of M5-brane on K3
winding number r on S^1_M
M2 brane on curve $\mathbf{Q}$ on K3

Type IIA string on X_3
D4-D2-D0 bound state
of charge vector
$\gamma = (r\Sigma_{\mathrm{K3}}, \mathbf{Q}, n)$

Can view these states as winding modes of heterotic string from M5 on K3 at KK level n and charge vector $\mathbf{Q}$

Special case $r = 1$:
By level matching: $n = n_L =$ (left-moving) excitation level of single heterotic string

4d BPS states of charge vector $\gamma = (\Sigma_{\mathrm{K3}}, \mathbf{Q}, n)$	$\Longrightarrow$	existence of non-BPS string excitations in 5d at level $n_L = n$ and charge $\mathbf{Q}$

For simplicity focus on $\mathbf{Q} \in \Lambda^*_-$
Goal: Show existence of super-extremal state at $n = -\frac{1}{2}\mathbf{Q}^2$

K3 tower counting

Modified elliptic genus of wrapped heterotic string [Gaiotto,Strominger,Yin'06]:

$$Z_{\mathbf{S}}^{(r=1)}(\tau,\bar\tau,\mathbf{z}) = \mathrm{Tr}_{\mathrm{RR}} F_{\mathrm{R}}^2 (-1)^{F_{\mathrm{R}}} q^{L_0-\frac{c_{\mathrm{L}}}{24}} \bar q^{\bar L_0 - \frac{c_{\mathrm{R}}}{24}} e^{2\pi i z^i Q_i}$$
$$= \sum_{n_L,n_R} N(n_L,n_R,\mathbf{Q})\, q^{n_L}\, \bar q^{n_R} e^{2\pi i z^i Q_i}$$

Expression in terms of 4d BPS numbers:

$$Z_{\mathbf{S}}^{(r=1)}(\tau,\bar\tau,\mathbf{z}) = \sum_{\boldsymbol{\mu}\in\Lambda^*/\Lambda} Z_{\boldsymbol{\mu}}(\tau)\Theta^*_{\boldsymbol{\mu}}(\tau,\bar\tau,\mathbf{z})$$

$$Z_{\boldsymbol{\mu}}(\tau) = \sum_{n=0}^{\infty} \Omega(\gamma) q^{n+\mathbf{Q}^2/2-1}, \quad \Theta^*_{\boldsymbol{\mu}}(\tau,\bar\tau,\mathbf{z}) = \sum_{\boldsymbol{\lambda}\in\boldsymbol{\mu}+\Lambda} q^{-\frac12(\boldsymbol{\lambda})^2_-}\, \bar q^{+\frac12(\boldsymbol{\lambda})^2_+}\, e^{2\pi i(\boldsymbol{\lambda})\cdot z}$$

$\Omega(\gamma)$: 4d BPS index for D4-D2-D0 states (Donaldson-Thomas invariants)

Focus on $\mathbf{Q}=\mathbf{Q}_- \in \Lambda^*_-$

If $\Omega(\gamma)\neq 0$ for $n=-\frac{\mathbf{Q}^2}{2}>0$, then have states at excitaton level n in 5d

Recall:

n: KK number on S^1_M, but by level matching identified with excitation level in 5d

K3 tower counting

$$\Omega(\gamma) \neq 0 \quad \text{for} \quad \gamma = (\Sigma_{\rm K3}, \mathbf{Q}, n) \quad \text{at} \quad n = -\frac{\mathbf{Q}^2}{2}\,, \; \mathbf{Q} \in \Lambda^*_-$$

Key insight: [Bouchard,Creutzig,Diaconescu,Doran,Quigley,Sheshmani16]

$$\begin{aligned} Z_{\boldsymbol{\mu}}(\tau) &= \sum_{n=0}^{\infty} \Omega(\gamma) q^{n+\mathbf{Q}^2/2-1} \\ &= \eta^{-24}(\tau)\Phi_{\boldsymbol{\mu}}(\tau) \; = \left[q^{-1} + 24 + \mathcal{O}(q)\right] \Phi_{\boldsymbol{\mu}}(\tau) \end{aligned}$$

for $\Phi_{\boldsymbol{\mu}}(\tau)$ a component of a vector-valued modular form

Expansion coefficients related to Noether-Lefschetz numbers [Maulik,Pandharipande'13]
[Pandharipande,Thomas'16]

$$NL_{(h,\mathbf{Q})} = \text{Coeff}\left(\Phi_{\boldsymbol{\mu}}, q^{\Delta_{\rm NL}}\right)\,, \qquad \Delta_{\rm NL} = \frac{1}{2}\eta^{ij}Q_iQ_j + 1 - h$$

- If $\Delta_{\rm NL} < 0$, then $NL_{(h,\mathbf{Q})} = 0$
- If $\Delta_{\rm NL} = 0$, then $NL_{(h,\mathbf{Q})} = -2$
- If $\Delta_{\rm NL} > 0$, then $NL_{(h,\mathbf{Q})} \in \mathbb{Z}$

$\Longrightarrow$ States with $n = -\frac{\mathbf{Q}^2}{2}$ appear at order q^{-1} in

$$Z_{\mathbf{0}}(\tau) = \eta^{-24}(\tau)\Phi_{\mathbf{0}}(\tau) = \left[q^{-1} + 24 + \mathcal{O}(q)\right]\left[-2 + \mathcal{O}(q)\right] = -2q^{-1} + \mathcal{O}(q^0)$$

Super-extremality

$$F_{\text{Coulomb}} \overset{!}{\geq} F_{\text{grav}} + F_{\text{Yukawa}}$$

$$\frac{(M_{\mathsf{Pl}} g_5^2)(Q_\alpha f^{\alpha\beta} Q_\beta)}{M_k^2/M_{\mathsf{Pl}}^2} \overset{!}{\geq} \left.\frac{d-3}{d-2}\right|_{d=5} + \frac{1}{2}\frac{M_{\mathsf{Pl}}^4}{M_k^4}\left(f^{\alpha\beta} - \frac{1}{3}\hat{v}^\alpha \hat{v}^\beta\right)\partial_\alpha\left(\frac{M_k^2}{M_{\mathsf{Pl}}^2}\right)\partial_\beta\left(\frac{M_k^2}{M_{\mathsf{Pl}}^2}\right)$$

Explicitly check this for states at excitation level $n_k = -\frac{1}{2}\mathbf{Q}^2$

Input from string theory:

$$M_k^2 = 8\pi(n_k - 1)T_s + \Delta_{\mathsf{CB}}$$

- First term: Contribution from string oscillators, with string tension

$$T_s = 2\pi \mathcal{V}_{\mathbf{S}} M_{11\mathrm{d}}^2 = 2\pi(4\pi)^{-2/3}\hat{\mathcal{V}}_{\mathbf{S}} M_{\mathsf{Pl}}^2$$

- Δ_{CB}: contribution from Coulomb branch in 5d

$$\Delta_{\mathsf{CB}} = 4\pi^2(4\pi)^{-2/3} Q_i Q_j \hat{v}^i \hat{v}^j \, M_{\mathsf{Pl}}^2 \qquad \hat{v}^i \text{ : Kähler moduli of K3 fiber}$$

In the asymptotic limit a number of simplifications occur

$\Longrightarrow$ **Together with $n_k = -\frac{1}{2}\mathbf{Q}^2$ the equality is marginally obeyed**

Summary

Exemplified two related Swampland Conjectures in M-theory on CY3

Emergent String Conjecture in Kähler moduli sector	$\Longleftrightarrow$	Natural fibration structure for infinite distance limits
Asymptotic Tower Weak Gravity Conjecture	$\Longleftrightarrow$	Kähler geometry and (non-) BPS counting

Related studies for theories in 6d N=1 [Lee,Lerche,TW'18]
and 4d N=1 [Lee,Lerche,TW'18] [Kläwer,Lee,TW,Wiesner'20] [Cota,Mininno,TW,Wiesner'22]

Mysterious:
Tower WGC for theories without weak coupling limits?