Quantum Gravity Conjectures and Hodge Theory

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Based on:

First part: works with Irene Valenzuela, Eran Palti, Damian van de Heisteeg, Chongchuo Li, Brice Bastian, Jeroen Monnee, Fabian Ruehle

Second part: 2112.06995 with Ben Bakker, Christian Schnell, Jacob Tsimerman

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Simons Collaboration on Special Holonomy in Geometry, Analysis, and Physics 2023

Introduction

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Identify the general principles that have to be satisfied in any four-dimensional theory compatible with quantum gravity.

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→ test or ‘prove’ them in as many as possible instances

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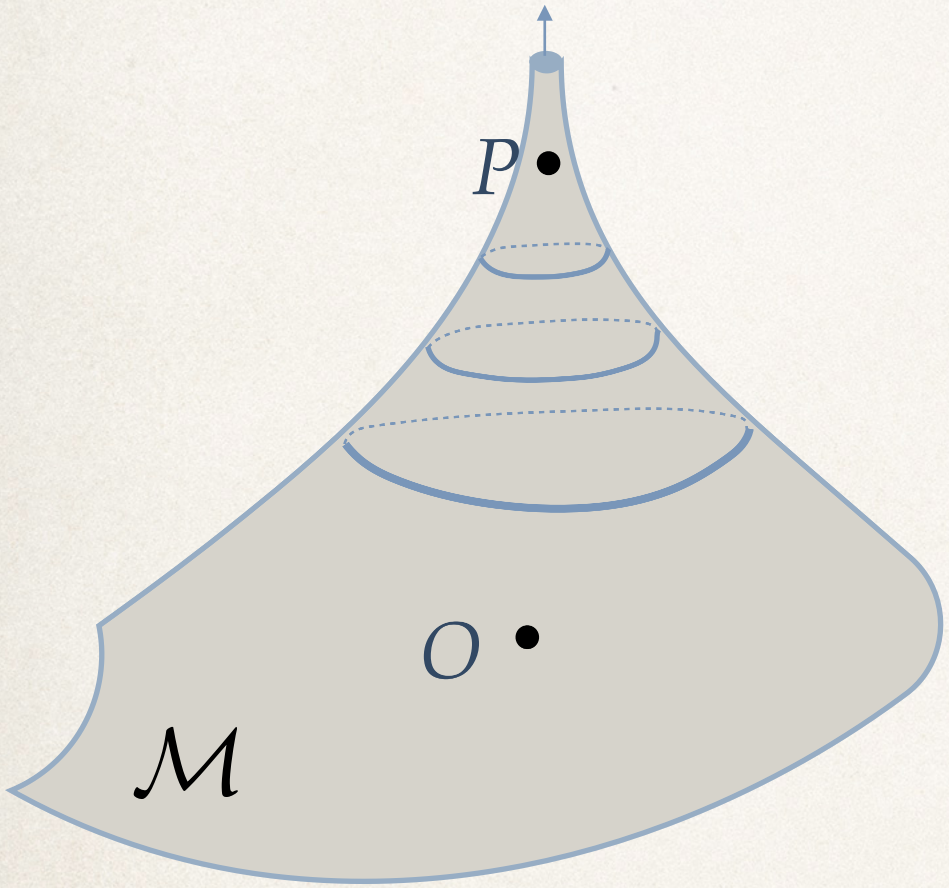
  → more D-branes might become relevant when changing complex structure → new light particles

Distance Conjecture

[Ooguri,Vafa '06]

- Consider field space $\mathcal{M}$ in a four-dimensional theory (e.g. moduli space)



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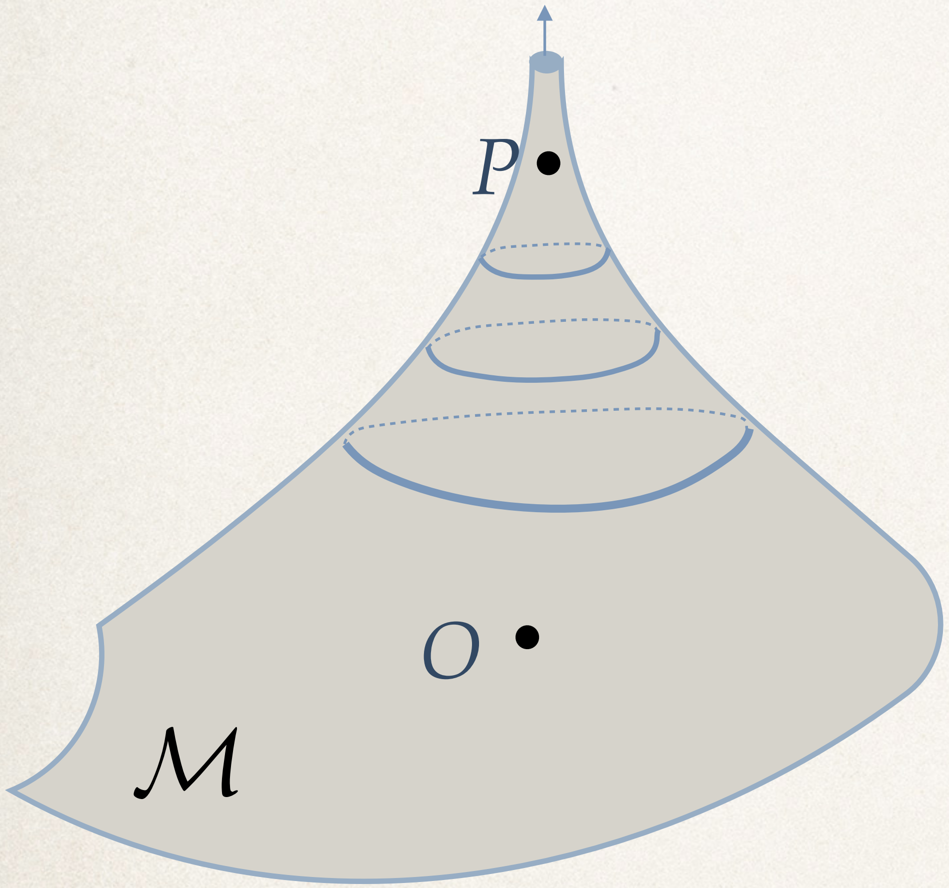

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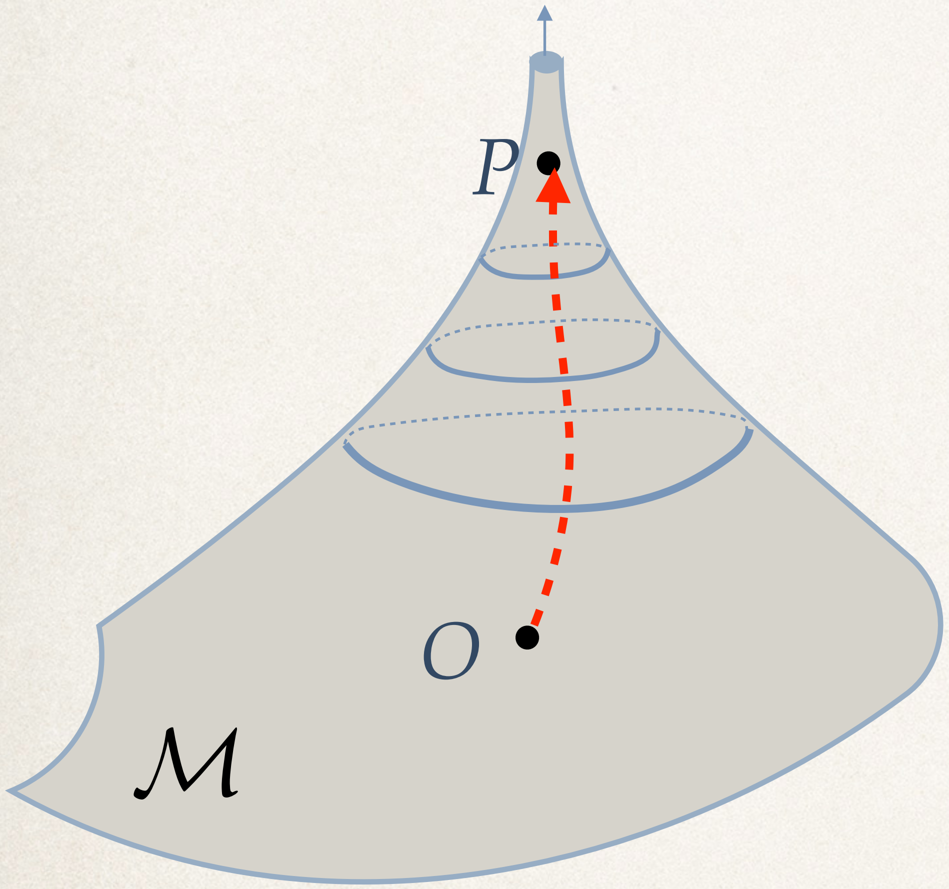

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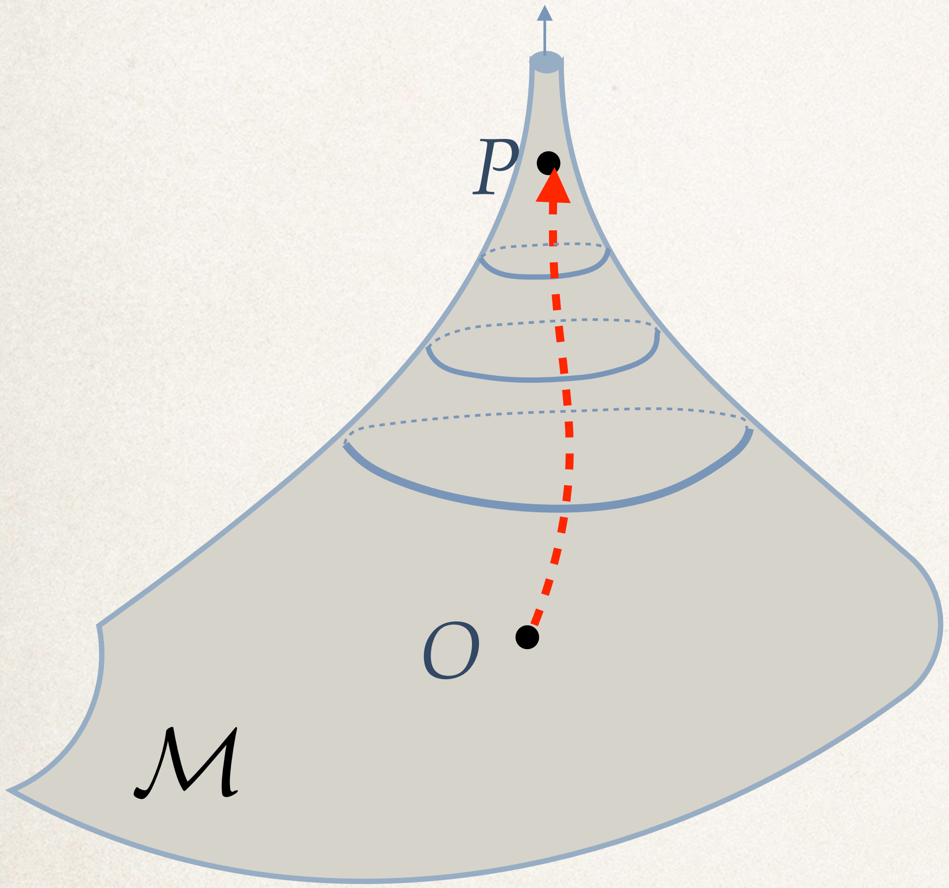

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Conjecture:

Infinite number of particles (states) become massless approaching P:

$$m(P) \propto M_P \, e^{-\gamma \, d(P,O)} \text{ as } d(P,O) \gg 1$$

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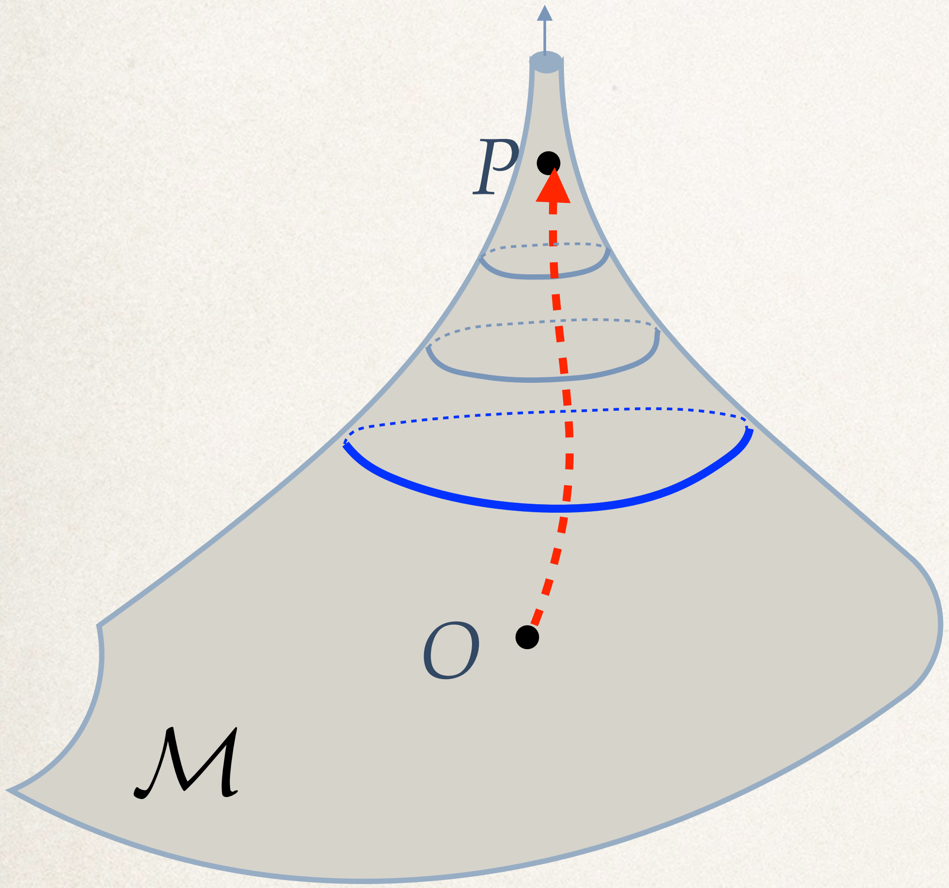

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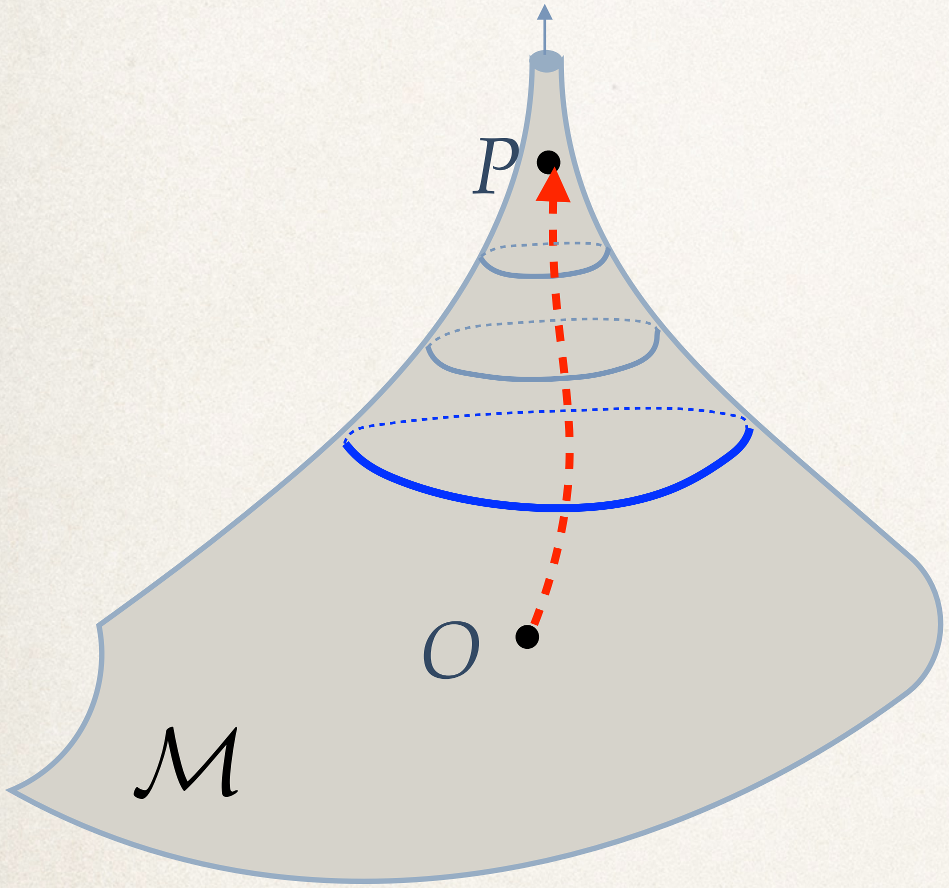

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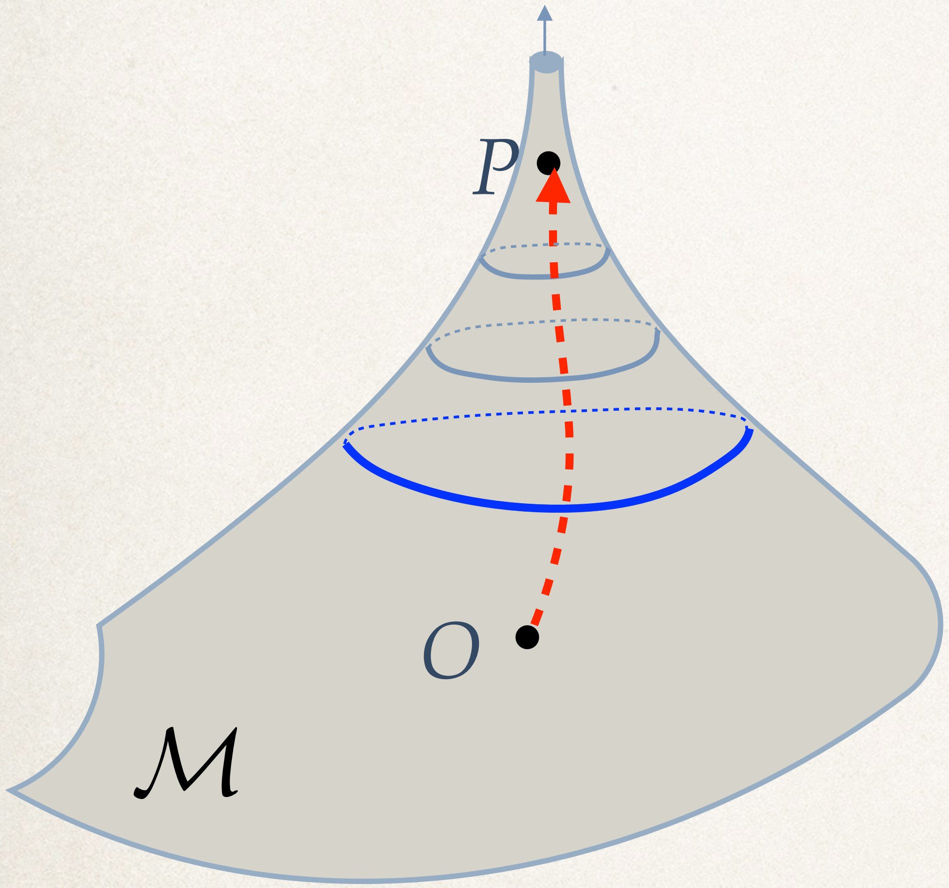

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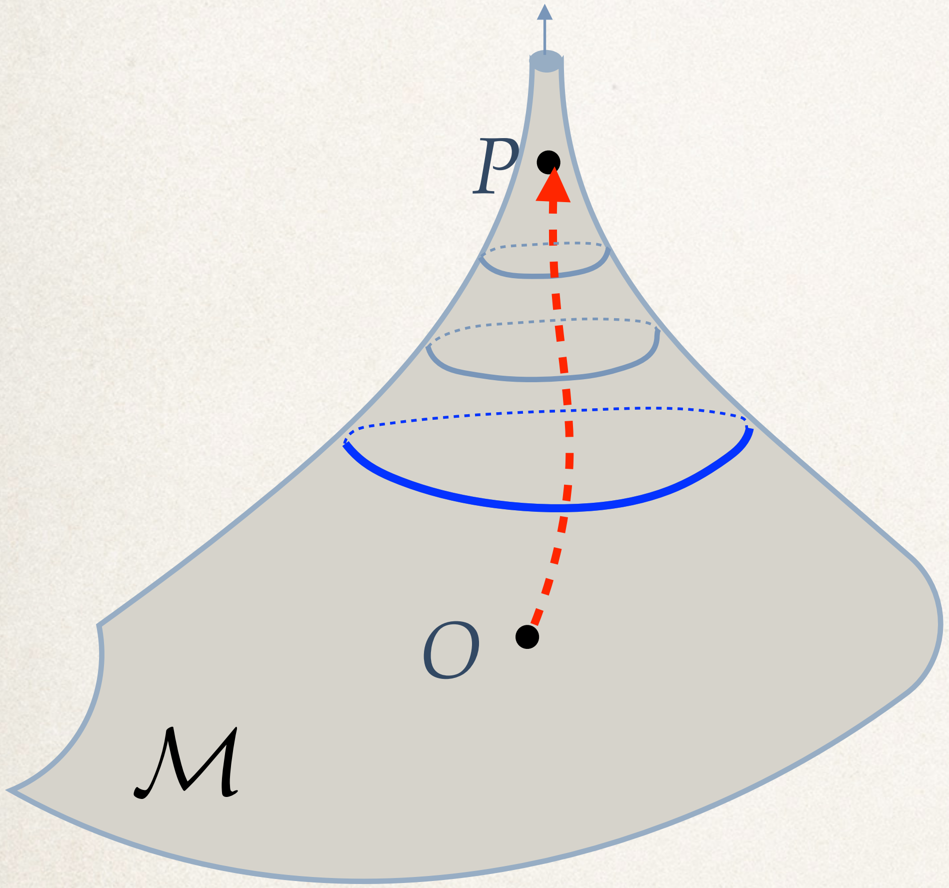

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→ no proof yet, but significant evidence

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- Distances are determined by: $d_\gamma(P,O) = \int_\gamma \sqrt{g_{IJ}\dot{x}^I \dot{x}^J} ds$

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Kähler metric: $g_{I\bar{J}} = \partial_{z^I}\partial_{\bar{z}^J} K$

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<u>Question 1:</u> What are the points P for which $d_\gamma(P,O)$ is infinite for every path?

States and their masses

- Candidate states to consider: BPS - D3 branes wrapping three-cycles

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Question 2: Is there an infinite set of lattice sites in $H^3(X,\mathbb{Z})$

such that $m(z) \propto e^{-d(z_0,z)}$ $d(z_0,z) \to \infty$?

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<u>Question 3:</u> Are there BPS states at these sites, are they stable?

$\Rightarrow$ counting problem, study walls of stability…

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We used general results from asymptotic Hodge theory for Question 1 & 2.

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What is the physics of the limits?

emergence proposal [TG,Palti,Valenzuela][Heidenreich,Reece,Rudelius][Palti]…

emergent strings [Lee,Lerche,Weigand]…

Asymptotic Hodge Theory

Structure of complex structure moduli space

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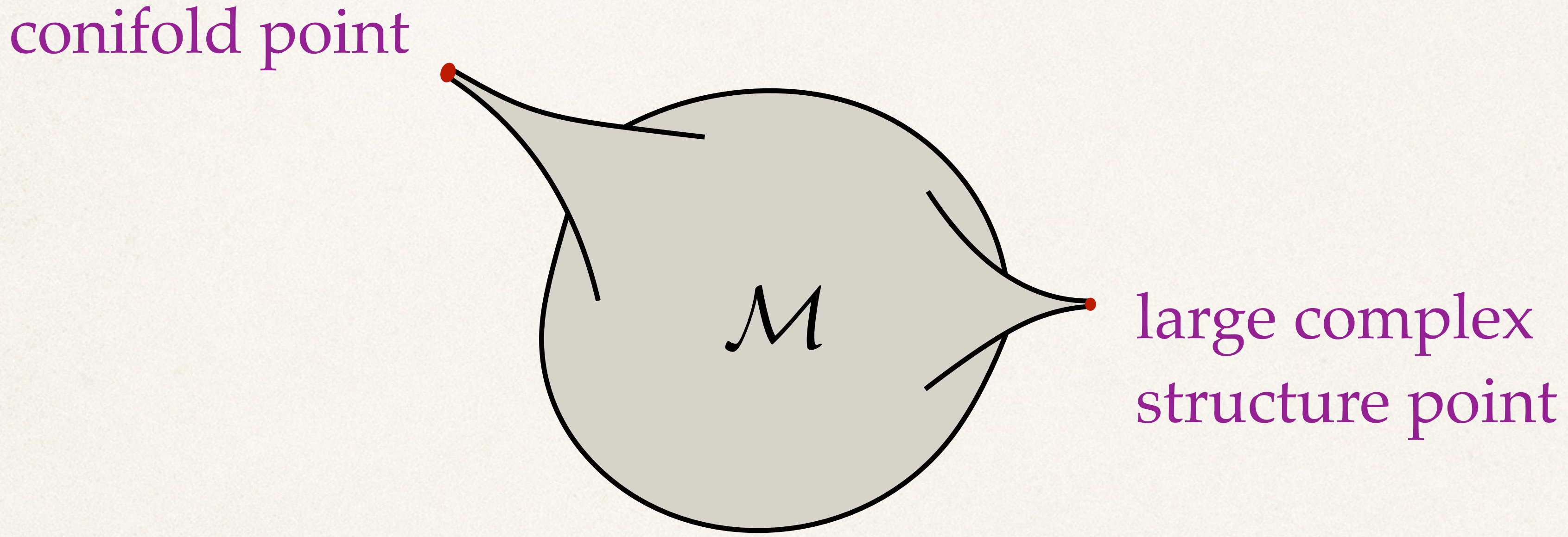
  Example: Calabi-Yau threefold (such as mirror quintic)



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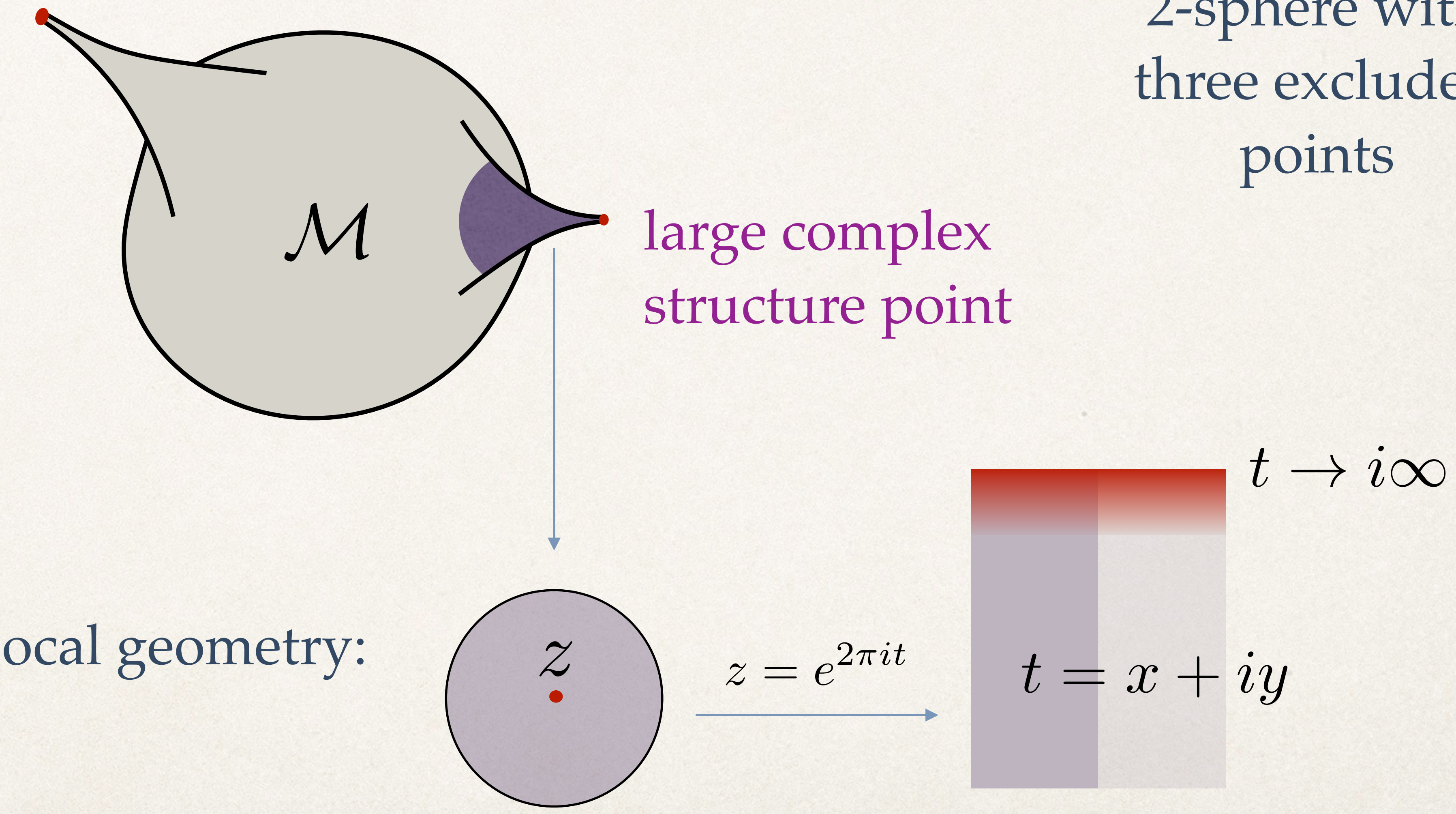

conifold point

2-sphere with
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$\mathcal{M}$

large complex
structure point

$t \to i\infty$

Local geometry:

z

$z = e^{2\pi i t}$

$t = x + iy$

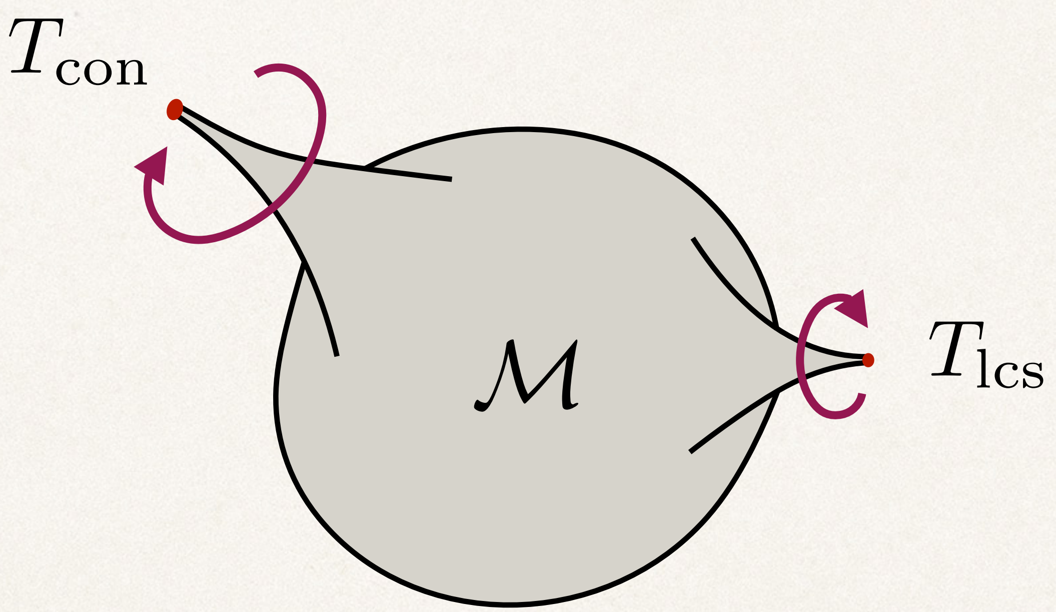
punctured disc

upper half-plane

Boundaries in complex structure moduli space

- Boundaries are the points where Calabi-Yau manifold degenerates

  $\Rightarrow$ associate a monodromy T around singular loci



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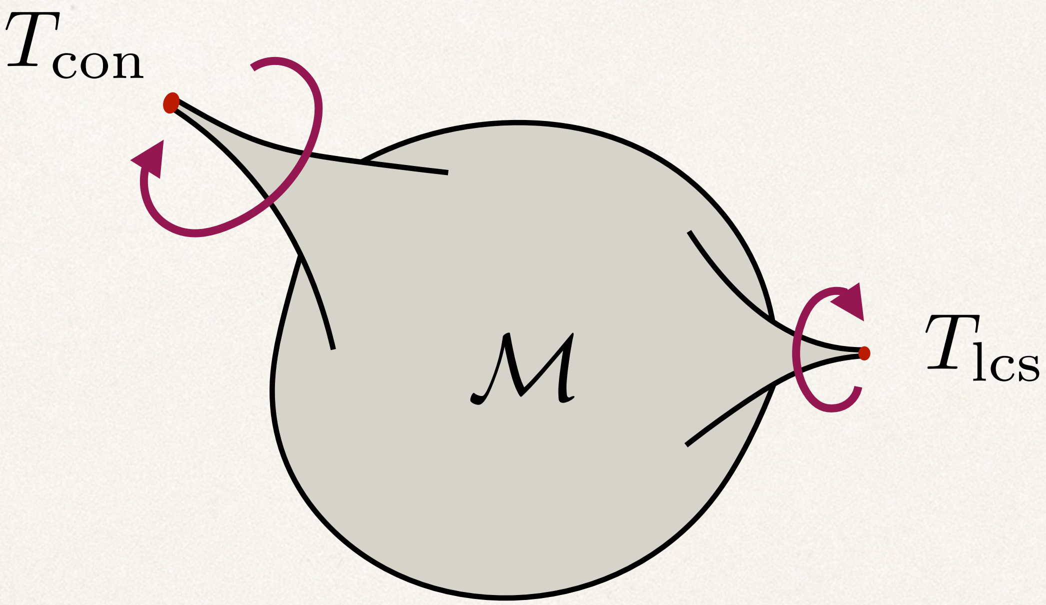

each boundary: $\boxed{\text{nilpotent} \quad N = \log T^{\text{u}}}$ $\exists k : \;\; N^k = 0$

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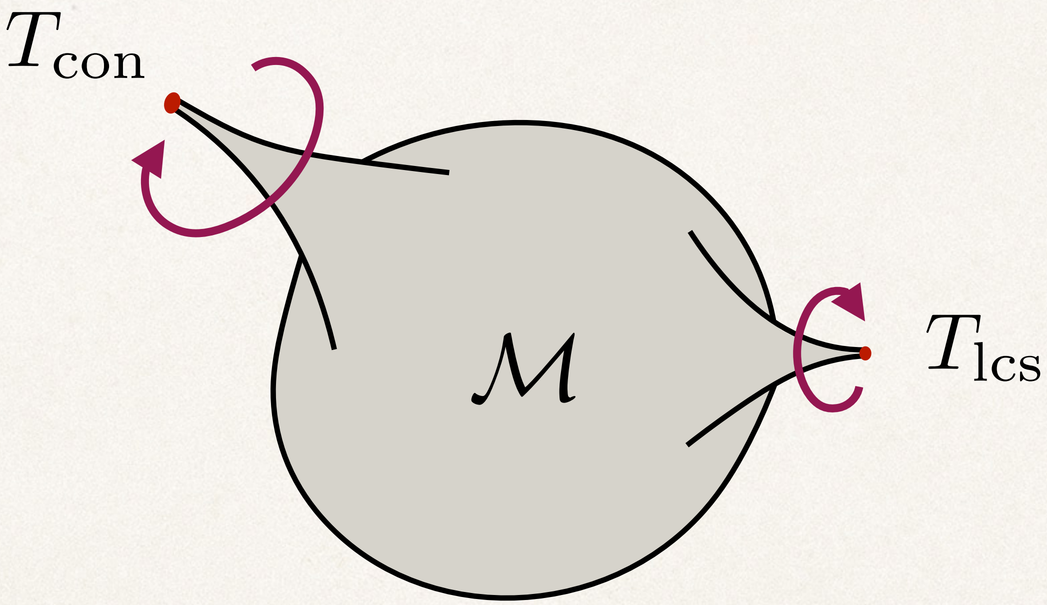

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higher-dimensional situation: N_i boundaries at normal intersection

Asymptotic behavior of (p,q)-decomposition

- naturally combine: $$F^p = \bigoplus_{r \geq p} H^{r,D-r}$$ holomorphic over $\mathcal{M}$ [Griffiths]

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- Nilpotent orbit theorem: [Schmid]
  limiting behavior of F^p near boundary $t^i \equiv x^i + iy^i \to x^i_0 + i\infty$

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Polynomial in t^i

nilpotent orbit

small near boundary $\Longrightarrow$ neglect

Comments on examples

- Large complex structure Calabi-Yau threefold (near MUM point): [TG,Li,Palti]

$$N_A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\delta_{AI} & 0 & 0 & 0 \\ -\frac{1}{2}\mathcal{K}_{AAI} & -\mathcal{K}_{AIJ} & 0 & 0 \\ \frac{1}{6}\mathcal{K}_{AAA} & \frac{1}{2}\mathcal{K}_{AJJ} & -\delta_{AJ} & 0 \end{pmatrix} \qquad F_0^3 = \begin{pmatrix} 1 \\ 0 \\ -c_{2\,I} \\ \frac{i\zeta(3)\chi}{8\pi^3} \end{pmatrix}$$

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⇒ Mirror side: Classification of Calabi-Yau threefolds into equivalence classes using infinite distance limits in Kähler cone [TG,Heisteeg,Ruehle '19]

Type classification and distances

- Calabi-Yau threefolds: $4\,h^{2,1}$ types of data [Kerr,Pearlstein,Robles]

Types: $\mathrm{I}_a,\ \mathrm{II}_b,\ \mathrm{III}_c, \mathrm{IV}_d$

Type	rank of N	rank of N^2	rank of N^3	eigenvalues of ηN
I_a	a	0	0	a negative
II_b	$2+b$	0	0	2 positive, b negative
III_c	$4+c$	2	0	not needed
IV_d	$2+d$	2	1	not needed

wedge product

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- Calabi-Yau fourfolds: $8\, h^{3,1}$ types of data [TG,Li,Valenzuela]

Types: $\mathrm{I}_{a,a'},\ \mathrm{II}_{b,b'},\ \mathrm{III}_{c,c'},\ \mathrm{IV}_{d,d'},\ \mathrm{V}_{e,e'}$

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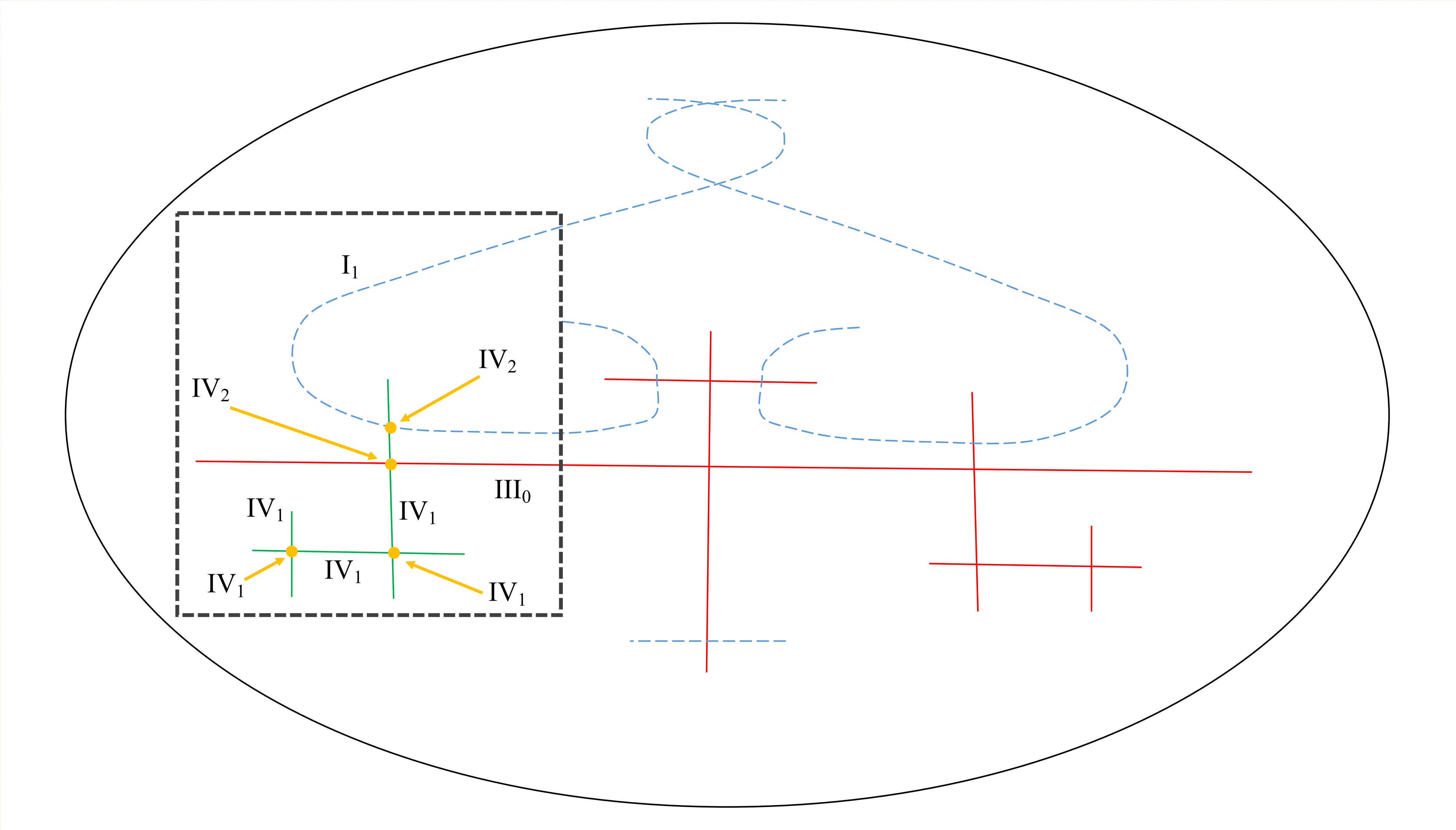
- Use nilpotent orbit theorem to compute asymptotic K

Question 1: Infinite distance boundaries $\mathrm{II}_b,\ \mathrm{III}_c, \mathrm{IV}_d$

An example moduli space

- An explicit example: $\mathbb{P}^{1,1,1,6,9}[18]$ [Candelas,Font,Katz,Morrison] [Candelas,De La Ossa,Font,Katz,Morrison]



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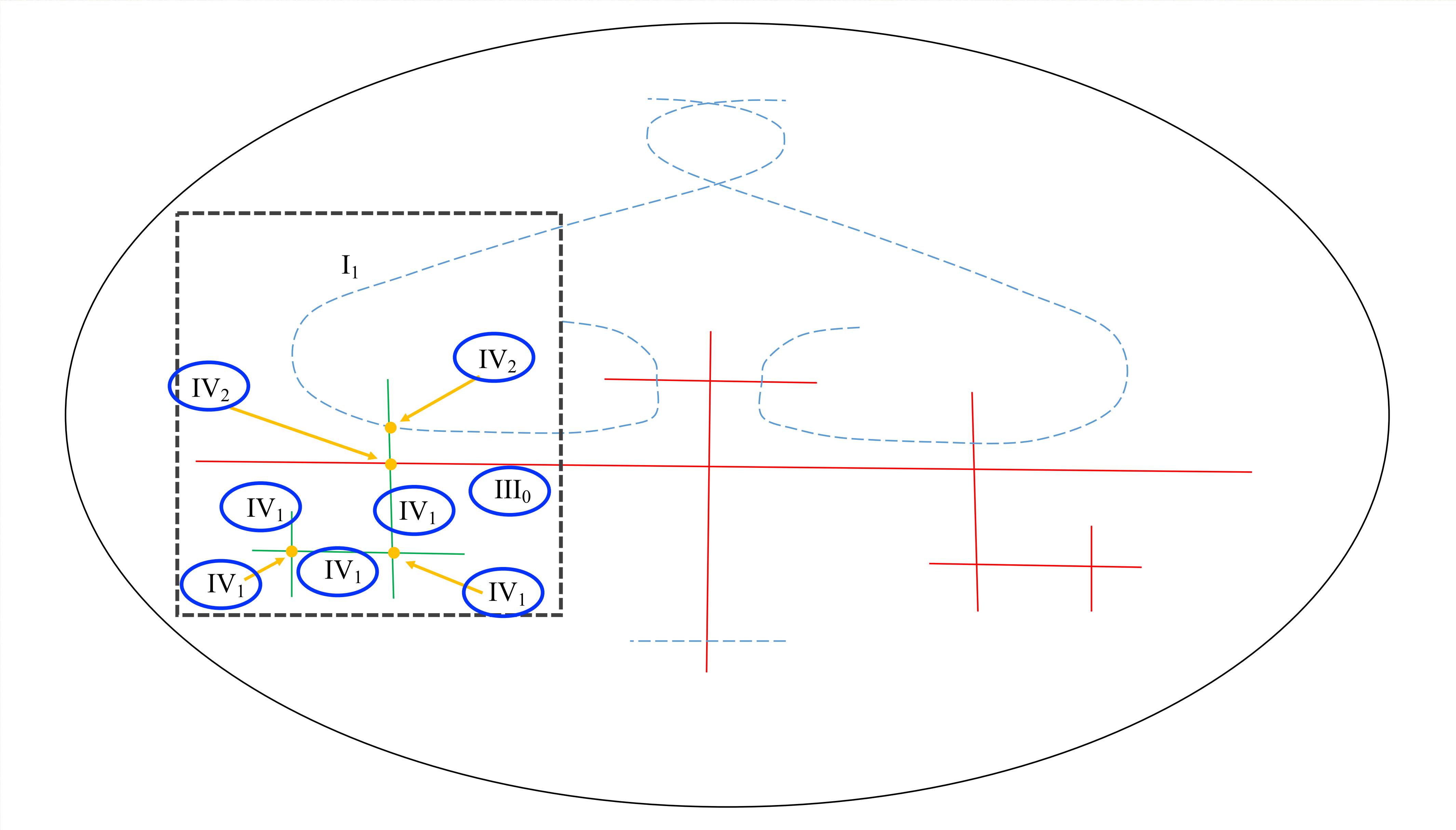

Asymptotic masses of states

- upper bound on asymptotic masses:

$$m(z, H) \leq \int_X H \wedge *H \equiv \|H\|^2$$

⇒ growth theorems for the Hodge norm by [Schmid][Kashiwara]

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$\rightarrow$ W_n, F^0_m can be used to define mixed Hodge structures

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- growth on sectors: $\{y_1 \geq y_2 \geq ... \geq y_n > 1,\ 1 > x_i \geq 0\}$

$$\|H\|^2 \sim (y^1)^{l_1 - 3} (y^2)^{l_2 - l_1} ... (y^n)^{l_{n-1} - l_n}$$

$H \in W_{l_1}(N_1) \cap W_{l_2}(N_{(2)}) \cap ... \cap W_{l_n}(N_{(n)})$ $\qquad N_{(i)} = N_1 + ... + N_i$

(smallest $\{l_i\}$ for which this is true)

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Asymptotic masses of states

- upper bound on asymptotic masses:

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Question 2: All infinite distance boundaries have $H \in V_{\text{light}}$
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- Construct near-boundary periods $\Pi(z)$ starting from this boundary data

Recovering asymptotic periods

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Final step: constructing the periods from F^p_{nil}

→ Many applications in swampland program and model building: general tests to asymptotic conjectures, new models away from large complex structure (MUM point)

A Finiteness Conjecture and Theorem

Solutions with background fields

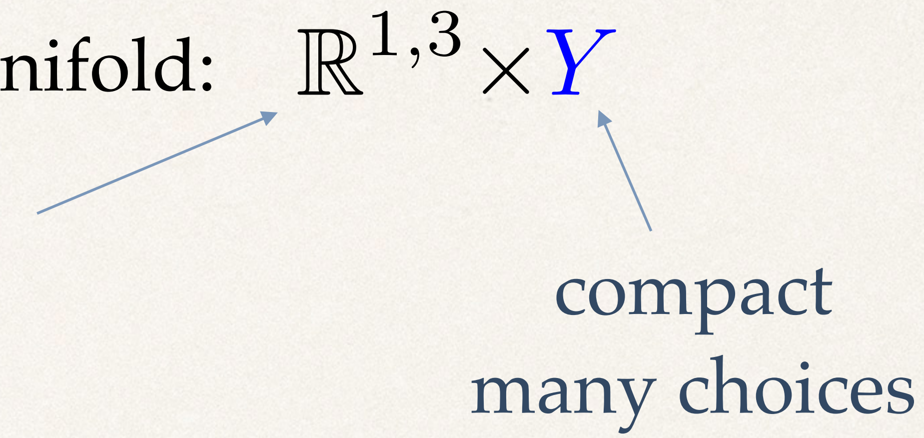
Recall: higher-dimensional space-time manifold: $\mathbb{R}^{1,3} \times Y$

our 4-dimensional space-time

compact
many choices

$\rightarrow$ Four-dimensional physics depends on choice of Y

Solutions with background fields



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Problem: deformations of Y can correspond to massless fields
→ fifths force → immediate contradiction with experiment

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…[Becker,Becker ’96],[Gukov,Vafa,Witten ’99],[Giddings,Kachru, Polchiski ’03],[TG,Louis ’04]…

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rough idea: - Y is Calabi-Yau fourfold

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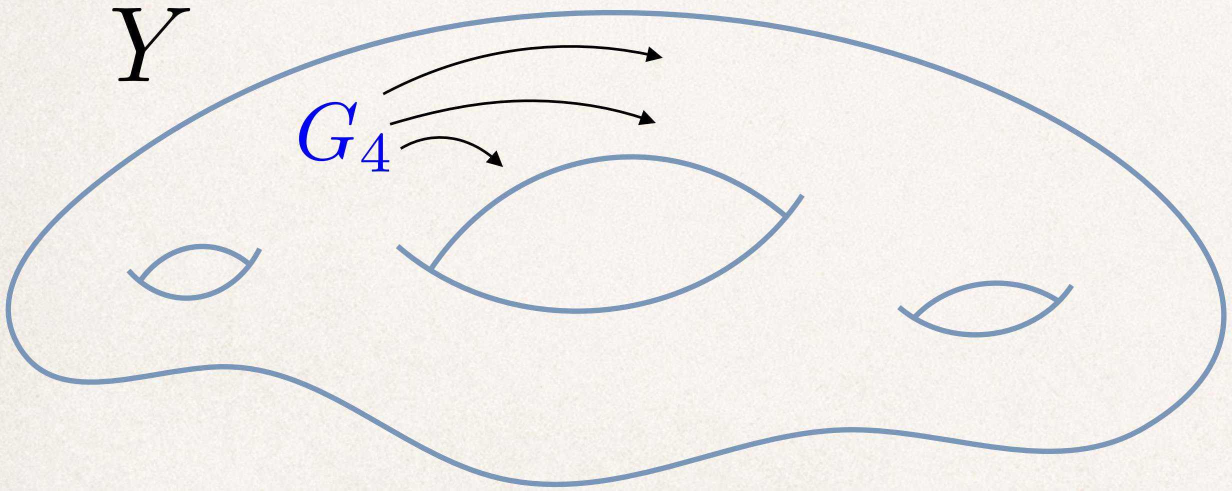

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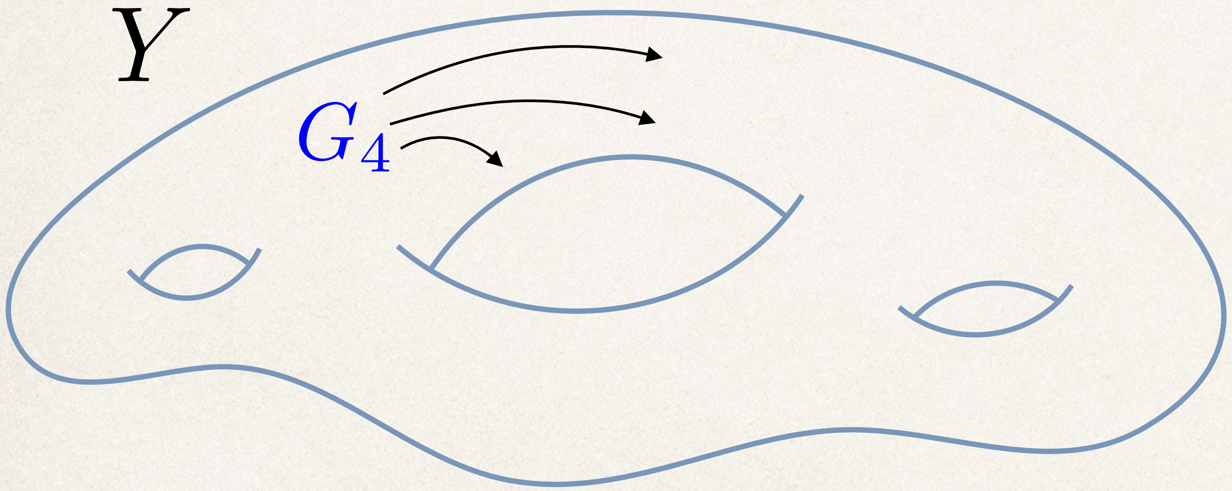

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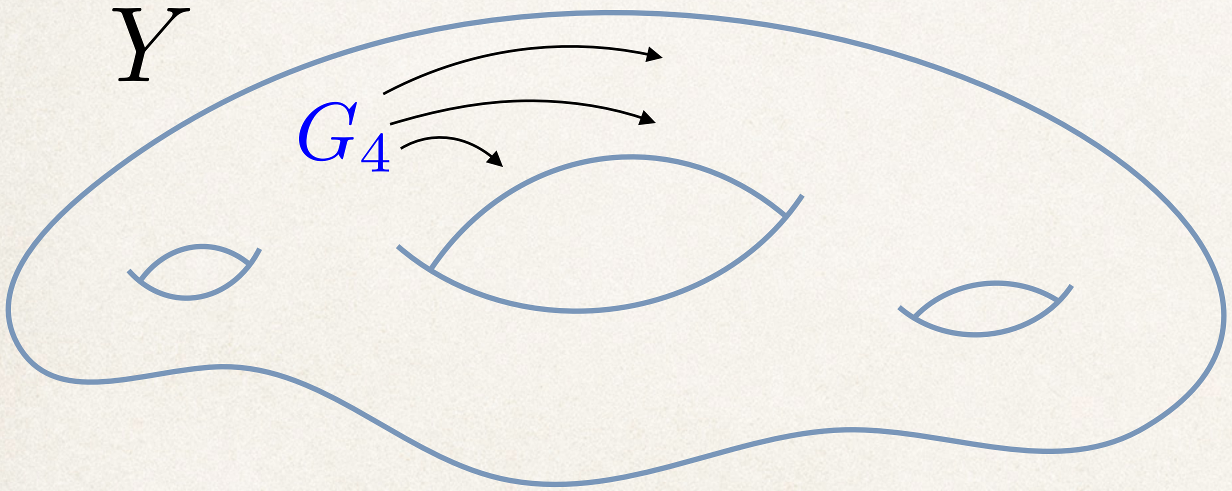

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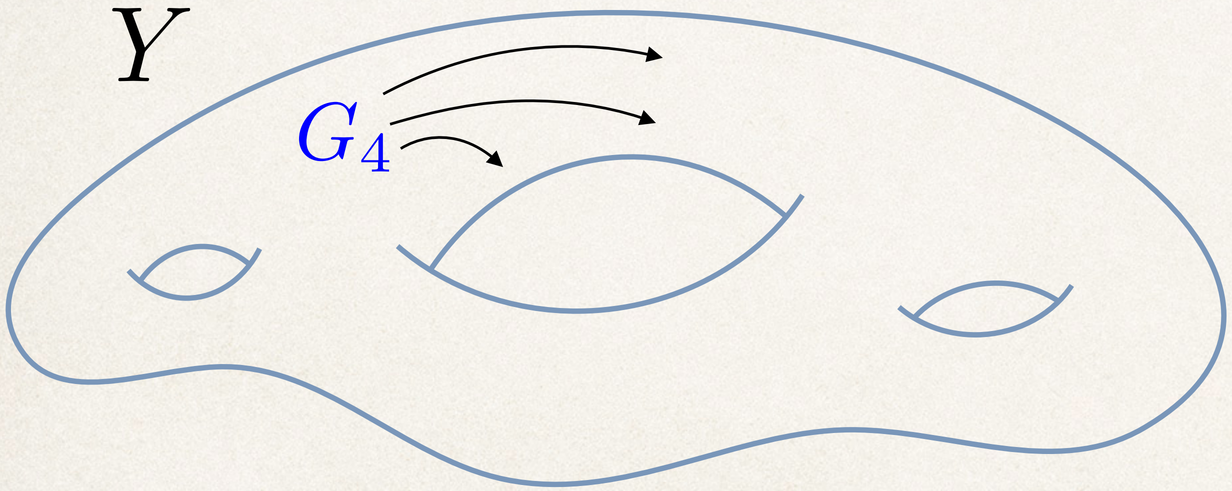

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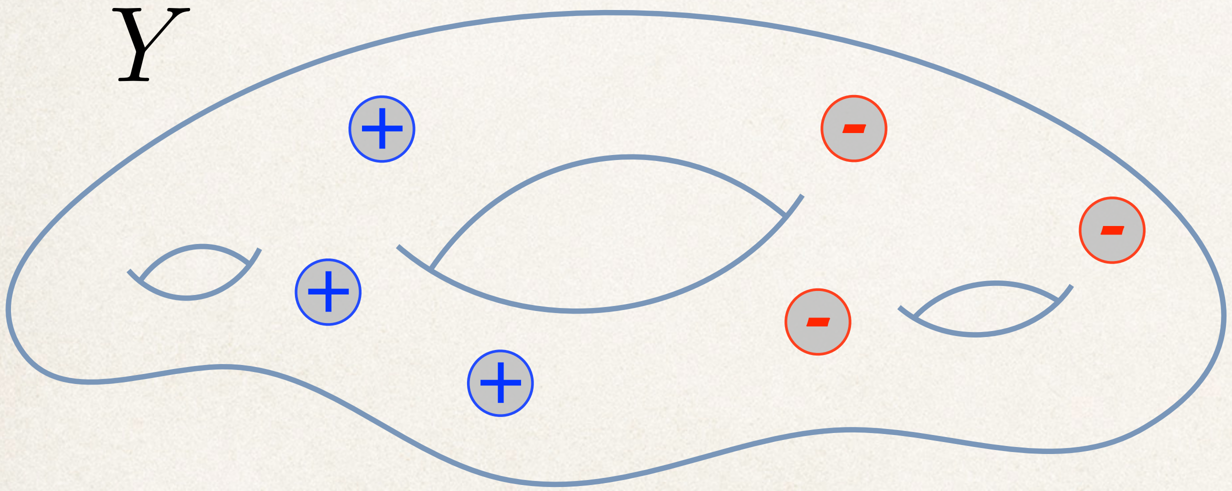

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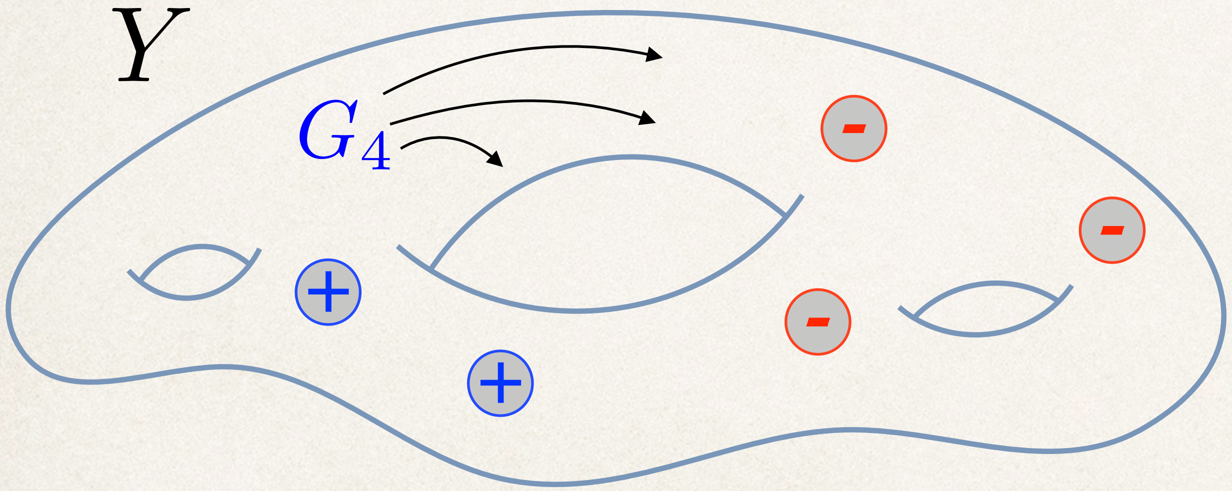

Assume Y is compact:

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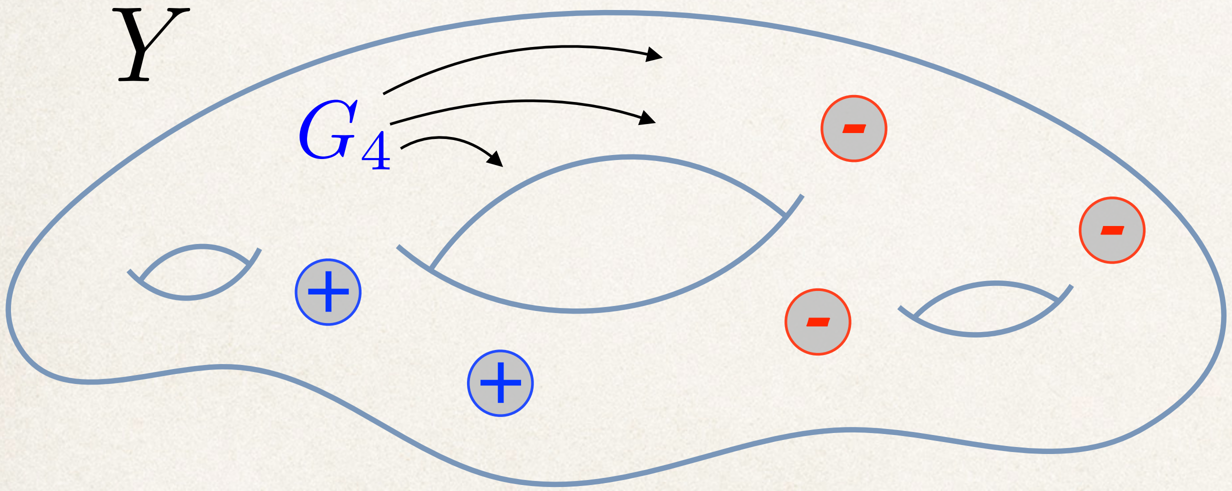

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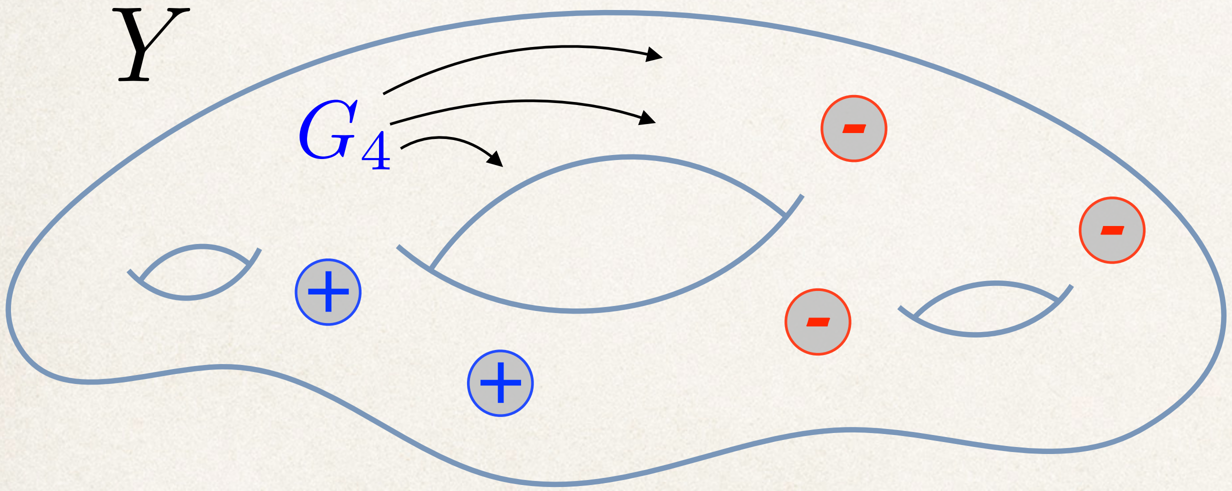

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- Solution to 12-dimensional theory (F-theory) of the form:

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$\Rightarrow$ should be read as a condition on the choice of complex structure and Kähler structure $\Rightarrow$ fix deformations

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- Concrete conjecture: The number of solutions in the described setting finite. Finitely many choices for G_4.

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Recently: Finiteness is part of a stronger property 'Tameness'

Tameness conjectures of set of UV completable quantum field theories and all their physical observables.

[TG '21][Douglas,TG,Schlechter] I & II

Connection with Hodge theory

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- Study how $H^{p,q}_x$ changes over the moduli space $\mathcal{M}$

  ⇒ variations of Hodge structures

Why is finiteness non-trivial?

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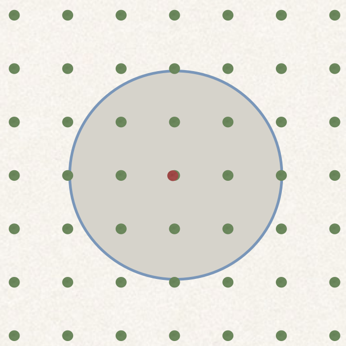

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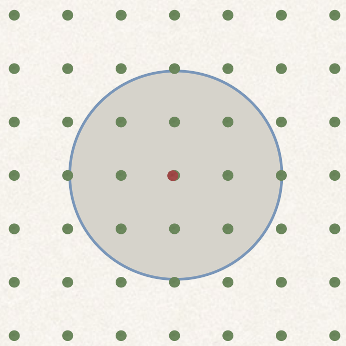

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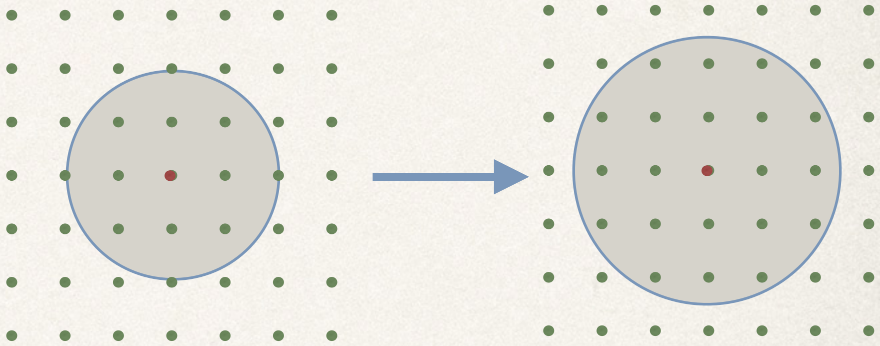

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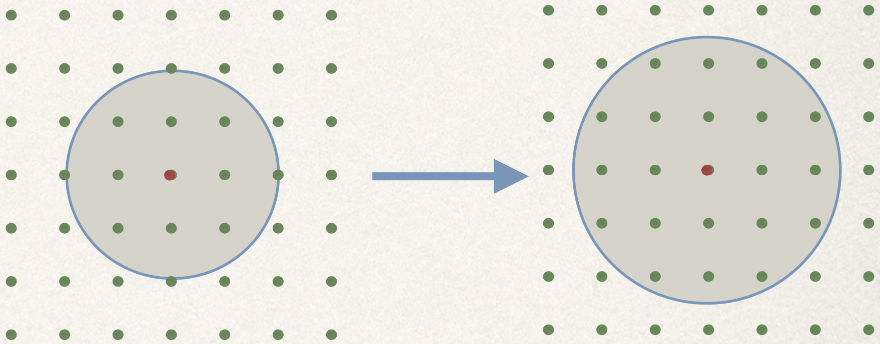

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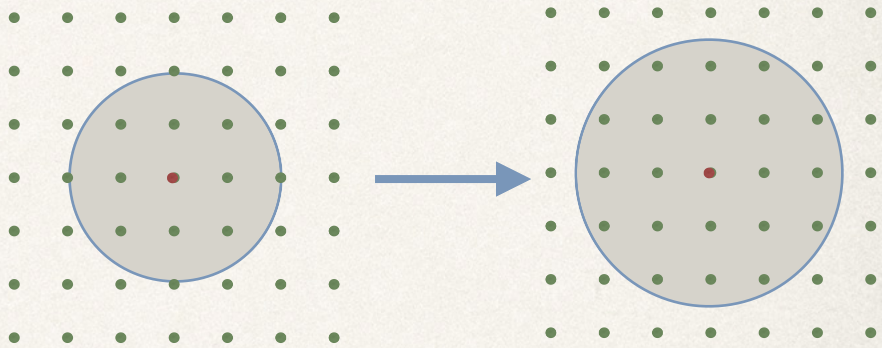

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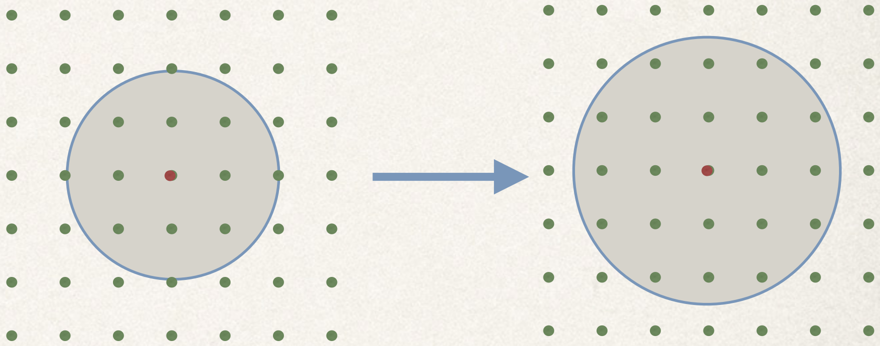

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$\Rightarrow$ works well for one-parameter limits [Schnell] [TG] '20

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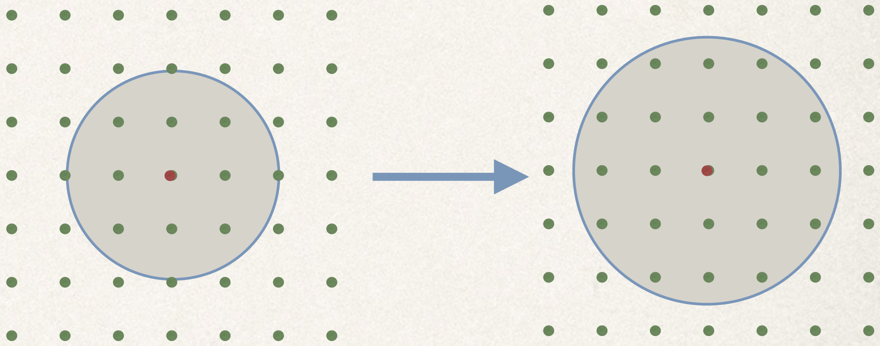

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$\to$ Weil operator (Hodge star) can degenerate on boundaries of $\mathcal{M}$
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- Idea: use asymptotic Hodge theory: nilpotent orbit theorem [Schmid] , sl(2)-orbit theorem [Schmid][Cattani,Kaplan,Schmid]

$\Rightarrow$ using multi-variable Sl(2)-orbit theorem too involved

Theorems in Abstract Variations of Hodge Structures

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Theorem [Cattani,Deligne,Kaplan ’95]: For integer $\ell > 0$, locus of integral Hodge classes

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- remarkable theorem which follows partly from the Hodge conjecture for Hodge structures associated to families of projective Kähler manifolds Y

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- covers the finiteness of the special case: $G_4 \;\in\; H^4(Y_4, \mathbb{Z}) \cap H^{2,2}$

(supersymmetric fluxes)

Generalization to self-dual classes

- recall Weil operator C (e.g. Hodge star): $Cv = i^{p-q}v \qquad v \in H^{p,q}$

Theorem [Bakker,TG,Schnell,Tsimerman]: For integer $\ell > 0$, the locus of integral self-dual classes

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is $\mathbb{R}_{\text{an,exp}}$- definable, closed real-analytic subspace of E and the restriction of p to this set is proper with finite fibers.

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locus is definable in the o-minimal structure $\mathbb{R}_{\mathrm{an,exp}}$

→ finitely many connected components

general fact
of tame geometry

Some remarks on the proof

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- Basic idea: specify space of allowed (definable) sets $S_n \subset \mathbb{R}^n$
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- Crucial criterium: definable sets in $\mathbb{R}$ are finitely many points and intervals + every higher-dimensional set linearly projects to such sets on $\mathbb{R}$

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  - use orbit theorems of Hodge theory to show that period map is definable in $\mathbb{R}_{\text{an,exp}}$
  - new proof of the theorem of [Cattani,Deligne,Kaplan] for Hodge classes
    <u>but:</u> uses holomorphicity - self-dual world is real!

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- Step 4: Prove finiteness in a single orbit: $\Gamma a,\ a \in H_{\mathbb{Z}}$

  Proof: some computations and definablity of $\Gamma_a\backslash G_a/K_a \to \Gamma\backslash G/K$ [BKT]

  groups fixing a

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Solutions have near boundary expansion: $y \gg 1$

$$h(x,y) = e^{xN}\left(1 + \frac{g_1}{y} + \frac{g_2}{y^2} + \dots\right)y^{-\frac{1}{2}N^0}$$

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- Reconstruct the near boundary solution near boundaries $\mathrm{I}_1,\ \mathrm{II}_0,\ \mathrm{IV}_1$ from simple sl(2)-data and compatible δ

An example

[TG,vd Heisteeg,Monnee]

- Simple boundary data for I_1 boundary

product $\langle a, b \rangle$ $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$

charge operator: $Q_\infty = \begin{pmatrix} 0 & 0 & 0 & -\frac{3i}{2} \\ 0 & 0 & -\frac{i}{2} & 0 \\ 0 & \frac{i}{2} & 0 & 0 \\ \frac{3i}{2} & 0 & 0 & 0 \end{pmatrix}$

$$\mathfrak{sl}(2,\mathbb{R}) : \qquad N^+ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad N^0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad N^- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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phase operator: $$\delta = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -c & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ $\Rightarrow$ most general form

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- write in geometric terms (period vector):

$$z = e^{2\pi i t} \qquad \Pi^T(z) = \Big(1, z, \frac{1}{2\pi i} z\log(z) - i(c+2\pi)z, i - \frac{i}{4\pi}z^2\Big)$$

$\Rightarrow$ conifold period

Approach 2: constructing two-cube periods

- General form of the periods: $\mathbf{\Pi}(t) = e^{i\delta} e^{-\zeta} e^{t^i N_i} e^{\Gamma(z)}\, \tilde{\mathbf{a}}_0$

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on the boundary
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- Step 1: construct most general N_i from $(N_i^{\pm}, N_i^0)$ [Brosnan,Pearlstein,Robles]

$$N_i = N_i^- + \sum_{l \leq -2} N_{i,l} \longleftarrow \text{weight under } N_i^0 - N_{i-1}^0$$

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- General form of the periods: $\mathbf{\Pi}(t) = e^{i\delta} e^{-\zeta} e^{t^i N_i} e^{\Gamma(z)} \tilde{\mathbf{a}}_0$

- Step 2: construct most general δ compatible with $(N_i^{\pm}, N_i^0)$

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- Step 3: get ζ from δ :

$$\zeta_{-1,-1} = \zeta_{-2,-2} = 0\,, \qquad \zeta_{-1,-2} = -\frac{i}{2}\delta_{-1,-2}\,, \qquad \zeta_{-1,-3} = -\frac{3i}{4}\delta_{-1,-3}\,,$$

$$\zeta_{-2,-3} = -\frac{3i}{8}\delta_{-2,-3} - \frac{1}{8}\left[\delta_{-1,-1},\, \delta_{-1,-2}\right], \qquad \zeta_{-3,-3} = -\frac{1}{8}\left[\delta_{-1,-1},\, \delta_{-2,-2}\right]$$

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- General form of the periods: $\mathbf{\Pi}(t) = e^{i\delta} e^{-\zeta} e^{t^i N_i} \boxed{e^{\Gamma(z)}}\, \tilde{\mathbf{a}}_0$

- Step 4: derive most general $\Gamma(z)$ using horizontality

  $\Longrightarrow$ solve differential conditions on Γ [Cattani,Fernandez]

Approach 2: constructing two-cube periods

- Results for two-cubes: I_2 class : $\langle \mathrm{I}_1|\mathrm{I}_2|\mathrm{I}_1\rangle\,,\ \langle \mathrm{I}_2|\mathrm{I}_2|\mathrm{I}_1\rangle\,,\ \langle \mathrm{I}_2|\mathrm{I}_2|\mathrm{I}_2\rangle$

$$\Pi = \begin{pmatrix} 1 - \frac{a^2}{8\pi k_2} z_1^{2k_1} z_2^{2k_2} - \frac{b^2}{8\pi m_1} z_1^{2m_1} z_2^{2m_2} \\ a z_1^{k_1} z_2^{k_2} \\ b z_1^{m_1} z_2^{m_2} \\ i + \frac{ia^2}{8\pi k_2} z_1^{2k_1} z_2^{2k_2} + \frac{ib^2}{8\pi m_1} z_1^{2m_1} z_2^{2m_2} \\ -\frac{a}{2\pi i} z_1^{k_1} z_2^{k_2} \big(n_1 \log[z_1] + \log(z_2) - 1/k_1\big) + ib\delta_1 z_1^{m_1} z_2^{m_2} \\ -\frac{b}{2\pi i} z_1^{m_1} z_2^{m_2} \big(\log(z_1) + n_2 \log[z_2] - 1/m_2\big) + ia\delta_1 z_1^{k_1} z_2^{k_2} \end{pmatrix}$$

parameters	$\langle \mathrm{I}_1\vert\mathrm{I}_2\vert\mathrm{I}_1\rangle$	$\langle \mathrm{I}_2\vert\mathrm{I}_2\vert\mathrm{I}_1\rangle$	$\langle \mathrm{I}_2\vert\mathrm{I}_2\vert\mathrm{I}_2\rangle$
log-monodromies n_1, n_2	$n_1 = n_2 = 0$	$n_1 \in \mathbb{Q}_{>0},\ n_2 = 0$	$n_1, n_2 \in \mathbb{Q}_{>0},\ n_1 n_2 \neq 1$
instanton orders k_1, k_2	$k_1 = 0, k_2 = 1$	$k_1 = n_1 k_2$	$k_1 = n_1 k_2$
instanton orders m_1, m_2	$m_1 = 1, m_2 = 0$	$m_1 = 1, m_2 = 0$	$m_2 = n_2 m_1$
instanton coefficients a, b	$a, b \in \mathbb{R} - \{0\}$		
phase operator δ	$\delta_1 \in \mathbb{R}$		

Approach 2: constructing two-cube periods

- Results for two-cubes:

Coni-LCS class : $\langle \mathrm{I}_1|\mathrm{IV}_2|\mathrm{IV}_1\rangle\,,\ \langle \mathrm{I}_1|\mathrm{IV}_2|\mathrm{IV}_2\rangle$

$$\Pi = \begin{pmatrix} 1 \\ a z_1 \\ \frac{\log[z_2]}{2\pi i} \\ -\frac{i\log[z_2]^3}{48\pi^3} - \frac{i a^2 n z_1^2 \log[z_2]}{4\pi} + \frac{a^2}{4\pi i} z_1^2 + i\delta_2 + i\delta_1 a z_1 \\ -a z_1 \frac{\log[z_1]+n\log[z_2]}{2\pi i} + i\delta_1 \\ -\frac{\log[z_2]^2}{8\pi^2} - \frac{1}{2} a^2 n z_1^2 \end{pmatrix}$$

parameters	$\langle \mathrm{I}_1\|\mathrm{IV}_2\|\mathrm{IV}_1\rangle$	$\langle \mathrm{I}_1\|\mathrm{IV}_2\|\mathrm{IV}_2\rangle$
log-monodromies n_1, n_2	$n = 0$	$n \in \mathbb{Q}_{>0}$
instanton coefficient a	$a \in \mathbb{R} - \{0\}$	
phase operator δ	$\delta_1, \delta_2 \in \mathbb{R}$	