Modularity of BPS indices on Calabi-Yau threefolds

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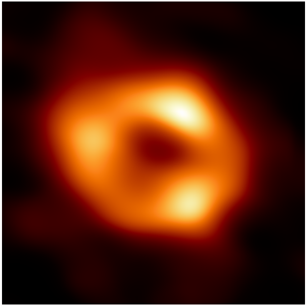
References

- "*"Indefinite theta series and generalized error functions"*, with S. Alexandrov, S. Banerjee, J. Manschot, Selecta Math. 24 (2018) 3927 [arXiv:1606.05495]
- *"Black holes and higher depth mock modular forms"*, with S. Alexandrov, Commun.Math.Phys. 374 (2019) 549 [arXiv:1808.08479]
- *"S-duality and refined BPS indices"*, with S. Alexandrov and J. Manschot, Commun.Math.Phys. 380 (2020) 755 [arXiv:1910.03098]
- *"Modular bootstrap for D4-D2-D0 indices on compact Calabi-Yau threefolds"*, with S. Alexandrov, N. Gaddam, J. Manschot [arXiv:2204.02207]
- S. Alexandrov, S. Feyzbakhsh, A. Klemm, BP, T. Schimannek, to appear.

Introduction

- A driving force in high energy theoretical physics has been the quest for a microscopic explanation of the entropy of black holes. Providing a derivation of the Bekenstein-Hawking formula is a benchmark test of any theory of quantum gravity.

$$S_{BH} = \frac{1}{4G_N} A$$

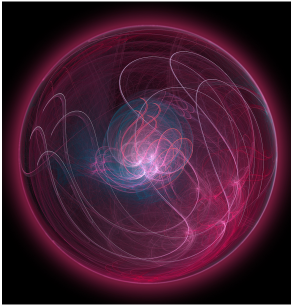


$$S_{BH} \stackrel{?}{=} \log \Omega$$

Sgr A, Event Horizon Telescope 2022*

Black hole microstates as wrapped D-branes

- Back in 1996, Strominger and Vafa argued that String Theory passes this test with flying colors ♫, at least in the context of BPS black holes in vacua with extended SUSY: micro-states can be understood as bound states of D-branes wrapped on calibrated cycles of the internal manifold, and counted efficiently.



Calabi-Yau black hole, courtesy F. Le Guen

BPS indices and Donaldson-Thomas invariants

- More precisely, in the context of type IIA strings compactified on a Calabi-Yau three-fold $\mathfrak{Y}$, BPS states are described by stable objects in the derived category of coherent sheaves $\mathcal{C} = D^b\mathrm{Coh}\mathfrak{Y}$. The Chern character $\gamma = (\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2, \mathrm{ch}_3)$ is identified as the electromagnetic charge, or D6-D4-D2-D0-brane charge.
- The problem becomes a question in enumerative geometry: for fixed $\gamma \in K(\mathfrak{Y})$, compute the Donaldson-Thomas invariant $\Omega_z(\gamma)$ counting (semi)stable objects of class γ in $\mathcal{C}$ with respect to $z \in \mathrm{Stab}\,\mathcal{C}$, and determine its growth as $|\gamma| \to \infty$.
- Physical arguments predict that suitable generating series of rank 0 DT invariants (counting D4-D2-D0 bound states) should have definite modular properties. This gives very good control on their asymptotic growth, and confirms that $\Omega_z(\gamma) \simeq e^{S_{BH}(\gamma)}$ as $|\gamma| \to \infty$.

Simplest example: Abelian three-fold

- For $\mathfrak{Y} = T^6$, $\Omega_z(\gamma)$ depends only on a certain quartic polynomial $m = I_4(\gamma)$ in the charges, and is moduli independent. It is given by the Fourier coefficient $c(m)$ of a weak modular form,

$$\frac{\theta_3(2\tau)}{\eta^6(4\tau)} = \sum_{m\geq -1} c(m)\, \mathrm{q}^m = \frac{1}{\mathrm{q}} + 2 + 8\mathrm{q}^3 + 12\mathrm{q}^4 + 39\mathrm{q}^7 + 56\mathrm{q}^8 + \dots$$

Moore Maldacena Strominger 1999, BP 2005, Shih Strominger Yin 2005

Bryan Oberdieck Pandharipande Yin'15

- Recall that $f(\tau) := \sum_{n\geq 0} c(n) \mathrm{q}^{n-\Delta}$ (with $\mathrm{q} = e^{2\pi \mathrm{i}\tau}$, $\mathrm{Im}\tau > 0$) is a modular form of weight k if $\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$,

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k\, f(\tau) \quad \Rightarrow \quad c(n) \sim \exp\left(4\pi\sqrt{\Delta n}\right)$$

Wall-crossing and mock modularity

- For a general CY3, the story is more involved and interesting. First, $\Omega_z(\gamma)$ depends on the Kähler parameters z (more generally, on the stability condition), with a complicated chamber structure.
- Second, the generating series of rank 0 invariants $\Omega_\star(\gamma)$ in the large volume attractor chamber are generally not modular but rather mock modular of higher depth.
- A (depth one) mock modular form of weight w transforms inhomogeneously under $SL(2,\mathbb{Z})$,

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k \left[f(\tau) - \int_{-d/c}^{\mathrm{i}\infty} \overline{g(-\bar\rho)} (\tau+\rho)^{-w} \mathrm{d}\rho \right]$$

where $g(\tau)$ is an ordinary modular form of weight $2-w$, known as the shadow.

Wall-crossing and mock modularity

- Equivalently, the non-holomorphic completion

$$\widehat{f}(\tau,\bar\tau) := f(\tau) + \int_{-\bar\tau}^{i\infty} \overline{g(-\bar\rho)}(\tau+\rho)^{-w}\mathrm{d}\rho$$

  transforms like a modular form of weight w, and satisfies $\tau_2^w \partial_{\bar\tau}\widehat{f}(\tau,\bar\tau) \propto \overline{g(\tau)}$

  Ramanujan'1920, Hirzebruch-Zagier'1973, Zwegers'02

- The Fourier coefficients still grow as $c(n) \sim \exp\left(4\pi\sqrt{\Delta n}\right)$ but subleading corrections are markedly different.

Outline

- Explain the formalism of (weak) stability conditions on $\mathcal{C} = D^b\mathrm{Coh}\mathfrak{Y}$
- Spell out the modularity properties of rank 0 DT invariants for general CY threefold
- Check modularity for non-compact $\mathfrak{Y} = K_S$ with S a Fano surface, where rank 0 DT invariants reduce to Vafa-Witten invariants.
- Test modularity for compact CY threefolds with $b_2(\mathfrak{Y}) = 1$, using recent results of S. Feyzbakhsh and R. Thomas
- Obtain new constraints on higher genus GW invariants from modularity

Mathematical preliminaries

- Let $\mathfrak{Y}$ a compact CY threefold, and $\mathcal{C} = D^b\mathrm{Coh}\mathfrak{Y}$ the bounded derived category of coherent sheaves. Objects $E \in \mathcal{C}$ are bounded complexes

$$E = (\cdots \overset{d_{-2}}{\to} \mathcal{E}^{-1} \overset{d_{-1}}{\to} \mathcal{E}^{0} \overset{d_{0}}{\to} \mathcal{E}^{1} \overset{d_{1}}{\to} \dots),$$

  of coherent sheaves $\mathcal{E}^k$ on $\mathfrak{Y}$, with morphisms $d_k : \mathcal{E}^k \to \mathcal{E}^{k+1}$ such that $d_{k+1} d_k = 0$. Physically, $\mathcal{E}^k$ describe D6-branes for k even, or anti D6-branes for k odd, and d_k are open strings .
- $\mathcal{C}$ is graded by the (numerical) Grothendieck group $K(\mathcal{C})$. Let $\Gamma \subset H^{\mathrm{even}}(\mathfrak{Y}, \mathbb{Q})$ be the image of $K(\mathcal{C})$ under the Chern character $E \mapsto \mathrm{ch}\, E = \sum_k (-1)^k \,\mathrm{ch}\, \mathcal{E}_k$. Γ is the lattice of electromagnetic charges, equipped with the Dirac-Schwinger symplectic pairing

$$\langle E, E' \rangle = \chi(E', E) = \int_{\mathfrak{Y}} (\mathrm{ch}\, E')^\vee \,\mathrm{ch}(E)\, \mathrm{Td}(T\mathfrak{Y}) \in \mathbb{Z}$$

Bridgeland stability conditions

- Let $\mathcal{S} = \mathrm{Stab}(\mathcal{C})$ be the space of Bridgeland stability conditions $\sigma = (Z, \mathcal{A})$, where
  1. $Z : \Gamma \to \mathbb{C}$ is a linear map, known as the central charge. Let $Z(E) := Z(\mathrm{ch}(E))$.
  2. $\mathcal{A} \subset \mathcal{C}$ is the heart of a bounded t-structure on $\mathcal{C}$;
  3. For any non-zero $E \in \mathcal{A}$, $(i)\, \mathrm{Im} Z(E) \geq 0$ and $(ii)\, \mathrm{Im} Z(E) = 0 \Rightarrow \mathrm{Re} Z(E) < 0$. Relax (ii) for weak stability conditions.
  4. Harder-Narasimhan filtration + support property
- If $\mathcal{S}$ is not empty, then it is a complex manifold of dimension $\mathrm{rk}\,\Gamma = b_{\mathrm{even}}(\mathfrak{Y})$, locally parametrized by $Z(\gamma_i)$ with γ_i a basis of Γ.
- Stability conditions are known to exist only for a handful of CY threefolds, including the quintic in $\mathbb{P}^4$. Their construction depends on the conjectural Bayer-Macrì-Toda (BMT) inequality. Weak stability conditions are much easier to construct.

Physical stability conditions

- Physics/Mirror symmetry conjecturally selects a subspace $\Pi \subset \mathcal{C}$, known as 'physical slice' or slice of Π-stability conditions, parametrized by complexified Kähler structure of $\mathfrak{Y}$, or complex structure of $\widehat{\mathfrak{Y}}$. Hence $\dim \Pi = b_2(\mathfrak{Y}) + 1 = b_3(\widehat{\mathfrak{Y}})$.
- Along this slice, the central charge is given by the period

$$Z(\gamma) = \int_{\hat{\gamma}} \Omega_{3,0}$$

  where $\Omega_{3,0}$ is the holomorphic 3-form on $\widehat{\mathfrak{Y}}$ and $\hat{\gamma} \in H_3(\hat{\mathfrak{Y}}, \mathbb{Z})$ is the 3-cycle dual to $\gamma \in \Gamma$.
- Near the large volume point in $\mathcal{M}_K(\mathfrak{Y})$, or MUM point in $\mathcal{M}_{cx}(\widehat{\mathfrak{Y}})$,

$$Z(E) = -\int_{\mathfrak{Y}} e^{-z^a H_a} \sqrt{Td(T\mathfrak{Y})}\, \mathrm{ch}(E)$$

  where H_i is a basis of $H^2(\mathfrak{Y}, \mathbb{Z})$, and $z^a = b^a + \mathrm{i} t^a$ are the complexified Kähler moduli.

Generalized Donaldson-Thomas invariants

- Given a (weak) stability condition $\sigma = (Z, \mathcal{A})$, an object $F \in \mathcal{A}$ is called σ-semi-stable if $\arg Z(F') \le \arg Z(F)$ for every non-zero subobject $F' \subset F$ (where $0 < \arg Z \le 1$).
- Let $\mathcal{M}_\sigma(\gamma)$ be the moduli stack of σ-semi-stable objects of class γ in $\mathcal{A}$. Following *[Joyce-Song'08]* one can associate the DT invariant $\bar\Omega_\sigma(\gamma) \in \mathbb{Q}$. When γ is primitive and $\mathcal{M}_\sigma(\gamma)$ is smooth and projective, then $\bar\Omega_\sigma(\gamma) = (-1)^{\dim \mathcal{M}_\sigma(\gamma)} \chi(\mathcal{M}_\sigma(\gamma))$.
- Conjecturally, the generalized DT invariant defined by

$$\Omega_\sigma(\gamma) = \sum_{m|\gamma} \frac{\mu(m)}{m^2} \bar\Omega_\sigma(\gamma/m)$$

is integer for any γ, and reduces to the BPS index along Π.

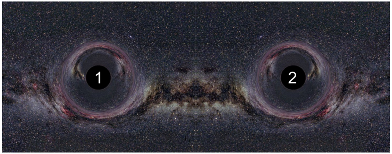
Wall-crossing

- The invariants $\bar{\Omega}_\sigma(\gamma)$ are locally constant on $\mathcal{S}$, but jump across walls of instability (or marginal stability), where the central charge $Z(\gamma)$ aligns with $Z(\gamma')$ where $\gamma' = \mathrm{ch}\, E'$ for a subobject $E' \subset E$. The jump is governed by a universal wall-crossing formula.

Joyce Song'08; Kontsevich Soibelman'08

- Physically, the jump corresponds to the (dis)appearance of multi-centered black hole bound states. In the simplest case,

$$\Delta\Omega(\gamma_1+\gamma_2) = (-1)^{\langle\gamma_1,\gamma_2\rangle+1} |\langle\gamma_1,\gamma_2\rangle|\, \Omega(\gamma_1)\, \Omega(\gamma_2)$$



S-duality constraints on DT invariants

- Constraints on DT invariants can be derived by studying instanton corrections to the moduli space in IIA/$\mathfrak{Y} \times S^1(R)$=M/$\mathfrak{Y} \times T^2(\tau)$.
[Alexandrov, Banerjee, Manschot, BP, Robles-Llana, Rocek, Saueressig, Theis, Vandoren '06-19]
- The moduli space $\mathcal{M}_3$ factorizes into $\mathcal{M}_H \times \widetilde{\mathcal{M}}_V$ where
  1. $\mathcal{M}_H$ parametrizes the complex structure of $\mathfrak{Y}$+ dilaton ϕ + Ramond gauge fields in $H^{\rm odd}(\mathfrak{Y})$
  2. $\widetilde{\mathcal{M}}_V$ parametrizes the Kähler structure of $\mathfrak{Y}$ + radius R + Ramond gauge fields in $H^{\rm odd}(\mathfrak{Y})$
- Both factors carry a quaternion-Käler metric. $\mathcal{M}_H$ is largely irrelevant for this talk, but note that it is exchanged with $\widetilde{\mathcal{M}}_V$ under mirror symmetry.

S-duality constraints on DT invariants

- At large R, $\widetilde{\mathcal{M}}_V$ is a flat torus bundle over $\mathbb{R}^+ \times \mathcal{M}_K$, but the QK metric receives $\mathcal{O}(e^{-R|Z(\gamma)|})$ corrections from Euclidean black holes winding around S^1.
- These corrections are determined from the DT invariants $\Omega_z(\gamma)$ by a twistorial construction à la Gaiotto-Moore-Neitzke *[Alexandrov BP Saueressig Vandoren'08]*
- Since type IIA/S^1 is the same as M-theory on T^2, $\widetilde{\mathcal{M}}_V$ must have an isometric action of $SL(2,\mathbb{Z})$. This strongly constrains the DT invariants in the large volume limit.

S-duality constraints on BPS indices

Requiring that $\widetilde{\mathcal{M}}_V$ admits an isometric action of $SL(2,\mathbb{Z})$ near large volume, one can show that DT invariants $\Omega_z(\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2, \mathrm{ch}_3)$ satisfy

- For n D0-branes, $\Omega_z(0,0,0,n) = -\chi_{\mathfrak{Y}}$ (independent of n)
- For D2-branes supported on a curve of class $q_a\gamma^a \in \Lambda^* = H_2(\mathfrak{Y},\mathbb{Z})$, $\Omega_z(0,0,q_a,n) = N^{(0)}_{q_a}$ is given by the genus-zero GV invariant (independent of n)
- For D4-branes supported on an irreducible divisor $\mathcal{D}$ of class $p^a\gamma_a \in \Lambda = H_4(\mathfrak{Y},\mathbb{Z})$, the generating series

$$h_{p^a,q_a}(\tau) := \sum_n \Omega_\star(0,p^a,q_a,n)\, \mathrm{q}^{n+\frac12 q_a\kappa^{ab}q_b+\frac12 q_a\kappa^{ab}q_b-\frac12 p^a q_a-\frac{\chi(\mathcal{D})}{24}}$$

should be a vector-valued weakly holomorphic modular form of weight $w = -\frac12 b_2(\mathfrak{Y}) - 1$ and prescribed multiplier system.

S-duality constraints on D4-D2-D0 indices

$$h_{p^a,q_a}(\tau) = \sum_n \bar\Omega_\star(0,p^a,q_a,n)\, \mathrm{q}^{n+\frac12 q_a\kappa^{ab}q_b+\frac12 p^a q_a-\frac{\chi(\mathcal{D})}{24}}$$

- Here, $\bar\Omega_\star(0,p^a,q_a,n)$ is the index in the large volume attractor chamber (aka MSW index)

$$\bar\Omega_\star(\gamma) = \lim_{\lambda\to+\infty} \bar\Omega_{\left(z^a=-\kappa^{ab}q_b+\mathrm{i}\lambda p^a\right)}(\gamma)$$

  where κ^{ab} is the inverse of the quadratic form $\kappa_{ab}=\kappa_{abc}p^c$ with Lorentzian signature $(1, b_2(\mathfrak{Y})-1)$.
- For CY threefolds with $\mathrm{Pic}\,\mathfrak{Y}=\mathbb{Z}H$, $\bar\Omega_\star(\gamma)$ coincides with the DT invariant $\bar\Omega_H(\gamma)$ counting H-Gieseker stable sheaves.
- The Bogolomov-Gieseker inequality guarantees that n is bounded from below, $n \geq -\frac12 q_a\kappa^{ab}q_b - \frac12 p^a q_a$.

S-duality constraints on D4-D2-D0 indices

- By construction, $\Omega_\star(0, p^a, q_a, n)$ is invariant under tensoring with a line bundle $\mathcal{O}(\epsilon^a H_a)$ (aka spectral flow)

$$q_a \to q_a - \kappa_{ab}\epsilon^b \;, \quad n \mapsto n - \epsilon^a q_a + \frac{1}{2}\kappa_{ab}\epsilon^a\epsilon^b$$

  Thus, the D2-brane charge q_a can be restricted to the finite set Λ^*/Λ, of cardinal $|\det(\kappa_{ab})|$.

- h_{p^a,q_a} transforms under the Weil representation for Λ, e.g.

$$h_{p^a,q_a}(-1/\tau) = \sum_{q'_a \in \Lambda^*/\Lambda} \frac{e^{-2\pi \mathrm{i} \kappa^{ab} q_a q'_b + \frac{\mathrm{i}\pi}{4}(b_2(\mathfrak{Y}) + 2\chi(\mathcal{O}(\mathcal{D})) - 2)}}{\sqrt{|\det(\kappa_{ab})|}} h_{p^a,q'_a}(\tau)$$

D4-D2-D0 indices from elliptic genus

- Summing over all D2-brane charges and using spectral flow invariance, one gets

$$
\begin{aligned}
Z_p(\tau, v) &:= \sum_{q\in\Lambda, n} \Omega_\star(0, p^a, q_a, n)\, \mathrm{q}^{n+\frac{1}{2} q_a \kappa^{ab} q_b} e^{2\pi \mathrm{i} q_a v^a} \\
&= \sum_{q\in\Lambda^*/\Lambda} h_{p,q}(\tau) \Theta_q(\tau, v)
\end{aligned}
$$

  where $\Theta_q(\tau, v)$ is the (non-holomorphic) Siegel theta series for the indefinite lattice (Λ, κ_{ab}). S-duality then requires that Z_p should transform as a (non-holomorphic) Jacobi form.
- The Jacobi form Z_p can be interpreted as the elliptic genus of the $(0, 4)$ superconformal field theory obtained by wrapping an M5-brane on the divisor $\mathcal{D}$ *[Maldacena Strominger Witten '98]*.

Mock modularity constraints on D4-D2-D0 indices

- For γ supported on a reducible divisor $\mathcal{D} = \sum_{i=1}^{n\geq 2} \mathcal{D}_i$, the generating series h_p (omitting q index for simplicity) is no longer expected to be modular. Rather, it should be a vector-valued mock modular form of depth $n-1$ and same weight/multiplier system.

Alexandrov Banerjee Manschot BP '16-19

- There exists explicit non-holomorphic theta series such that

$$\widehat{h}_p(\tau,\bar{\tau}) = h_p(\tau) + \sum_{p=\sum_{i=1}^{n\geq 2} p_i} \Theta_n(\{p_i\},\tau,\bar{\tau}) \prod_{i=1}^{n} h_{p_i}(\tau)$$

transforms as a modular form of weight $-\frac{1}{2}b_2(\mathfrak{Y}) - 1$. Moreover the completion satisfies an explicit holomorphic anomaly equation,

$$\partial_{\bar{\tau}} \widehat{h}_p(\tau,\bar{\tau}) = \sum_{p=\sum_{i=1}^{n\geq 2} p_i} \widehat{\Theta}_n(\{p_i\},\tau,\bar{\tau}) \prod_{i=1}^{n} \widehat{h}_{p_i}(\tau,\bar{\tau})$$

Crash course on Indefinite theta series

- Θ_n and $\widehat{\Theta}_n$ belongs to the class of indefinite theta series

$$\vartheta_{\Phi,q}(\tau,\bar\tau) = \tau_2^{-\lambda} \sum_{k\in\Lambda+q} \Phi\left(\sqrt{2\tau_2}k\right)\, e^{-\mathrm{i}\pi\tau Q(k)}$$

  where (Λ, Q) is an even lattice of signature $(r, d-r)$, $q \in \Lambda^*/\Lambda$, $\lambda \in \mathbb{R}$. The series converges if $f(x) \equiv \Phi(x) e^{\frac{\pi}{2}Q(x)} \in L_1(\Lambda \otimes \mathbb{R})$.
- Theorem (Vignéras, 1978): $\{\vartheta_{\Phi,q}, q \in \Lambda^*/\Lambda\}$ transforms as a vector-valued modular form of weight $(\lambda + \frac{d}{2}, 0)$ provided
  - $R(x)f, R(\partial_x)f \in L_2(\Lambda \otimes \mathbb{R})$ for any polynomial $R(x)$ of degree ≤ 2
  - $\left[\partial_x^2 + 2\pi(x\partial_x - \lambda)\right]\Phi = 0$ [*]
- The relevant lattice $\Lambda = H^2(\mathfrak{Y}, \mathbb{Z})^{\oplus n-1}$ has signature $(r, d-r) = (n-1)(1, b_2(\mathfrak{Y}) - 1)$.

Indefinite theta series

- Example 1 (Siegel): $\Phi = e^{\pi Q(x_+)}$, where x_+ is the projection of x on a fixed plane of dimension r, satisfies [*] with $\lambda = -n$. ϑ_Φ is then the usual (non-holomorphic) Siegel-Narain theta series.
- Example 2 (Zwegers): In signature $(1, d-1)$, choose C, C' two vectors such that $Q(C), Q(C'), (C, C') > 0$, then

$$\widehat{\Phi}(x) = \mathrm{Erf}\left(\frac{(C,x)\sqrt{\pi}}{\sqrt{Q(C)}}\right) - \mathrm{Erf}\left(\frac{(C',x)\sqrt{\pi}}{\sqrt{Q(C')}}\right)$$

satisfies [*] with $\lambda = 0$. As $|x| \to \infty$, or if $Q(C) = Q(C') = 0$,

$$\widehat{\Phi}(x) \to \Phi(x) := \mathrm{sgn}(C, x) - \mathrm{sgn}(C', x)$$

- The theta series $\Theta_2(\{p_1, p_2\})$, $\widehat{\Theta}_2(\{p_1, p_2\})$ fall in this class. The generalization to $n > 2$ involves generalized error functions.

Alexandrov Banerjee Manschot BP 2016; Nazaroglu 2016

Indefinite theta series

- For $r > 1$, one can construct solutions of $[*]$ which asymptote to $\prod_i \mathrm{sgn}(C_i, x)$ as $|x| \to \infty$: the generalized error functions

$$E_r(C_1, \dots C_r; x) = \int_{\langle C_1, \dots, C_r \rangle} \mathrm{d}x' \, e^{-\pi Q(x_+ - x')} \prod_i \mathrm{sgn}(C_i, x')$$

  where x_+ is the projection of x on the positive plane $\langle C_1, \dots, C_r \rangle$.
- Taking suitable linear combinations of $E_r(C_1, \dots C_r; x)$, one can construct a kernel Φ which leads to a convergent, modular (but non-holomorphic) theta series ϑ_Φ.

Alexandrov Banerjee Manschot BP 2016; Nazaroglu 2016

- More geometrically, ϑ_Φ arises by integrating the r-form valued Kudla-Millson theta series on a suitable polyhedron in $Gr(r, d - r)$

Kudla Funke 2016-17

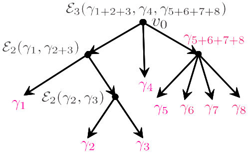
Explicity modular completions

- The series $\widehat{\Theta}_n$ appearing in the holomorphic anomaly equation

$$\partial_{\bar\tau}\widehat{h}_p(\tau,\bar\tau) = \sum_{p=\sum_{i=1}^{n\geq 2} p_i} \widehat{\Theta}_n(\{p_i\},\tau,\bar\tau) \prod_{i=1}^{n} \widehat{h}_{p_i}(\tau,\bar\tau)$$

are exactly of that type, with kernel given by a sum over rooted trees,

$$\widehat{\Phi}_n = \mathrm{Sym} \sum_{T\in\mathbb{T}_n^S} (-1)^{n_T-1}\mathcal{E}_{v_0} \prod_{v\in V_T\setminus\{v_0\}} \mathcal{E}_v$$



Explicity modular completions

- The series Θ_n appearing in the modular completion

$$\widehat{h}_p(\tau,\bar\tau) = h_p(\tau) + \sum_{p=\sum_{i=1}^{n\geq 2} p_i} \Theta_n(\{p_i\},\tau,\bar\tau) \prod_{i=1}^{n} h_{p_i}(\tau)$$

are not modular, but their anomaly cancels against that of h_p:

$$\Phi_n = \mathrm{Sym} \sum_{T\in\mathbb{T}_n^S} (-1)^{n_T-1} \mathcal{E}_{v_0}^{(+)} \prod_{v\in V_T\setminus\{v_0\}} \mathcal{E}_v^{(0)}$$

where $\mathcal{E}_v = \mathcal{E}_v^{(0)} + \mathcal{E}_v^{(+)}$ with $\mathcal{E}_v^{(0)}(x) = \lim_{\lambda\to\infty} \mathcal{E}_v(\lambda x)$.

- NB: these formulae hold for generating series of refined invariants, otherwise derivatives of error functions appear.

Alexandrov Manschot BP 18-19

Simplifications in one-divisor case

- On a threefold with $b_4(\mathfrak{Y}) = 1$, the D4-brane charge $p^a = N p_0^a$ is a multiple of the class p_0 of the primitive divisor $\mathcal{D}$, which we assume to be ample, with self-intersection $\kappa := [\mathcal{D}]^3 = |\Lambda^*/\Lambda|$. The sum over $p = \sum_i p_i$ reduces to a sum over partitions $N = \sum_{i=1}^n N_i$.
- Remarkably, only partitions of length two contribute to the holomorphic anomaly. In terms of the 'elliptic genus' $Z_N = \sqrt{\frac{\kappa}{N}} \sum_q \widehat{h}_{N,q}(\tau,\bar\tau)\Theta_q(\bar\tau, v)$, this reduces to

$$\mathcal{D}_{\bar\tau} Z_N = \frac{\sqrt{2\tau_2}}{32\pi\mathrm{i}} \sum_{N=N_1+N_2} N_1 N_2 Z_{N_1} Z_{N_2}$$

Minahan Nemeschansky Vafa Warner'98; Alexandrov Manschot BP'19

- In contrast, the modular completion involves a sum over partitions of arbitrary length.

Mock modularity for local CY

- A class of (non-compact) CY threefolds with $b_4(\mathfrak{Y})=1$ is obtained by taking the total space $\mathfrak{Y}=K_S$ of the canonical bundle over a complex Fano surface S.
- The BPS index $\Omega_z(\gamma)$ for $\gamma=(0,N,\mu,n)$ coincides with the Vafa-Witten invariant, given (up to sign) by the Euler number of the moduli space $\mathcal{M}_{N,\mu,n,}$ of J-Gieseker semi-stable sheaves of rank N on S.
- Since $b_2^+(S)=1$, Vafa-Witten invariants depend on the Kähler form J on S. The large volume attractor point corresponds to the canonical polarization $J\propto c_1(S)$. Denote by $\bar\Omega_\star(0,N,\mu,n)$ the corresponding DT invariants.

Mock modularity for local CY

- The generating series

$$h_{N,\mu} = \sum_n \bar{\Omega}_\star(0, N, \mu, n)\, \mathrm{q}^{n - \frac{N-1}{2N}\mu^2 - N\frac{\chi(S)}{24}}$$

  is invariant under $\mu \mapsto \mu + N$, and should transform as a vv mock modular form of weight $w = -1 - \frac{b_2(S)}{2}$ and depth $N - 1$.
- For $N = 1$, the moduli space reduces to the Hilbert scheme of n points on S, and the generating series is manifestly modular *[Goettsche'90]*,

$$h_{1,\mu}(\tau) = \frac{1}{\eta^{b_2(S)+2}}$$

- For $N > 1$, one expects non-holomorphic contributions from the boundary of the space of flat connections where the holonomy becomes reducible *[Vafa Witten 94; Dabholkar Putrov Witten '20]*.

Mock modularity for local CY

- For $S = \mathbb{P}^2$, rank 2 Vafa-Witten invariants are related to Hurwitz class numbers *[Klyachko'91, Yoshioka'94]*

$$h_{2,\mu}(\tau) = \frac{3H_\mu(\tau)}{\eta^6} \qquad \begin{cases} H_0(\tau) = & -\frac{1}{12} + \frac{1}{2}\mathrm{q} + \mathrm{q}^2 + \frac{4}{3}\mathrm{q}^3 + \frac{3}{2}\mathrm{q}^4 + \dots \\ H_1(\tau) = & \mathrm{q}^{\frac{3}{4}}\left(\frac{1}{3} + \mathrm{q} + \mathrm{q}^2 + 2\mathrm{q}^3 + \mathrm{q}^4 + \dots\right) \end{cases}$$

- This is the simplest example of depth 1 mock modular form, with completion *[Hirzebruch Zagier'75-76]*

$$\widehat{h}_{2,\mu}(\tau) = h_{2,\mu}(\tau) + \frac{3(1+\mathrm{i})}{8\pi\eta^6} \int_{-\bar{\tau}}^{\mathrm{i}\infty} \frac{\sum_{m\in\mathbb{Z}+\frac{\mu}{2}} e^{2\mathrm{i}\pi m^2 u}\mathrm{d}u}{(\tau+u)^{3/2}}$$

consistent with our general prescription.

Mock modularity for local CY

- For $S = \mathbb{P}^2$, $\mathbb{F}_0$ or any other del Pezzo surface, the VW invariants can be obtained in principle for any rank N by a sequence of blow ups and wall-crossings *[Yoshioka'95-96, Manschot'10-14]*. Alternatively, one can relate them to DT invariants for a suitable quiver associated to an exceptional collection on S. *[Beaujard Manschot BP'20]*
- Using our general prescription, one easily obtains the modular completion of the generating series. Moreover, with some ingenuity one can produce explicit solutions for all N, which (conjecturally) provide VW invariants for any del Pezzo surface and any N *[Alexandrov'20]*.
- Having the modular completion, one can apply Rademacher's circle method to extract the asymptotics of VW invariants as the instanton number n goes to infinity *[Bringmann Manschot'13, Bringmann Nazaroglu'18]*

Modularity for one-modulus compact CY

- We now specialize to compact CY threefolds with $b_2(\mathfrak{Y}) = 1$ and $p = N[\mathcal{D}]$ where $\mathcal{D}$ is an ample divisor with $[\mathcal{D}]^3 := \kappa$.
- We focus on smooth complete intersections in weighted projective space (CICY), $\mathfrak{Y} = X_{d_i}(w_j)$ with $\sum d_i = \sum w_j$. There are 13 such models, with Kähler moduli space $\mathcal{M}_K = \mathbb{P}^1 \backslash \{0, 1, \infty\}$, with a large volume point at $z = 0$ and a conifold singularity at $z = 1$.
- The central charge $Z_z(\gamma)$ is expressed in terms of hypergeometric functions. GV invariants $N_q^{(g)}$ are known up to high genus by direct integration method *[Huang Klemm Quackenbush'06]*
- I will concentrate on $N = 1$, and discuss $N = 2$ if time permits.

Gaiotto Strominger Yin '06-07; Alexandrov Gaddam Manschot BP'22

Modularity for one-modulus compact CY

CICY	$\chi(\mathfrak{Y})$	κ	$c_2(T\mathfrak{Y})$	$\chi(\mathcal{O}_\mathcal{D})$	n_1	C_1
$X_5(1^5)$	-200	5	50	5	7	0
$X_6(1^4,2)$	-204	3	42	4	4	0
$X_8(1^4,4)$	-296	2	44	4	4	0
$X_{10}(1^3,2,5)$	-288	1	34	3	2	0
$X_{4,3}(1^5,2)$	-156	6	48	5	9	0
$X_{4,4}(1^4,2^2)$	-144	4	40	4	6	1
$X_{6,2}(1^5,3)$	-256	4	52	5	7	0
$X_{6,4}(1^3,2^2,3)$	-156	2	32	3	3	0
$X_{6,6}(1^2,2^2,3^2)$	-120	1	22	2	1	0
$X_{3,3}(1^6)$	-144	9	54	6	14	1
$X_{4,2}(1^6)$	-176	8	56	6	15	1
$X_{3,2,2}(1^7)$	-144	12	60	7	21	1
$X_{2,2,2,2}(1^8)$	-128	16	64	8	33	3

Computing the polar terms

- For $N = 1$, the generating series

$$h_{1,q} = \sum_{n\in\mathbb{Z}} \Omega(0,1,q,n)\, \mathrm{q}^{n+\frac{q^2}{2\kappa}+\frac{q}{2}-\frac{\chi(\mathcal{D})}{24}}$$

  should transform as a vector-valued modular form of weight $-\frac{3}{2}$ in the Weil representation of $\mathbb{Z}[\kappa]$. In particular $q \in \mathbb{Z}/\kappa\mathbb{Z}$.

- An overcomplete basis is given for κ even by

$$\frac{E_4^a E_6^b}{\eta^{4\kappa+c_2}} D^\ell(\vartheta_q^{(\kappa)}) \quad \text{with} \quad \vartheta_q^{(\kappa)} = \sum_{k\in\mathbb{Z}+\frac{q}{\kappa}+\frac{1}{2}} \mathrm{q}^{\frac{1}{2}\kappa k^2}$$

  where $D = \mathrm{q}\partial_{\mathrm{q}} - \frac{w}{12}E_2$, is the Serre derivative (Alternatively, one may use Rankin-Cohen brackets).

- For κ odd, the same works with an extra insertion of $(-1)^{\kappa k}\, k^2$.

A naive Ansatz for the polar terms

- $h_{1,q}$ is uniquely determined by the polar terms $n < \frac{\chi(\mathcal{D})}{24} - \frac{q^2}{2\kappa} - \frac{q}{2}$, but the dimension $d_1 = n_1 - C_1$ of the space of modular forms may be smaller than the number n_1 of polar terms !
- Physically, we expect that polar coefficients arise as bound states of D6-brane and anti D6-branes *[Denef Moore'07]*
- Earlier studies *[Gaiotto Strominger Yin'06]* suggest that only bound states of the form $(D6 + qD2 + nD0, \overline{D6(-1)})$ contribute to polar coeffs:

$$\Omega(0,1,q,n) = (-1)^{\chi(\mathcal{O}_\mathcal{D})-q-n+1} \left(\chi(\mathcal{O}_\mathcal{D}) - q - n\right) DT(q,n)$$

where $DT(q,n)$ counts ideal sheaves with $\mathrm{ch}_2 = q$ and $\mathrm{ch}_3 = n$ *[Alexandrov Gaddam Manschot BP'22]*

GV/DT/PT relation

- For a single D6-brane, the DT-invariant $DT(q,n) = \Omega(1,0,q,n)$ at large volume can be computed via the GV/DT relation

$$\sum_{Q,n} DT(Q,n)\, \mathrm{q}^n v^Q = M(-\mathrm{q})^{\chi_{\mathfrak{Y}}} \prod_{Q,g,\ell} \left(1-(-\mathrm{q})^{g-\ell-1} v^Q\right)^{(-1)^{g+\ell}\binom{2g-2}{\ell} N_Q^{(g)}}$$

where $M(-\mathrm{q}) = \prod_{n\geq 1}(1-(-\mathrm{q})^n)^{-n}$ is the Mac-Mahon function.

Maulik Nekrasov Okounkov Pandharipande'06

- Pandharipande-Thomas invariants $PT(Q,n)$ counting stable pairs $E = (\mathcal{O}_{\mathfrak{Y}} \xrightarrow{s} F)$ with $[F] = Q$ and $\chi(F) = n$ satisfy a similar relation without the Mac-Mahon factor $M(-\mathrm{q})^{\chi_{\mathfrak{Y}}}$.
- The topological string partition function is given by

$$\Psi_{\mathrm{top}}(z,\lambda) = M(-\mathrm{q})^{-\chi_{\mathfrak{Y}}/2} Z_{DT}\ , \quad \mathrm{q} = e^{\mathrm{i}\lambda}, v = e^{2\pi\mathrm{i}z/\lambda}$$

can be computed by the direct integration method.

Modular predictions for D4-D2-D0 indices (naive)

- Remarkably, there exists a vv modular form with integer Fourier coefficients matching these polar terms for almost all CICY (except $X_{4,2}, X_{3,2,2}, X_{2,2,2,2}$), reproducing earlier results *[Gaiotto Strominger Yin]* by for X_5, X_6, X_8, X_{10} and $X_{3,3}$
- X_5 (Quintic in $\mathbb{P}^4$):

$$h_{1,0} = q^{-\frac{55}{24}} \left(\underline{5 - 800q + 58500q^2} + 5817125q^3 + \dots\right)$$
$$h_{1,\pm 1} = q^{-\frac{55}{24}+\frac{3}{5}} \left(\underline{0 + 8625q} - 1138500q^2 + 3777474000q^3 + \dots\right)$$
$$h_{1,\pm 2} = q^{-\frac{55}{24}+\frac{2}{5}} \left(\underline{0 + 0q} - 1218500q^2 + 441969250q^3 + \dots\right)$$

- X_{10} (Decantic in $W\mathbb{P}^{5,2,1,1,1}$):

$$h_{1,0} \stackrel{?}{=} \frac{541 E_4^4 + 1187 E_4 E_6^2}{576\, \eta^{35}}$$
$$= q^{-\frac{35}{24}} \left(\underline{3 - 576q} + 271704q^2 + 206401533q^3 + \cdots\right)$$

Rank 0 DT invariants from GV invariants

- Our Ansatz for polar terms was just an educated guess. Fortunately, recent progress in Donaldson-Thomas theory allows to compute D4-D2-D0 indices in a rigorous fashion, and compare with modular predictions.

Bayer Macri Toda'11; Toda'11; Feyzbakhsh Thomas'20-22

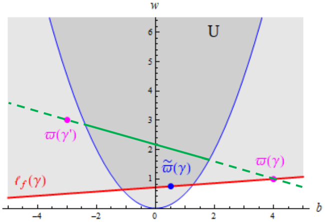
- The key idea is to consider a family of weak stability conditions on the boundary of Stab $\mathcal{C}$, called tilt stability, with degenerate central charge

$$Z_{b,t}(E) = \frac{\mathrm{i}}{6} t^3 \, \mathrm{ch}_0(E) - \frac{1}{2} t^2 \, \mathrm{ch}_1^b(E) - \mathrm{i} t \, \mathrm{ch}_2^b(E) + 0 \, \mathrm{ch}_3^b(E)$$

with $\mathrm{ch}_k^b = \int_{\mathfrak{Y}} H^{3-k} e^{-bH} \, \mathrm{ch}$. The heart $\mathcal{A}_b$ is generated by length-two complexes $\mathcal{E} \xrightarrow{d} \mathcal{F}$ with $\mathrm{ch}_1^b(\mathcal{E}) > 0, \mathrm{ch}_1^b(\mathcal{F}) \leq 0$.

Rank 0 DT invariants from GV invariants

- Tilt stability agrees with physical stability at large volume, but the chamber structure is much simpler: walls are straight lines in the plane spanned by $(b, w = \frac{1}{2}b^2 + \frac{1}{6}t^2)$, with $w > \frac{1}{2}b^2$.



$$\nu_{b,w}(E) = \frac{\mathrm{ch}_2 .H - w\,\mathrm{ch}_0 .H^3}{\mathrm{ch}_1 .H^2 - b\,\mathrm{ch}_0 .H^3}$$

$$\varpi(E) = \left(\frac{\mathrm{ch}_1 .H^2}{\mathrm{ch}_0 .H^3}, \frac{\mathrm{ch}_2 .H}{\mathrm{ch}_0 .H^3} \right)$$

$$\tilde\varpi(E) = \left(\frac{2\,\mathrm{ch}_2 .H}{\mathrm{ch}_1 .H^2}, \frac{3\,\mathrm{ch}_3}{\mathrm{ch}_1 .H^2} \right)$$

- Importantly, for any $\nu_{b,w}$-semistable object E there is a conjectural inequality on Chern classes $C_i := \int_{\mathfrak{Y}} \mathrm{ch}_i(E).H^{3-i}$ *[Bayer Macri Toda'11; Bayer Macri Stellari'16]*

$$(C_1^2 - 2C_0C_2)w + (3C_0C_3 - C_1C_2)b + (2C_2^2 - 3C_1C_3) \geq 0$$

Rank 0 DT invariants from GV invariants

- By studying wall-crossing between the empty chamber provided by BMT bound and large volume, *[Feyzbakhsh Thomas]* show that D4-D2-D0 indices can be computed from rank 1 DT or PT invariants, which are in turn related to GV invariants.
- In particular for (q, n) large enough, the PT invariant counting tilt-stable objects of class $(-1, 0, q, n)$ is given by *[Feyzbakhsh'22]*

$$PT(q,n) = (-1)^{\langle \overline{D6(1)}, \gamma \rangle + 1} \langle \overline{D6(1)}, \gamma \rangle \, \Omega(\gamma)$$

with $\overline{D6(1)} := \mathcal{O}_{\mathfrak{Y}}(H)[1]$ and $\gamma = (0, 1, q, n)$. Using spectral flow invariance, one finds for all $m \geq m_0(q, n)$

$$\boxed{\Omega(\gamma) = \frac{(-1)^{\langle \overline{D6(1-m)}, \gamma \rangle + 1}}{\langle \overline{D6(1-m)}, \gamma \rangle} PT(q', n')} \qquad \begin{cases} q' = q + \kappa m \\ n' = n - mq - \frac{\kappa}{2} m(m+1) \end{cases}$$

Modular predictions for D4-D2-D0 (rigorous)

- Using this idea, we have computed most of the polar terms (and many non-polar ones) for all models except $X_{3,2,2}, X_{2,2,2,2}$ – for those the required GV invariants are currently out of reach.

Alexandrov, Feyzbakhsh, Klemm, BP, Schimannek'23

- We find that our educated guess is correct for $X_5, X_6, X_8, X_{3,3}, X_{4,4}, X_{6,6}$ ☺ , but (as anticipated by *[van Herck Wyder'09]*) misses some $\mathcal{O}(1)$ contributions for $X_{10}, X_{6,2}, X_{6,4}, X_{4,3}$ ☹   E.g. for X_{10},

$$h_{1,0} = \frac{203 E_4^4 + 445 E_4 E_6^2}{216\,\eta^{35}} = \mathrm{q}^{-\frac{35}{24}} \left(\underline{3 - 575\mathrm{q}} + 271955\mathrm{q}^2 + \cdots \right)$$

In all cases, modularity holds with flying colors ! ☼ ♫ ☺

- Note that *[Toda'13, Feyzbakhsh'22]* also prove a version of our $D6 - \overline{D6}$ ansatz, but under very restrictive conditions which are only satisfied by the most polar terms.

Mock modularity for non-Abelian D4-D2-D0 indices

- Let us consider D4-D2-D0 indices with $N = 2$ units of D4-brane charge. In that case, $\{h_{2,q}, q \in \mathbb{Z}/(2\kappa\mathbb{Z})\}$ should transform as a vv mock modular form with modular completion

$$\widehat{h}_{2,q}(\tau,\bar{\tau}) = h_{2,q}(\tau) + \sum_{q_1,q_2=0}^{\kappa-1} \delta^{(\kappa)}_{q_1+q_2-q}\, \Theta^{(\kappa)}_{q_2-q_1+\kappa}\, h_{1,q_1}\, h_{1,q_2}$$

where
$$\Theta^{(\kappa)}_q = \frac{(-1)^q}{8\pi} \sum_{k\in 2\kappa\mathbb{Z}+q} |k|\, \beta\left(\frac{\tau_2 k^2}{\kappa}\right) e^{-\frac{\pi i \tau}{2\kappa}k^2},$$

and $\beta(x^2) = 2|x|^{-1} e^{-\pi x^2} - 2\pi \mathrm{Erfc}(\sqrt{\pi}|x|)$.

- For $\kappa = 1$, the series $\Theta^{(1)}_q$ is the one appearing in the modular completion of rank 2 Vafa-Witten invariants on $\mathbb{P}^2$!

Mock modularity for non-Abelian D4-D2-D0 indices

- The series $\Theta_q^{(\kappa)}$ is convergent but not modular invariant. Suppose there exists a holomorphic function $g_q^{(\kappa)}$ such that $\Theta_q^{(\kappa)} + g_q^{(\kappa)}$ transforms as a vv modular form. Then

$$\widetilde{h}_{2,q}(\tau,\bar\tau) = h_{2,q}(\tau) - \sum_{q_1,q_2=0}^{\kappa-1} \delta^{(\kappa)}_{q_1+q_2-q}\, g^{(\kappa)}_{q_2-q_1+\kappa}\, h_{1,q_1}\, h_{1,q_2}$$

  will be an ordinary weakly holomorphic vv modular form, hence uniquely determined by its polar part.
- To construct $g_q^{(\kappa)}$, notice that for κ prime, $\Theta_q^{(\kappa)}$ is obtained from $\Theta_q^{(1)}$ by acting with the Hecke-type operator *[Bouchard Creutzig Diaconescu Doran Quigley Sheshmani'16]*

$$(\mathcal{T}_\kappa[\phi])_q(\tau) = \frac{1}{\kappa} \sum_{\substack{a,d>0\\ ad=\kappa}} \left(\tfrac{\kappa}{d}\right)^{w+\frac12} \delta_\kappa(q,d) \sum_{b=0}^{d-1} e^{-\pi \mathrm{i}\, \frac{b}{a}\, q^2}\, \phi_{dq}\left(\tfrac{a\tau+b}{d}\right),$$

  with $q \in \Lambda^*/\Lambda(\kappa)$ and $\delta_\kappa(q,d) = 1$ if $q \in \Lambda^*/\Lambda(d)$ and 0 otherwise.

Mock modularity for non-Abelian D4-D2-D0 indices

- For $\kappa = 1$, a candidate for $g_q^{(1)}$ is well-known: the generating series of Hurwitz class numbers *[Hirzebruch Zagier 1973]*

$$H_0(\tau) = -\frac{1}{12} + \frac{1}{2}\mathrm{q} + \mathrm{q}^2 + \frac{4}{3}\mathrm{q}^3 + \frac{3}{2}\mathrm{q}^4 + \dots$$
$$H_1(\tau) = \mathrm{q}^{\frac{3}{4}} \left(\frac{1}{3} + \mathrm{q} + \mathrm{q}^2 + 2\mathrm{q}^3 + \mathrm{q}^4 + \dots \right)$$

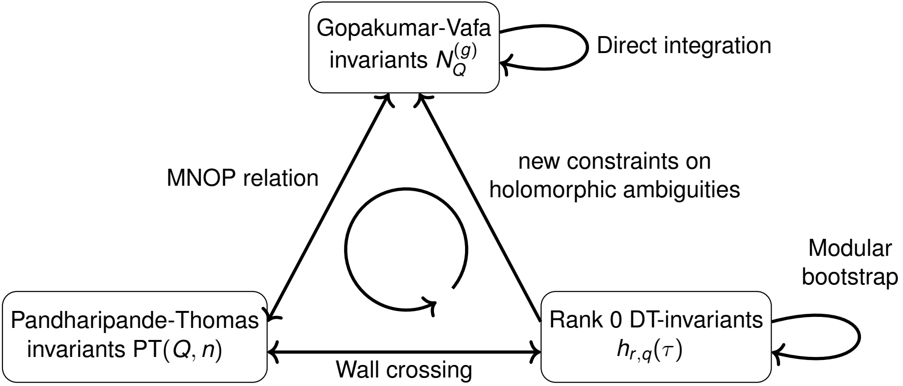
  For any κ, we can thus choose $g_q^{(\kappa)} = \mathcal{T}_\kappa(H)_q$.
- The vv modular form $\widetilde{h}_{2,q}$ is uniquely specified by its polar terms but those must satisfy constraints for such a form to exist, and integrality is not guaranteed !
- Mathematical results by Feyzbakhsh in principle allow to compute polar terms from DT/PT invariants, hence GV invariants, but the required degree and genus is prohibitive so far.

Mock modularity for non-Abelian D4-D2-D0 indices

CICY	χ	κ	c_2	$\chi(\mathcal{O}_{2\mathcal{D}})$	n_2	C_2
$X_5(1^5)$	-200	5	50	15	36	1
$X_6(1^4,2)$	-204	3	42	11	19	1
$X_8(1^4,4)$	-296	2	44	10	14	1
$X_{10}(1^3,2,5)$	-288	1	34	7	7	0
$X_{4,3}(1^5,2)$	-156	6	48	16	42	0
$X_{4,4}(1^4,2^2)$	-144	4	40	12	25	1
$X_{6,2}(1^5,3)$	-256	4	52	14	30	1
$X_{6,4}(1^3,2^2,3)$	-156	2	32	8	11	1
$X_{6,6}(1^2,2^2,3^2)$	-120	1	5	2	5	0
$X_{3,3}(1^6)$	-144	9	54	21	78	3
$X_{4,2}(1^6)$	-176	8	56	20	69	3
$X_{3,2,2}(1^7)$	-144	12	60	26	117	0
$X_{2,2,2,2}(1^8)$	-128	16	64	32	185	4

Quantum geometry from stability and modularity

Conversely, we can use our knowledge of Abelian D4-D2-D0 invariants to compute GV invariants and push the direct integration method to higher genus !



Alexandrov Feyzbakhsh Klemm BP Schimannek'23

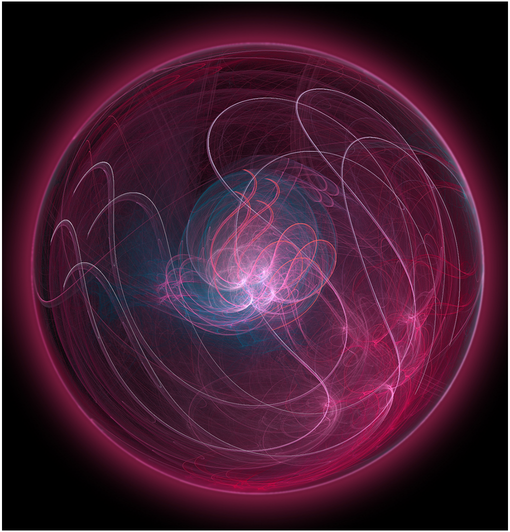
Quantum geometry from stability and modularity

CICY	χ	κ	type	g_{integ}	g_{mod}	g_{avail}
$X_5(1^5)$	-200	5	F	53	69	55
$X_6(1^4,2)$	-204	3	F	48	57	31
$X_8(1^4,4)$	-296	2	F	60	80	48
$X_{10}(1^3,2,5)$	-288	1	F	50	70	47
$X_{4,3}(1^5,2)$	-156	6	F	20	24	24
$X_{6,4}(1^3,2^2,3)$	-156	2	F	14	17	17
$X_{6,6}(1^2,2^2,3^2)$	-120	1	K	18	22	22
$X_{4,4}(1^4,2^2)$	-144	4	K	26	34	34
$X_{3,3}(1^6)$	-144	9	K	29	33	33
$X_{4,2}(1^6)$	-176	8	C	50	66	43
$X_{6,2}(1^5,3)$	-256	4	C	63	78	42

Conclusion

- The existence of an isometric action of S-duality on the vector-multiplet moduli space in $D = 3$, leads to strong modularity constraints on rank 0 DT invariants in the large volume limit.
- For $p = \sum_{i=1}^{n} p_i$ the sum of n irreducible divisors, the generating function h_p is a mock modular form of depth $n - 1$ with an explicit shadow, thus it is uniquely determined by its polar coefficients.
- While modularity is clear physically, its mathematical origin is mysterious. Perhaps Noether-Lefschetz theory or VOAs can help *[Bouchard Creutzig Diaconescu Doran Quigley Sheshmani'16]*
- Using modularity and GV/DT/PT relations, we can not only compute D4D2-D0 indices, but also push $\Psi_{\rm top}$ to higher genus !
- Mock modularity affects the growth of Fourier coefficients, hence the microscopic entropy of supersymmetric black holes. It should have an imprint on the macroscopic side as well...

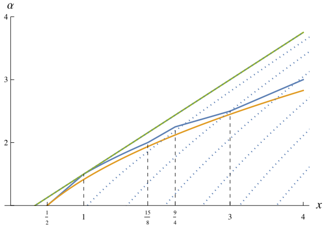
Thanks for your attention !



A new explicit formula (S. Feyzbakhsh'23)

- Let $(\mathfrak{Y}, H)$ be a smooth polarised CY threefold with $\mathrm{Pic}(\mathfrak{Y}) = \mathbb{Z}.H$ satisfying the BMT conjecture.
- Fix $m \in \mathbb{Z}, \beta \in H_2(\mathfrak{Y}, \mathbb{Z})$ and define $x = \frac{\beta.H}{H^3}, \quad \alpha = -\frac{3m}{2\beta.H}$

$$f(x) := \begin{cases} x + \frac{1}{2} & \text{if } 0 < x < 1 \\ \sqrt{2x + \frac{1}{4}} & \text{if } 1 < x < \frac{15}{8} \\ \frac{2}{3}x + \frac{3}{4} & \text{if } \frac{15}{8} \leq x < \frac{9}{4} \\ \frac{1}{3}x + \frac{3}{2} & \text{if } \frac{9}{4} \leq x < 3 \\ \frac{1}{2}x + 1 & \text{if } 3 \leq x \end{cases}$$



A new explicit formula (S. Feyzbakhsh'23)

Theorem (wall-crossing for class $(-1, 0, \beta, -m)$):

- If $f(x) < \alpha$ then the stable pair invariant $\mathrm{PT}_{m,\beta}$ equals

$$\sum_{(m',\beta')} (-1)^{\chi_{m',\beta'}} \chi_{m',\beta'}\, \mathrm{PT}_{m',\beta'}\ \Omega\left(0,\ H,\ \frac{H^2}{2} - \beta' + \beta\ ,\ \frac{H^3}{6} + m' - m - \beta'.H\right),$$

where $\chi_{m',\beta'} = \beta.H + \beta'.H + m - m' - \frac{H^3}{6} - \frac{1}{12} c_2(\mathfrak{Y}).H$.

- The sum runs over $(m',\ \beta') \in H_0(\mathfrak{Y}, \mathbb{Z}) \oplus H_2(\mathfrak{Y}, \mathbb{Z})$ such that

$$0 \leq \beta'.H \leq \frac{H^3}{2} + \frac{3mH^3}{2\beta.H} + \beta.H$$

$$-\frac{(\beta'.H)^2}{2H^3} - \frac{\beta'.H}{2} \leq\ m'\ \leq \frac{(\beta.H - \beta'.H)^2}{2H^3} + \frac{\beta.H + \beta'.H}{2} + m$$

In particular, $\beta'.H < \beta.H$.

Corollary (Castelnuovo bound): $\mathrm{PT}_{m,\beta} = 0$ unless $m \geq -\frac{(\beta.H)^2}{2H^3} - \frac{\beta.H}{2}$