Vinberg $\theta$-pairs and Higgs bundles

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On the occasion of Alastair King’s 60th birthday
Guanajuato, 2006 (photo by Leticia Brambila)
Symposium on Vector Bundles in Algebraic Geometry
Durham, 1993 (photo by Steve Bradlow)
Vinberg $\theta$-pairs

- $G$ semisimple complex Lie group with Lie algebra $\mathfrak{g}$
- $\theta : G \to G$ order $m > 0$ holomorphic automorphism
- $\theta$ defines an automorphism of $\mathfrak{g}$ (denoted also by $\theta$), determining a $\mathbb{Z}/m$-grading of $\mathfrak{g}$ (we write $\mathbb{Z}/m$ instead of $\mathbb{Z}/m\mathbb{Z}$):

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}/m} \mathfrak{g}_i \text{ with } \mathfrak{g}_i = \{ x \in \mathfrak{g} \text{ such that } \theta(x) = \zeta^ix \},$$

where $\zeta$ is a primitive $m$-th root of unity. One has

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}.$$
Let $\mu_m = \{ z \in \mathbb{C}^* \text{ such that } z^m = 1 \}$. Having $\mathbb{Z}/m$-grading on $\mathfrak{g}$ is equivalent to having a homomorphism

$$\tilde{\theta} : \mu_m \to \text{Aut}(\mathfrak{g}).$$

Then $\mathfrak{g}_i = \{ x \in \mathfrak{g} \text{ such that } \tilde{\theta}(z)x = z^ix \}$ for every $z \in \mu_m$.

Let $G^\theta < G$ be the **fixed point subgroup**. $G^\theta$ is a reductive group with Lie algebra $\mathfrak{g}_0$.

Since $[\mathfrak{g}_0, \mathfrak{g}_i] \subset \mathfrak{g}_i$, all the subspaces $\mathfrak{g}_i$ are stable under the adjoint action of $G^\theta$.

The pairs $(G^\theta, \mathfrak{g}_i)$ are called **Vinberg $\theta$-pairs** (also $\theta$-groups or **Vinberg $\theta$-representations**).

Sometimes this term is used for the pairs $(G_0, \mathfrak{g}_i)$, where $G_0 = G^\theta_0$ is the connected component of the identity.
Example 1: Adjoint representation. If $m = 1$, then $\theta$ is the identity automorphism and $(G, \mathfrak{g})$ is the only Vinberg $\theta$-pair.

Example 2: Symmetric pairs. Let $m = 2$: $\mathbb{Z}/2$-grading

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

theory of symmetric spaces and real forms of $\mathfrak{g}$ and $G$.

A real form of $G^\sigma < G$ is the fixed point subgroup of a conjugation (antiholomorphic involution) $\sigma$ of $G$.

Cartan: Given a holomorphic involution $\theta$ of $G$ there is a compact conjugation $\tau$ of $G$, so that $\sigma := \tau \theta = \theta \tau$ is a conjugation of $G$. This gives a bijection

$$\text{Aut}_2(G)/\sim \leftrightarrow \text{Conj}(G)/\sim,$$

where equivalence is conjugation by an inner automorphism of $G$. 

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**Example 3: Cyclic quivers.** Let $m \geq 2$. Let $V$ be a complex vector space equipped with a $\mathbb{Z}/m$-grading

$$V = \bigoplus_{i \in \mathbb{Z}/m} V_i.$$ 

Let $G = \text{SL}(V)$. Define on $\mathfrak{g} = \mathfrak{sl}(V)$ the $\mathbb{Z}/m$-grading given by

$$\mathfrak{g}_i = \{ A \in \mathfrak{sl}(V) \text{ such that } A(V_j) \subset V_{j+i} \text{ for every } j \in \mathbb{Z}/m \}.$$ 

In this situation

$$G^\theta = \text{S}( \prod_{i \in \mathbb{Z}/m} \text{GL}(V_i)), $$

and

$$\mathfrak{g}_1 = \bigoplus_{i \in \mathbb{Z}/m} \text{Hom}(V_i, V_{i+1}).$$
Define the quiver $Q$ with $m$ vertices indexed by $\mathbb{Z}/m$ and arrows $i \mapsto i + 1$ for each $i \in \mathbb{Z}/m$.

Then $\mathfrak{g}_1$ is the space of representations of $Q$ where we put $V_i$ at the vertex $i$:

This can be represented by the diagramme

$$
V_0 \xrightarrow{f_0} V_1 \xrightarrow{f_1} \ldots \xrightarrow{f_{m-2}} V_{m-1}.
$$

For other classical groups the action of $G^\theta$ on $\mathfrak{g}_1$ can be interpreted in terms of a cyclic quiver with some extra structure.
For a moment we will consider Vinberg $\theta$-pairs $(G_0, g_1)$ where $G_0 = G^\theta_0$. The study for other $g_i$ can be reduced to the case of $g_1$.

Recall that if $t \subset g$ is a Cartan subalgebra, and $W(t)$ is the Weyl group, the Chevalley restriction theorem establishes an isomorphism

$$g \parallel G \cong t/W(t).$$

Similarly, if $\theta$ is an involution of $G$ and $g = g_0 \oplus g_1$ is the Cartan decomposition defined by $\theta$, and $W(a)$ is the little Weyl group defined by a maximal abelian subalgebra $a \subset g_1$, there is also a Chevalley restriction theorem studied by Kostant–Rallis (1971):

$$g_1 \parallel G_0 \cong a/W(a).$$
A main result of Vinberg’s theory (1976) is a version of the Chevalley restriction theorem for Vinberg θ-pairs. Key concept is that of Cartan subspace: linear subspace $a \subset g_1$ which is abelian as a Lie algebra, consisting of semisimple elements, and maximal with these two properties.

The little Weyl group

$$W(a) = N_{G_0}(a)/C_{G_0}(a)$$

is a finite linear group generated by semisimple transformations of $a$ fixing a hyperplane. Hence $\mathbb{C}[a]^{W(a)}$ is a polynomial ring, and the restriction of polynomial functions $\mathbb{C}[g_1] \to \mathbb{C}[a]$ induces an isomorphism of invariant polynomial rings $\mathbb{C}[g_1]^{G_0} \to \mathbb{C}[a]^{W(a)}$, or equivalently,

$$g_1 // G_0 \cong a/W(a).$$
Vinberg $\theta$-pairs

- The fact that $W(a)$ is a finite linear group generated by complex reflections implies that $\mathbb{C}[a]^{W(a)} = \mathbb{C}[f_1, \ldots, f_r]$ is a polynomial algebra generated by $r$ algebraically independent polynomials $f_1, \ldots, f_r$ whose degrees $d_1, \ldots, d_r$ are determined by the grading. Here $r$ is the dimension of $a$, an invariant called the rank of $(G_0, g_1)$.

- Kostant (1963) showed that the quotient map $g \to g \parallel G$ has a section, known as the Kostant section.

- This was extended by Kostant–Rallis (1971) to obtain the Kostant–Rallis section in the symmetric pair case for the quotient map $g_1 \to g_1 \parallel G_0$.

- The existence of a similar section for Vinberg’s $\theta$-pairs for $\theta$ of higher order was conjectured by Popov (1976), but only proved more recently in full generality by Reeder–Levy–Yu–Gross (2012). In this context, such a section is referred as a Kostant–Weirstrass section.
Vinberg $\theta$-pairs

The theory of Vinberg $\theta$-pairs has been extended to general fields of 0 characteristic and good positive characteristic not dividing the order of $\theta$ (Levy).

Applications of Vinberg’s theory include the following:

- **Classification of trivectors** of 9-dimensional space $\mathbb{C}^9$ by Elashvili–Vinberg (1978) using a $\mathbb{Z}/3$-grading of $\mathfrak{e}_8$:
  \[ \mathfrak{e}_8 = \mathfrak{sl}(9, \mathbb{C}) \oplus \Lambda^3(\mathbb{C}^9) \oplus \Lambda^6(\mathbb{C}^9). \]

- Certain Vinberg $\theta$-pairs have interesting connections to the arithmetic theory of elliptic curves and Jacobians: Bhargava–Shankar (2010), Bhargava–Gross (2012), ...

- There is also a connection between the Vinberg $\theta$-pair ($\text{SL}(9, \mathbb{C})/\mu_3, \Lambda^3(\mathbb{C}^9)$) coming from the $\mathbb{Z}/3$-grading of $\mathfrak{e}_8$ and the moduli space of genus 2 curves with some additional data: Rains–Sam (2016, 2018).

- Vinberg pairs are also used in the description of the moduli space of vector bundles on curves of small genus ($g = 2, 3$).
Vinberg $\theta$-pairs

- Vinberg’s theory has also connections to physics related to del Pezzo surfaces: **mysterious duality** (Iqbal–Neitzke–Vafa, 2001)

Madrid, 2022
The goal of this talk is to discuss the role of Vinberg $\theta$-pairs in **Higgs bundle theory**.

Part of it will be a review of some well-known results put in the larger context of Vinberg’s theory.

Then I will introduce some new problems and work in progress joint with my student **Miguel González** in the study of the geometry of moduli space of **cyclic Higgs bundles**.
Higgs pairs

- $X$ compact Riemann surface of genus $g \geq 2$ with canonical line bundle $K$
- $G$ reductive complex Lie group with Lie algebra $\mathfrak{g}$
- $\rho : G \to \mathrm{GL}(V)$ a representation of $G$ in a complex vector space $V$
- A $(G, V)$-Higgs pair on $X$ is a pair $(E, \varphi)$ consisting of a holomorphic principal $G$-bundle $E \to X$ and $\varphi \in H^0(X, E(V) \otimes K)$, where $E(V) = E \times_G V$ is the vector bundle associated to the representation $\rho$.
- There are suitable notions of (semi,poly) stability. Consider the moduli space of polystable $(G, V)$-Higgs pairs:
  $$\mathcal{M}(G, V)$$
When $\rho$ is the adjoint representation $G \to \text{GL}(\mathfrak{g})$ $(G, \mathfrak{g})$-Higgs pairs are the $G$-Higgs bundles introduced by Hitchin (1987).

$\mathcal{M}(G)$ : moduli space of polystable $G$-Higgs bundles
• From now on we assume that $G$ is semisimple

• Let $\theta \in \text{Aut}(G)$ be of order $m$. Consider the $\mathbb{Z}/m$-grading defined by $\theta$:

\[ \mathfrak{g} = \bigoplus_{i \in \mathbb{Z}/m} \mathfrak{g}_i, \]

and the Vinberg $\theta$-pairs $(G^{\theta}, \mathfrak{g}_i)$.

• Let $\mathcal{M}(G^{\theta}, \mathfrak{g}_i)$ be the **moduli space of $(G^{\theta}, \mathfrak{g}_i)$-Higgs pairs over $X$.**

• The moduli spaces associated to Vinberg pairs do appear naturally inside the moduli space $\mathcal{M}(G)$ of $G$-Higgs bundles as fixed point subvarieties for a certain action of a cyclic group. This is studied in joint paper with S. Ramanan (2019).
Aut(G) acts on $\mathcal{M}(G)$: For $\alpha \in \text{Aut}(G)$ and a $G$-Higgs bundle $(E, \varphi)$ we defined

$$\alpha \cdot (E, \varphi) := (\alpha(E), \alpha(\varphi)) \text{ where } \alpha(E) = E \times_\alpha G.$$  

This descends to an action of $\text{Out}(G) = \text{Aut}(G)/\text{Int}(G)$.

- $\mathbb{C}^*$ acts on $\mathcal{M}(G)$ by rescaling the Higgs field.
- Let $\mu_m = \{z \in \mathbb{C}^* \text{ such that } z^m = 1\}$ and let $\zeta \in \mu_m$ be a primitive $m$-th root of unity. Consider the homomorphism $\mu_m \to \text{Aut}(G) \times \mathbb{C}^*$ defined by $\zeta \mapsto (\theta, \zeta)$. Let $\Gamma$ be the image.
- $\Gamma$ is isomorphic to $\mu_m$ and acts on $\mathcal{M}(G)$ by the rule

$$(E, \varphi) \mapsto (\theta(E), \zeta \theta(\varphi)).$$
Extension of structure group defines a finite map

\[ \mathcal{M}(G^\theta, g_1) \rightarrow \mathcal{M}(G). \]

Denote the image by \( \tilde{\mathcal{M}}(G^\theta, g_1) \), then

\[ \tilde{\mathcal{M}}(G^\theta, g_1) \subset \mathcal{M}(G)^\Gamma. \]

Since the action of \( \theta \) depends only on the class of \( \theta \) in \( \text{Out}(G) \), there are other subvarieties in \( \mathcal{M}(G)^\Gamma \).

Let \( \text{Aut}_m(G) \subset \text{Aut}(G) \) the be set of elements of order \( m \).

There is a map

\[ \text{cl} : \text{Aut}_m(G)/\sim \rightarrow \text{Out}_m(G) \]

called the clique map. For an element \( a \in \text{Out}_m(G) \) we refer to the set \( \text{cl}_m^{-1}(a) \) as the **clique** defined by \( a \).
To identify $\mathcal{C}_{m}^{-1}(a)$, let $Z = Z(G)$ be the centre of $G$ and

$$S_{\theta} := \{s \in G : s\theta(s) \cdots \theta^{m-1}(s) = z \in Z\}.$$ 

There is an action of $Z$ on $S_{\theta}$ by multiplication and of $G$ given by $s \cdot g := g^{-1}s\theta(g) = g \in G, s \in S_{\theta}$, and

$$S_{\theta}/(Z \times G) = H^{1}(\mathbb{Z}/m, \text{Ad}(G)),$$

where $H^{1}(\mathbb{Z}/m, \text{Ad}(G))$ is the first Galois cohomology set.

The map $S_{\theta} \to \text{Aut}_{m}(G)$ given by $s \mapsto \text{Int}_{s} \theta$ defines a bijection

$$S_{\theta}/(Z \times G) \longleftrightarrow \mathcal{C}_{m}^{-1}(a).$$
\( \mathcal{M}(G)^\Gamma \) contains also \( \widetilde{\mathcal{M}}(G_{\theta'}, g'_1) \) for \( \theta' \in S_\theta \), and the subvariety \( \widetilde{\mathcal{M}}(G_{\theta'}, g'_1) \subset \mathcal{M}(G) \) depends only on the class 
\( [\theta'] \in S_\theta/(Z \times G) \).

There is a converse to this result if we consider the smooth locus \( \mathcal{M}_*(G) \subset \mathcal{M}(G) \):

\[
\mathcal{M}_*(G)^\Gamma \subset \bigcup_{[\theta'] \in S_\theta/(Z \times G)} \widetilde{\mathcal{M}}(G^{\theta'}, g'_1).
\]

The elements in \( \widetilde{\mathcal{M}}(G^\theta, g_1) \) are called cyclic \( G \)-Higgs bundles and have been studied by many authors.

The moduli spaces \( \mathcal{M}(G^\theta, g_i) \) for general \( g_i \) in the \( \mathbb{Z}/m \)-grading of \( g \) do also show up as fixed point in \( \mathcal{M}(G) \). But now the homomorphism \( \mu_m \to \text{Aut}(G) \times \mathbb{C}^* \) is defined by \( \zeta \mapsto (\theta, \zeta^i) \) and consider the action of the image \( \Gamma \).
Recall that $\mathcal{R}(G)$ is the $G$-character variety of the fundamental group of $X$ defined as

$$\mathcal{R}(G) = \text{Hom}(\pi_1(X), G) \parallel G,$$

the non-abelian Hodge correspondence establishes a homeomorphism

$$\mathcal{M}(G) \cong \mathcal{R}(G).$$

If $m = 2$, $\theta$ defines the $\mathbb{Z}/2$-grading $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, and there is also a homeomorphism

$$\mathcal{M}(G^\theta, \mathfrak{g}_1) \cong \mathcal{R}(G^\sigma),$$

where $G^\sigma$ is the real form of $G$ defined the conjugation $\sigma = \theta \tau$, with $\tau$ a compact conjugation commuting with $\theta$.

The subvarieties $\widetilde{\mathcal{M}}(G^\theta, \mathfrak{g}_1)$ are in the fixed point locus for the action of $\mathbb{Z}/2$ on $\mathcal{M}(G)$ sending $(E, \varphi) \mapsto (\theta(E), -\theta(\varphi))$ and define Lagrangian subvarieties of $\mathcal{M}(G)$. 
Cyclic $\text{SL}(n, \mathbb{C})$-Higgs bundles for inner automorphisms

**Involutive case ($m = 2$) — Bradlow-G-Gothen (2003)**

- $V = V_0 \oplus V_1$, $n = \dim V$, $n_0 = \dim V_0$, $n_1 = \dim V_1$
- $G = \text{SL}(n, \mathbb{C})$ and $G^\theta = S(\text{GL}(V_0) \times \text{GL}(V_1))$
- $g_1 = \text{Hom}(V_0, V_1) \oplus \text{Hom}(V_1, V_0)$.

The pair $(G^\theta, g_1)$ is described by the representations of the quiver

$$
\begin{array}{ccc}
V_0 & \xrightarrow{f_0} & V_1 \\
\xleftarrow{f_1} & & \\
\end{array}
$$

- A $(G^\theta, g_1)$-Higgs pair over $X$ is equivalent to a 4-tuple
  $(E_0, E_1, \varphi_0, \varphi_1)$, consisting of **holomorphic vector bundles** $E_0$ and $E_1$ over $X$ of ranks $n_0$ and $n_1$, respectively with $\det E_1 = (\det E_0)^{-1}$, and **homomorphisms**

$$
\varphi_0 : E_0 \to E_1 \otimes K \quad \text{and} \quad \varphi_1 : E_1 \to E_0 \otimes K.
$$
This is represented by the \( K \)-twisted quiver bundle

\[
E_0 \xrightarrow{\varphi_0} E_1, \\
E_1 \xrightarrow{\varphi_1} E_0
\]

To \((E_0, E_1, \varphi_0, \varphi_1)\) be we can associate the \( \text{SL}(n, \mathbb{C}) \)-Higgs bundle \((E, \varphi)\) with

\[
E = E_0 \oplus E_1 \quad \text{and} \quad \varphi = \begin{pmatrix} 0 & \varphi_1 \\ \varphi_0 & 0 \end{pmatrix}.
\]

This defines a 2 : 1 morphism \( \mathcal{M}(G^\theta, g_1) \to \mathcal{M}(G) \), whose image is in the fixed point locus of the involution of \( \mathcal{M}(G) \) defined by \((E, \varphi)\) to \((E, -\varphi)\)

\( \mathcal{M}(G^\theta, g_1) \) is homeomorphic to the \( \text{SU}(n_0, n_1) \)-character variety of \( \pi_1(X) \).
Toledo invariant associated to \((E_0, E_1, \varphi_0, \varphi_1)\):

\[ d = \deg E_0 = -\deg E_1. \]

A main result ([BGG], 2003) is that if \((E_0, E_1, \varphi_0, \varphi_1)\) is semistable then

\[ -\text{rank}(\varphi_1)(g - 1) \leq d \leq \text{rank}(\varphi_0)(g - 1) \]

which implies the Milnor–Wood inequality

\[ |d| \leq \min\{n_0, n_1\}(g - 1). \]

This is proved by Hitchin (1987) for \(G = \text{SL}(2, \mathbb{C})\).
Higher order case \((m > 2)\)

- A \((G^\theta, g_1)\)-Higgs pair over \(X\) is equivalent to a \(K\)-twisted quiver bundle over \(X\)

\[
\begin{array}{c}
E_0 & \xleftarrow{\varphi_0} & E_1 & \xrightarrow{\varphi_1} & \ldots & \xrightarrow{\varphi_{m-2}} & E_{m-1}, \\
& & \hspace{1.5cm} \varphi_{m-1} & & \end{array}
\]

- To this we can associate a \(G\)-Higgs bundle \((E, \varphi)\) with

\[
E = E_0 \oplus \cdots \oplus E_{m-1} \quad \text{and} \quad \varphi = \begin{pmatrix} 0 & 0 & \ldots & \varphi_{m-1} \\ \varphi_0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & \varphi_{m-2} & 0 \end{pmatrix}.
\]

- This defines a finite morphism \(\mathcal{M}(G^\theta, g_1) \to \mathcal{M}(G)\).
The image is in the fixed point locus of the action of the group $\mu_m$ of $m$-th roots of unity on $\mathcal{M}(G)$ defined by $(E, \varphi) \mapsto (E, \zeta \varphi)$, where $\zeta$ is a primitive $m$-th root of unity.

**Question:** Are there in this situation analogues of the Toledo invariant and the Milnor–Wood inequality like in the $m = 2$ case?

To approach this question we first observe that when $\varphi_{m-1} = 0$ we obtain what is known as a **chain**. These are fixed points of $\mathcal{M}(G)$ for the action of $\mathbb{C}^*$. Their moduli have been extensively studied (Álvarez-Cónsul–G–Schmitt, 2006; G–Heinloth, 2013, G–Heinloth–Schmitt, 2014).

For chains there is indeed a Toledo invariant and a Milnor–Wood type inequality. To explain this we will take a general point of view, considering **Hodge bundles** for any semisimple $G$. This has been studied recently by Biquard–Collier–G–Toledo (2021).
- $G$ semisimple complex Lie group with Lie algebra $\mathfrak{g}$ and Killing form $B$.
- A $\mathbb{Z}$-grading of $\mathfrak{g}$ is a decomposition
  $$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$$
  such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$.

There is an element $\zeta \in \mathfrak{g}_0$ (grading element) such that
$$\mathfrak{g}_i = \{X \in \mathfrak{g} \mid [\zeta, x] = ix\}$$

- Having a $\mathbb{Z}$-grading on $\mathfrak{g}$ is equivalent to having a homomorphism $\psi : \mathbb{C}^* \rightarrow \text{Aut}(\mathfrak{g})$, defined by
  $$\psi(z)|_{\mathfrak{g}_i} = z^i I.$$  

- Let $G_0 < G$ be the centralizer of $\zeta$; $G_0$ acts on each $\mathfrak{g}_i$.
  Important result due to Vinberg (1975): For $i \neq 0$, $\mathfrak{g}_i$ is a prehomogeneous vector space for $G_0$. This means that $\mathfrak{g}_i$ (for $i \neq 0$) has a unique open dense $G_0$-orbit.
Without loss of generality, we can consider the prehomogeneous vector space \((G_0, \mathfrak{g}_1)\). Let \(\Omega \subset \mathfrak{g}_1\) be the open \(G_0\)-orbit.

Since \(\mathfrak{g}_0\) is the centralizer of \(\zeta\), \(B(\zeta, -): \mathfrak{g}_0 \to \mathbb{C}\) defines a character. The **Toledo character** \(\chi_T : \mathfrak{g}_0 \to \mathbb{C}\) is defined by

\[
\chi_T(x) = B(\zeta, x)B(\gamma, \gamma),
\]

where \(\gamma\) is the longest root such that \(\mathfrak{g}_\gamma \subset \mathfrak{g}_1\).

Let \(e \in \mathfrak{g}_1\) and \((h, e, f)\) be an \(\mathfrak{sl}_2\)-triple with \(h \in \mathfrak{g}_0\). We define the **Toledo rank** of \(e\) by

\[
\text{rk}_T(e) = \frac{1}{2} \chi_T(h),
\]

and the **Toledo rank** of \((G_0, \mathfrak{g}_1)\) by

\[
\text{rk}_T(G_0, \mathfrak{g}_1) = \text{rk}_T(e) \quad \text{for } e \in \Omega.
\]
For a \( \mathbb{Z} \)-grading we consider \((G_0, \mathfrak{g}_i)\)-Higgs pairs over \( X \). Let \((E, \varphi)\) be a \((G_0, \mathfrak{g}_i)\)-Higgs pair. Extending the structure group defines a \( G \)-Higgs bundle \((E_G, \varphi)\), where \( E_G = E \times_{G_0} G \), and we use \( E(\mathfrak{g}_i) \subset E_G(\mathfrak{g}) \).

A \( G \)-Higgs bundle \((E, \varphi)\) is called a Hodge bundle of type \((G_0, \mathfrak{g}_i)\) if it reduces to a \((G_0, \mathfrak{g}_i)\)-Higgs pair.

A result of Simpson (1992) states that the \( \mathbb{C}^* \)-fixed points in the moduli space of \( G \)-Higgs bundles (under the action of rescaling the Higgs field) are Hodge bundles for some \( \mathbb{Z} \)-grading.

Via de non-abelian Hodge correspondence, Hodge bundles correspond to holonomies of complex variations of Hodge structure.
Let \((E, \varphi)\) be a \((G_0, \mathfrak{g}_1)\)-Higgs pair and \(\chi_T : \mathfrak{g}_0 \rightarrow \mathbb{C}\) be the Toledo character associated to \((G_0, \mathfrak{g}_1)\).

For a **rational number** \(q\) sufficiently large \(q\chi_T\) lifts to a character \(\tilde{\chi}_T : G_0 \rightarrow \mathbb{C}^*\).

The **Toledo invariant** \(\tau(E, \varphi)\) is defined by

\[
\tau(E, \varphi) = \frac{1}{q} \deg\tilde{\chi}_T(E).
\]

**Arakelov–Milnor inequality** ([BCGT], 2021): If \((E, \varphi)\) is semistable, then

\[-\text{rk}_T(\varphi)(2g - 2) \leq \tau(E, \varphi) \leq 0,\]

where \(\text{rk}_T(\varphi) = \text{rk}_T(\varphi(x))\) for a generic \(x \in X\).

In particular,

\[-\text{rk}_T(G_0, \mathfrak{g}_1)(2g - 2) \leq \tau(E, \varphi) \leq 0.\]
Let $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ be a $\mathbb{Z}$-grading with grading element $\zeta \in \mathfrak{g}_0$, and let $G_0 < G$ be the centralizer of $\zeta$.

Let $m > 1$ be the greatest integer in the for which $\mathfrak{g}_{m-1} \neq 0$ and $\mathfrak{g}_i = 0$ for every $i \geq m$. One has the $\mathbb{Z}/m$-grading

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}/m} \mathfrak{g}_i, \text{ with } \mathfrak{g}_i = \mathfrak{g}_i \oplus \mathfrak{g}_{i-m}.$$ 

Assume that the element of $\text{Aut}(\mathfrak{g})$ giving this $\mathbb{Z}/m$-grading lifts to an automorphism $\theta$ of $G$, and that $G^\theta = G_0$.

We want to consider the Vinberg $\theta$-pair $(G_0, \mathfrak{g}_1)$ and study $(G_0, \mathfrak{g}_1)$-Higgs pairs over $X$. These correspond to particular type of cyclic Higgs bundles related to Hodge bundles.
Let \((E, \varphi)\) be a \((G_0, \mathfrak{g}_1)\)-Higgs pair over \(X\). We have that 
\[E(\mathfrak{g}_1) = E(\mathfrak{g}_1) \oplus E(\mathfrak{g}_{1-m}),\]
and hence \(\varphi = \varphi^+ + \varphi^-\) with
\[\varphi^+ \in H^0(X, E(\mathfrak{g}_1) \otimes K) \quad \text{and} \quad \varphi^- \in H^0(X, E(\mathfrak{g}_{1-m}) \otimes K).
\]

By means of the Toledo characters of \((G_0, \mathfrak{g}_1)\) and \((G_0, \mathfrak{g}_{1-m})\) one has corresponding Toledo invariants \(\tau^+\) and \(\tau^-\) for \((E, \varphi^+, \varphi^-)\), and we have
\[
\tau^+ \geq -\text{rk}_T(\varphi^+)(2g-2) \quad \text{if} \quad \varphi^- = 0
\]
and
\[
\tau^- \geq -\text{rk}_T(\varphi^-)(2g-2) \quad \text{if} \quad \varphi^+ = 0.
\]

**Question:** Do we have these inequalities when both \(\varphi^+ \neq 0\) and \(\varphi^- \neq 0\)?
One has 

\[ \tau^- = (1 - m) \frac{B(\gamma^-, \gamma^-)}{B(\gamma^+, \gamma^+)} \tau^+ \]

where \( B \) is the Killing form of \( \mathfrak{g} \) and \( \gamma^+ \) and \( \gamma^- \) are the longest roots in \( \mathfrak{g}_1 \) and \( \mathfrak{g}_{1-m} \) respectively. Then if \( \tau := \tau^+ \) we are asking if the semistability of \((E, \varphi^+, \varphi^-)\) implies

\[ -\text{rk}_T(\varphi^+)(2g-2) \leq \tau \leq \frac{1}{m-1} \frac{B(\gamma^+, \gamma^+)}{B(\gamma^-, \gamma^-)} \text{rk}_T(\varphi^-)(2g-2). \]

This is indeed the case in the \textbf{Hermitian situation} where the \( \mathbb{Z} \)-grading is

\[ \mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \quad \text{and} \quad \mathfrak{g}_1 = \mathfrak{g}_1 \oplus \mathfrak{g}_{-1} \]

where one has the \textbf{Milnor–Wood inequality}

\[ -\text{rk}_T(\varphi^+)(2g-2) \leq \tau \leq \text{rk}_T(\varphi^-)(2g-2). \]

proved by \textbf{Biquard-G-Rubio} (2017)
Back to chains and cyclic quivers

- $G = \text{SL}(n, \mathbb{C})$ and consider the cyclic $K$-twisted quiver bundle

\[ E_0 \xleftarrow{\varphi_0} E_1 \xrightarrow{\varphi_1} \ldots \xrightarrow{\varphi_{m-2}} E_{m-1}. \]

Here $\mathfrak{g}_1$ of the $\mathbb{Z}$-grading is represented by the chain and $\mathfrak{g}_{1}$ of the corresponding $\mathbb{Z}/m$-grading is represented by the cyclic quiver, and $\varphi^+ = (\varphi_0, \ldots, \varphi_{m-2})$ and $\varphi^- = \varphi_{m-1}$.

- Joint with Miguel González we have shown a positive answer to our question when $\text{rk} \ E_i = k$ for every $i$ and hence $n = mk$. In this situation we have proved that the semistability of the quiver bundle with both $\varphi^+ \neq 0$ and $\varphi^- \neq 0$ implies

$$
\tau^+ \geq - \text{rk}_T(G_0, \mathfrak{g}_1)(2g - 2) = (2g - 2) \frac{km(m - 1)(m + 1)}{3}.
$$

Oscar García-Prada  ICMAT-CSIC, Madrid  Vinberg $\theta$-pairs and Higgs bundles
Let $\theta \in \text{Aut}(G)$ be of order $m$. Consider the $\mathbb{Z}/m$-grading defined by $\theta$:

$$g = \bigoplus_{i \in \mathbb{Z}/m} g_i,$$

and the Vinberg $\theta$-pair $(G^\theta, g_1)$.

Consider the moduli space $\mathcal{M}(G^\theta, g_1)$ of $(G^\theta, g_1)$-Higgs pairs over $X$.

Recall

$$\mathbb{C}[g_1]^{G^\theta} \to \mathbb{C}[a]^{W(a)} = \mathbb{C}[f_1, \ldots, f_r],$$

where $r = \dim a = \text{rk}(G^\theta, g_1)$. Let $d_i = \deg f_i$.

Evaluating the polynomials $f_i$ on the Higgs field we have the Hitchin map:

$$h : \mathcal{M}(G^\theta, g_i) \to B(G^\theta, g_1) \cong \bigoplus_{i=1}^r H^0(X, K^{d_i}),$$
Vinberg $\theta$-pairs and the Hitchin fibration

$m = 1$ (Adjoint representation).

- $h$ is the usual Hitchin map (Hitchin, 1987):

$$\mathcal{M}(G) \to B(G) \cong \bigoplus_{i=1}^{r} H^0(X, K^{d_i})$$

Here $r = \text{rk} G$ and $\{d_1, \cdots, d_r\}$ are the exponents of $G$.

- **Hitchin** (1987): Spectral curve description of the generic fibres for the classical groups (Jacobian/Prym varieties): Hitchin integrable system.


- The Donagi–Gaitsgory approach was reformulated by Ngô (2010) in his proof of the Fundamental Lemma.

- **Hitchin** (1992) constructed a section of the Hitchin map which can be identified with a connected component of the character variety for a split real form of $G$: Hitchin component (instance of higher Teichmüller space).
$m = 2$ (symmetric pairs).

- We get the Hitchin map for the moduli space of Higgs bundles for the real form $G^\sigma$ ($\sigma = \tau \theta$)
- **Schaposnik** (2013): Spectral curve approach for classical real forms.
- **Peón-Nieto** (2013): Cameral curve approach for arbitrary real forms.
- From both points of view one can see that the generic fibres are **abelian if and only if the real form is quasi-split**. Non-abelianization phenomenon very nicely illustrated for certain real forms by **Hitchin–Schaposnik** (2014).
- A construction of the **gerbe in the quasi-split** case following the Donagi–Gaitsgory approach given by **G–Peón-Nieto** (2021).
- A section of the Hitchin map in this case was constructed by **G–Peón-Nieto–Ramanan** (2018): Hitchin–Kostant–Rallis section.
$m > 2$ (general Vinberg $\theta$-pairs). Joint work in progress with Miguel González. At the moment there are mostly questions:

- When do we get abelianization? Generalization of the notion of quasi-split. We proposed that $C_{g_0}(\mathfrak{a})$ be abelian. Two days after a discussion with Alastair in Madrid last September on the particular case of cyclic quiver bundles

$$E_0 \xrightarrow{\varphi_0} E_1 \xrightarrow{\varphi_1} \ldots \xrightarrow{\varphi_{m-2}} E_{m-1},$$

Alastair suggested that if $k$ is the minimal rank of the $E_i$, then the ranks should be at most $k + 1$. Indeed this satisfied our condition!

- In the case when all the ranks are equal we have a spectral description of the generic fibres generalizing the one given by Schaposnik (2015) in the involutive case for $U(k,k)$. 

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Vinberg $\theta$-pairs and Higgs bundles
HAPPY BIRTHDAY, ALASTAIR!

Guanajuato, 2006 (photo by Leticia Brambila)