Vinberg θ-pairs and Higgs bundles

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On the occasion of Alastair King's 60th birthday

Guanajuato, 2006 (photo by Leticia Brambila)

Symposium on Vector Bundles in Algebraic Geometry
Durham, 1993 (photo by Steve Bradlow)

Vinberg θ-pairs

- G semisimple complex Lie group with Lie algebra $\mathfrak{g}$
- $\theta : G \to G$ **order** $m > 0$ **holomorphic automorphism**
- θ defines an automorphism of $\mathfrak{g}$ (denoted also by θ), determining a $\mathbb{Z}/m$-**grading** of $\mathfrak{g}$ (we write $\mathbb{Z}/m$ instead of $\mathbb{Z}/m\mathbb{Z}$) :

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}/m} \mathfrak{g}_i \quad \text{with} \quad \mathfrak{g}_i = \{x \in \mathfrak{g} \ \text{ such that } \ \theta(x) = \zeta^i x\},$$

where ζ is a primitive m-th root of unity. One has

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}.$$

Vinberg θ-pairs

- Let $\mu_m = \{z \in \mathbb{C}^* \text{ such that } z^m = 1\}$. Having $\mathbb{Z}/m$-grading on $\mathfrak{g}$ is equivalent to having a homomorphism

$$\tilde{\theta} : \mu_m \to \mathrm{Aut}(\mathfrak{g}).$$

  Then $\mathfrak{g}_i = \{x \in \mathfrak{g} \text{ such that } \tilde{\theta}(z)x = z^i x\}$ for every $z \in \mu_m$.
- Let $G^\theta < G$ be the **fixed point subgroup**. G^θ is a reductive group with Lie algebra $\mathfrak{g}_0$.
- Since $[\mathfrak{g}_0, \mathfrak{g}_i] \subset \mathfrak{g}_i$, all the subspaces $\mathfrak{g}_i$ are stable under the adjoint action of G^θ.
- The pairs $(G^\theta, \mathfrak{g}_i)$ are called **Vinberg θ-pairs** (also **θ-groups** or **Vinberg θ-representations**).
- Sometimes this term is used for the pairs $(G_0, \mathfrak{g}_i)$, where $G_0 = G_0^\theta$ is the connected component of the identity.

- **Example 1: Adjoint representation**. If $m = 1$, then θ is the identity automorphism and $(G, \mathfrak{g})$ is the only Vinberg θ-pair.
- **Example 2: Symmetric pairs**. Let $m = 2$: $\mathbb{Z}/2$**-grading**

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

theory of **symmetric spaces and real forms** of $\mathfrak{g}$ and G.

A **real form** of $G^\sigma < G$ is the fixed point subgroup of a **conjugation** (antiholomorphic involution) σ of G.

Cartan: Given a holomorphic involution θ of G there is a **compact conjugation** τ of G, so that $\sigma := \tau\theta = \theta\tau$ is a conjugation of G. This gives a bijection

$$\mathrm{Aut}_2(G)/\sim \longleftrightarrow \mathrm{Conj}(G)/\sim,$$

where equivalence is conjugation by an inner automorphism of G.

- **Example 3: Cyclic quivers.** Let $m \geq 2$. Let V be a complex vector space equipped with a $\mathbb{Z}/m$-grading

$$V = \bigoplus_{i \in \mathbb{Z}/m} V_i.$$

Let $G = \mathrm{SL}(V)$. Define on $\mathfrak{g} = \mathfrak{sl}(V)$ the $\mathbb{Z}/m$**-grading** given by

$$\mathfrak{g}_i = \{A \in \mathfrak{sl}(V) \text{ such that } A(V_j) \subset V_{j+i} \text{ for every } j \in \mathbb{Z}/m\}$$

In this situation

$$G^\theta = \mathrm{S}(\prod_{i \in \mathbb{Z}/m} \mathrm{GL}(V_i)),$$

and

$$\mathfrak{g}_1 = \bigoplus_{i \in \mathbb{Z}/m} \mathrm{Hom}(V_i, V_{i+1}).$$

Vinberg θ-pairs

- Define the **quiver** Q with m **vertices** indexed by $\mathbb{Z}/m$ and **arrows** $i \mapsto i+1$ for each $i \in \mathbb{Z}/m$.
  Then $\mathfrak{g}_1$ is the **space of representations of** Q where we put V_i at the vertex i:
  This can be represented by the diagramme

$$\begin{tikzcd} V_0 \arrow[r, "f_0"] & V_1 \arrow[r, "f_1"] & \dots \arrow[r, "f_{m-2}"] & V_{m-1}. \arrow[lll, bend left, "f_{m-1}"] \end{tikzcd}$$

- For other classical groups the action of G^θ on $\mathfrak{g}_1$ can be interpreted in terms of a cyclic quiver with some extra structure.

- For a moment we will consider Vinberg θ-pairs $(G_0, \mathfrak{g}_1)$ where $G_0 = G_0^\theta$.
  The study for other $\mathfrak{g}_i$ can be reduced to the case of $\mathfrak{g}_1$.
- Recall that if $\mathfrak{t} \subset \mathfrak{g}$ is a Cartan subalgebra, and $W(\mathfrak{t})$ is the Weyl group, the **Chevalley restriction theorem** establishes an isomorphism

$$\mathfrak{g} /\!\!/ G \cong \mathfrak{t}/W(\mathfrak{t}).$$

- Similarly, if θ is an **involution** of G and $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is the **Cartan decomposition** defined by θ, and $W(\mathfrak{a})$ is the **little Weyl group** defined by a maximal abelian subalgebra $\mathfrak{a} \subset \mathfrak{g}_1$, there is also a **Chevalley restriction theorem** studied by **Kostant–Rallis** (1971):

$$\mathfrak{g}_1 /\!\!/ G_0 \cong \mathfrak{a}/W(\mathfrak{a}).$$

- A **main result** of Vinberg's theory (1976) is a **version of the Chevalley restriction theorem** for Vinberg θ-pairs.
- Key concept is that of **Cartan subspace**: linear subspace $\mathfrak{a} \subset \mathfrak{g}_1$ which is abelian as a Lie algebra, consisting of semisimple elements, and maximal with these two properties.
- The **little Weyl group**

$$W(\mathfrak{a}) = N_{G_0}(\mathfrak{a})/C_{G_0}(\mathfrak{a})$$

is a finite linear group generated by semisimple transformations of $\mathfrak{a}$ fixing a hyperplane. Hence $\mathbb{C}[\mathfrak{a}]^{W(\mathfrak{a})}$ is a **polynomial ring**, and the restriction of polynomial functions $\mathbb{C}[\mathfrak{g}_1] \to \mathbb{C}[\mathfrak{a}]$ induces an **isomorphism of invariant polynomial rings** $\mathbb{C}[\mathfrak{g}_1]^{G_0} \to \mathbb{C}[\mathfrak{a}]^{W(\mathfrak{a})}$, or equivalently,

$$\mathfrak{g}_1 /\!\!/ G_0 \cong \mathfrak{a}/W(\mathfrak{a}).$$

- The fact that $W(\mathfrak{a})$ is a finite linear group generated by complex reflections implies that $\mathbb{C}[\mathfrak{a}]^{W(\mathfrak{a})} = \mathbb{C}[f_1, \cdots, f_r]$ is a polynomial algebra generated by r **algebraically independent polynomials** $f_1, \cdots, f_r$ whose degrees $d_1, \cdots, d_r$ are determined by the grading. Here r is the dimension of $\mathfrak{a}$, an invariant called the **rank of** $(G_0, \mathfrak{g}_1)$.
- Kostant (1963) showed that the quotient map $\mathfrak{g} \to \mathfrak{g} /\!\!/ G$ has a section, known as the **Kostant section**.
- This was extended by Kostant–Rallis (1971) to obtain the **Kostant–Rallis section** in the symmetric pair case for the quotient map $\mathfrak{g}_1 \to \mathfrak{g}_1 /\!\!/ G_0$.
- The existence of a similar section for Vinberg's θ-pairs for θ of higher order was conjectured by Popov (1976), but only proved more recently in full generality by Reeder–Levy–Yu–Gross (2012). In this context, such a section is referred as a **Kostant–Weirstrass section**.

Vinberg θ-pairs

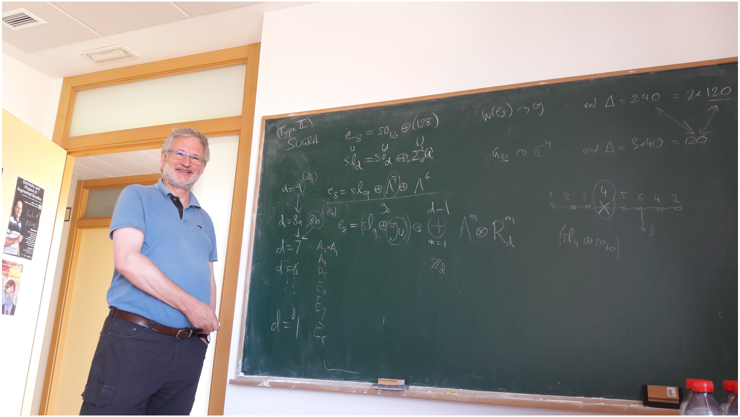
The theory of Vinberg θ-pairs has been extended to general fields of 0 characteristic and good positive characteristic not dividing the order of θ (Levy).

Applications of Vinberg's theory include the following:

- Classification of **trivectors** of 9-dimensional space $\mathbb{C}^9$ by Elashvili–Vinberg (1978) using a $\mathbb{Z}/3$-grading of $\mathfrak{e}_8$: $\mathfrak{e}_8 = \mathfrak{sl}(9,\mathbb{C}) \oplus \Lambda^3(\mathbb{C}^9) \oplus \Lambda^6(\mathbb{C}^9)$.
- Certain Vinberg θ-pairs have interesting connections to the arithmetic theory of elliptic curves and Jacobians: Bhargava–Shankar (2010), Bhargava–Gross (2012), ...
- There is also a connection between the Vinberg θ-pair $(\mathrm{SL}(9,\mathbb{C})/\mu_3, \Lambda^3(\mathbb{C}^9))$ coming from the $\mathbb{Z}/3$-grading of $\mathfrak{e}_8$ and the moduli space of genus 2 curves with some additional data: Rains–Sam (2016, 2018).
- Vinberg pairs are also used in the description of the moduli space of vector bundles on curves of small genus ($g = 2, 3$).

Vinberg θ-pairs

- Vinberg's theory has also connections to physics related to del Pezzo surfaces: **mysterious duality** (Iqbal–Neitzke–Vafa, 2001)



Madrid, 2022

- The goal of this talk is to discuss the role of Vinberg θ-pairs in **Higgs bundle theory**.
- Part of it will be a review of some well-known results put in the larger context of Vinberg's theory.
- Then I will introduce some new problems and work in progress joint with my student **Miguel González** in the study of the geometry of moduli space of **cyclic Higgs bundles**.

Higgs pairs

- X compact Riemann surface of genus $g \geq 2$ with canonical line bundle K
- G reductive complex Lie group with Lie algebra $\mathfrak{g}$
- $\rho : G \to \mathrm{GL}(V)$ a representation of G in a complex vector space V
- A **(G, V)-Higgs pair on** X is a pair (E, φ) consisting of a holomorphic principal G-bundle $E \to X$ and $\varphi \in H^0(X, E(V) \otimes K)$, where $E(V) = E \times_G V$ is the vector bundle associated to the representation ρ.
- There are suitable notions of (semi,poly) stability. Consider the **moduli space of polystable (G, V)-Higgs pairs**:

$$\mathcal{M}(G, V)$$

- When ρ is the **adjoint representation** $G \to \mathrm{GL}(\mathfrak{g})$ $(G, \mathfrak{g})$-Higgs pairs are the G-**Higgs bundles** introduced by **Hitchin** (1987).

  $\mathcal{M}(G)$: **moduli space of polystable** G**-Higgs bundles**

Vinberg θ-pairs and cyclic Higgs bundles

- From now on we assume that G is semisimple
- Let $\theta \in \mathrm{Aut}(G)$ be of order m. Consider the $\mathbb{Z}/m$-grading defined by θ:

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}/m} \mathfrak{g}_i,$$

  and the Vinberg θ-pairs $(G^\theta, \mathfrak{g}_i)$.
- Let $\mathcal{M}(G^\theta, \mathfrak{g}_i)$ be the **moduli space of $(G^\theta, \mathfrak{g}_i)$-Higgs pairs over X.**
- The moduli spaces associated to Vinberg pairs do appear naturally inside the moduli space $\mathcal{M}(G)$ of G-Higgs bundles as fixed point subvarieties for a certain action of a cyclic group.

  This is studied in joint paper with **S. Ramanan** (2019).

- $\mathrm{Aut}(G)$ acts on $\mathcal{M}(G)$: For $\alpha \in \mathrm{Aut}(G)$ and a G-Higgs bundle (E, φ) we defined

$$\alpha \cdot (E, \varphi) := (\alpha(E), \alpha(\varphi)) \text{ where } \alpha(E) = E \times_\alpha G.$$

  This descends to an action of $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Int}(G)$
- $\mathbb{C}^*$ acts on $\mathcal{M}(G)$ by rescaling the Higgs field.
- Let $\mu_m = \{z \in \mathbb{C}^* \text{ such that } z^m = 1\}$ and let $\zeta \in \mu_m$ be a **primitive m-th root of unity**.
  Consider the homomorphism $\mu_m \to \mathrm{Aut}(G) \times \mathbb{C}^*$ defined by $\zeta \mapsto (\theta, \zeta)$. Let Γ be the image.
- Γ is isomorphic to μ_m and acts on $\mathcal{M}(G)$ by the rule

$$(E, \varphi) \mapsto (\theta(E), \zeta\theta(\varphi)).$$

- **Extension of structure group** defines a finite map

$$\mathcal{M}(G^\theta, \mathfrak{g}_1) \to \mathcal{M}(G).$$

Denote the image by $\widetilde{\mathcal{M}}(G^\theta, \mathfrak{g}_1)$, then

$$\widetilde{\mathcal{M}}(G^\theta, \mathfrak{g}_1) \subset \mathcal{M}(G)^\Gamma.$$

- Since the action of θ depends only on the class of θ in $\mathrm{Out}(G)$, there are other subvarieties in $\mathcal{M}(G)^\Gamma$.
- Let $\mathrm{Aut}_m(G) \subset \mathrm{Aut}(G)$ the be set of elements of order m. There is a map

$$\mathcal{cl} : \mathrm{Aut}_m(G)/\sim \to \mathrm{Out}_m(G)$$

called the clique map. For an element $a \in \mathrm{Out}_m(G)$ we refer to the set $\mathcal{cl}_m^{-1}(a)$ as the **clique** defined by a.

- To identify $\mathcal{cl}_m^{-1}(a)$, let $Z = Z(G)$ be the centre of G and
$$S_\theta := \{s \in G \ : \ s\theta(s)\cdots\theta^{m-1}(s) = z \in Z\}.$$
There is an action of Z on S_θ by multiplication and of G given by $s \cdot g := g^{-1}s\theta(g) = \quad g \in G, s \in S_\theta$, and
$$S_\theta/(Z \times G) = H^1(\mathbb{Z}/m, \mathrm{Ad}(G)),$$
where $H^1(\mathbb{Z}/m, \mathrm{Ad}(G))$ is the first **Galois cohomology** set.
- The map $S_\theta \to \mathrm{Aut}_m(G)$ given by $s \mapsto \mathrm{Int}_s\,\theta$ defines a bijection
$$S_\theta/(Z \times G) \longleftrightarrow \mathcal{cl}^{-1}(a).$$

- $\mathcal{M}(G)^\Gamma$ contains also $\widetilde{\mathcal{M}}(G_{\theta'}, \mathfrak{g}_1')$ for $\theta' \in S_\theta$, and the subvariety $\widetilde{\mathcal{M}}(G_{\theta'}, \mathfrak{g}_1') \subset \mathcal{M}(G)$ depends only on the class $[\theta'] \in S_\theta/(Z \times G)$.
- There is a **converse** to this result if we consider the **smooth locus** $\mathcal{M}_*(G) \subset \mathcal{M}(G)$:

$$\mathcal{M}_*(G)^\Gamma \subset \bigcup_{[\theta'] \in S_\theta/(Z\times G)} \widetilde{\mathcal{M}}(G^{\theta'}, \mathfrak{g}_1').$$

- The elements in $\widetilde{\mathcal{M}}(G^\theta, \mathfrak{g}_1)$ are called **cyclic G-Higgs bundles** and have been studied by many authors.
- The moduli spaces $\mathcal{M}(G^\theta, \mathfrak{g}_i)$ for general $\mathfrak{g}_i$ in the $\mathbb{Z}/m$-grading of $\mathfrak{g}$ do also show up as fixed point in $\mathcal{M}(G)$. But now the homomorphism $\mu_m \to \mathrm{Aut}(G) \times \mathbb{C}^*$ is defined by $\zeta \mapsto (\theta, \zeta^i)$ and consider the action of the image Γ.

- Recall that is $\mathcal{R}(G)$ is the G-**character variety** of the fundamental group of X defined as

$$\mathcal{R}(G) = \mathrm{Hom}(\pi_1(X), G) /\!\!/ G,$$

the **non-abelian Hodge correspondence** establishes a homeomorphism

$$\mathcal{M}(G) \cong \mathcal{R}(G).$$

- If $m = 2$, θ defines the $\mathbb{Z}/2$-grading $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, and there is also a homeomorphism

$$\mathcal{M}(G^\theta, \mathfrak{g}_1) \cong \mathcal{R}(G^\sigma),$$

where G^σ is the real form of G defined the conjugation $\sigma = \theta\tau$, with τ a compact conjugation commuting with θ.
- The subvarieties $\widetilde{\mathcal{M}}(G^\theta, \mathfrak{g}_1)$ are in the fixed point locus for the action of $\mathbb{Z}/2$ on $\mathcal{M}(G)$ sending $(E, \varphi) \mapsto (\theta(E), -\theta(\varphi))$ and define **Lagrangian subvarieties** of $\mathcal{M}(G)$.

Involutive case $(m=2)$ — **Bradlow-G-Gothen** (2003)

- $V = V_0 \oplus V_1, \quad n = \dim V,\ n_0 = \dim V_0,\ n_1 = \dim V_1$
  $G = \mathrm{SL}(n,\mathbb{C}) \quad$ and $G^\theta = \mathrm{S}(\mathrm{GL}(V_0) \times \mathrm{GL}(V_1))$
  $\mathfrak{g}_1 = \mathrm{Hom}(V_0, V_1) \oplus \mathrm{Hom}(V_1, V_0).$
  The pair $(G^\theta, \mathfrak{g}_1)$ is described by the representations of the quiver

$$\begin{tikzcd} V_0 \arrow[r, bend left, "f_0"] & V_1 . \arrow[l, bend left, "f_1"] \end{tikzcd}$$

- A $(G^\theta, \mathfrak{g}_1)$-Higgs pair over X is equivalent to a 4-**tuple** $(E_0, E_1, \varphi_0, \varphi_1)$, consisting of **holomorphic vector bundles** E_0 and E_1 over X of ranks n_0 and n_1, respectively with $\det E_1 = (\det E_0)^{-1}$, and **homomorphisms**

$$\varphi_0 : E_0 \to E_1 \otimes K \quad \text{and} \quad \varphi_1 : E_1 \to E_0 \otimes K.$$

Cyclic $\mathrm{SL}(n,\mathbb{C})$-Higgs bundles for inner automorphisms

- This is represented by the K-**twisted quiver bundle**

$$\begin{tikzcd} E_0 \arrow[r, bend left, "\varphi_0"] & E_1, \arrow[l, bend left, "\varphi_1"] \end{tikzcd}$$

- To $(E_0, E_1, \varphi_0, \varphi_1)$ be we can associate the $\mathrm{SL}(n,\mathbb{C})$-**Higgs bundle** (E,φ) with

$$E = E_0 \oplus E_1 \quad \text{and} \quad \varphi = \begin{pmatrix} 0 & \varphi_1 \\ \varphi_0 & 0 \end{pmatrix}.$$

- This defines a $2:1$ morphism $\mathcal{M}(G^\theta, \mathfrak{g}_1) \to \mathcal{M}(G)$, whose image is in the fixed point locus of the involution of $\mathcal{M}(G)$ defined by (E,φ) to $(E,-\varphi)$
- $\mathcal{M}(G^\theta, \mathfrak{g}_1)$ is homeomorphic to the $\mathrm{SU}(n_0,n_1)$-**character variety** of $\pi_1(X)$.

- **Toledo invariant** associated to $(E_0, E_1, \varphi_0, \varphi_1)$:

$$d = \deg E_0 = -\deg E_1.$$

- A **main result** ([BGG], 2003) is that if $(E_0, E_1, \varphi_0, \varphi_1)$ is **semistable** then

$$-\operatorname{rank}(\varphi_1)(g-1) \leq d \leq \operatorname{rank}(\varphi_0)(g-1)$$

which implies the **Milnor–Wood inequality**

$$|d| \leq \min\{n_0, n_1\}(g-1).$$

- This is proved by Hitchin (1987) for $G = \mathrm{SL}(2,\mathbb{C})$.

Higher order case $(m > 2)$

- A $(G^\theta, \mathfrak{g}_1)$-Higgs pair over X is equivalent to a K-twisted quiver bundle over X

$$\begin{tikzcd} E_0 \arrow[r, "\varphi_0"] & E_1 \arrow[r, "\varphi_1"] & \dots \arrow[r, "\varphi_{m-2}"] & E_{m-1}, \arrow[lll, bend left, "\varphi_{m-1}"] \end{tikzcd}$$

- To this we can associate a G-Higgs bundle (E, φ) with

$$E = E_0 \oplus \cdots \oplus E_{m-1} \quad \text{and} \quad \varphi = \begin{pmatrix} 0 & 0 & \dots & \varphi_{m-1} \\ \varphi_0 & 0 & \dots & 0 \\ \vdots & \ddots & \dots & \vdots \\ 0 & \dots & \varphi_{m-2} & 0 \end{pmatrix}.$$

- This defines a finite morphism $\mathcal{M}(G^\theta, \mathfrak{g}_1) \to \mathcal{M}(G)$.

Cyclic SL($n,\mathbb{C}$)-Higgs bundles for inner automorphisms

- The image is in the fixed point locus of the action of the group μ_m of m-th roots of unity on $\mathcal{M}(G)$ defined by $(E,\varphi)\mapsto(E,\zeta\varphi)$, where ζ is a primitive m-th root of unity.
- **Question**: Are there in this situation analogues of the Toledo invariant and the Milnor–Wood inequality like in the $m=2$ case?
- To approach this question we first observe that when $\varphi_{m-1}=0$ we obtain what is known as a **chain**. These are fixed points of $\mathcal{M}(G)$ for the action of $\mathbb{C}^*$. Their moduli have been extensively studied (**Álvarez-Cónsul–G–Schmitt**, 2006; **G–Heinloth**, 2013, **G–Heinloth–Schmitt**, 2014).
- For chains there is indeed a Toledo invariant and a Milnor–Wood type inequality. To explain this we will take a general point of view, considering **Hodge bundles** for any semisimple G. This has been studied recently by **Biquard–Collier–G–Toledo** (2021).

- G semisimple complex Lie group with Lie algebra $\mathfrak{g}$ and Killing form B.
- A $\mathbb{Z}$-**grading** of $\mathfrak{g}$ is a decomposition

$$\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i \qquad \text{such that} \qquad [\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}.$$

  There is an element $\zeta \in \mathfrak{g}_0$ (**grading element**) such that $\mathfrak{g}_i = \{X \in \mathfrak{g} \mid [\zeta, x] = ix\}$
- Having a $\mathbb{Z}$-grading on $\mathfrak{g}$ is equivalent to having a homomorphism $\psi : \mathbb{C}^* \to \mathrm{Aut}(\mathfrak{g})$, defined by

$$\psi(z)|_{\mathfrak{g}_i} = z^i I.$$

- Let $G_0 < G$ be the centralizer of ζ; G_0 acts on each $\mathfrak{g}_i$. **Important result** due to **Vinberg** (1975): For $i \neq 0$, $\mathfrak{g}_i$ is a **prehomogeneous vector space** for G_0. This means that $\mathfrak{g}_i$ (for $i \neq 0$) has a unique open dense G_0-orbit.

- Without loss of generality, we can consider the prehomogeneous vector space $(G_0, \mathfrak{g}_1)$. Let $\Omega \subset \mathfrak{g}_1$ be the open G_0-orbit.
- Since $\mathfrak{g}_0$ is the centralizer of ζ, $B(\zeta, -) : \mathfrak{g}_0 \to \mathbb{C}$ defines a character. The **Toledo character** $\chi_T : \mathfrak{g}_0 \to \mathbb{C}$ is defined by

$$\chi_T(x) = B(\zeta, x)B(\gamma, \gamma) \ ,$$

  where γ is the longest root such that $\mathfrak{g}_\gamma \subset \mathfrak{g}_1$.
- Let $e \in \mathfrak{g}_1$ and (h, e, f) be an $\mathfrak{sl}_2$-triple with $h \in \mathfrak{g}_0$. We define the **Toledo rank** of e by

$$\mathrm{rk}_T(e) = \frac{1}{2}\chi_T(h),$$

  and the **Toledo rank** of $(G_0, \mathfrak{g}_1)$ by

$$\mathrm{rk}_T(G_0, \mathfrak{g}_1) = \mathrm{rk}_T(e) \quad \text{for } e \in \Omega.$$

$\mathbb{Z}$-gradings and Hodge bundles

- For a $\mathbb{Z}$-grading we consider $(G_0, \mathfrak{g}_i)$-**Higgs pairs** over X. Let (E, φ) be a $(G_0, \mathfrak{g}_i)$-Higgs pair. Extending the structure group defines a G-Higgs bundle (E_G, φ), where $E_G = E \times_{G_0} G$, and we use $E(\mathfrak{g}_i) \subset E_G(\mathfrak{g})$.
- A G-Higgs bundle (E, φ) is called a **Hodge bundle** of type $(G_0, \mathfrak{g}_i)$ if it reduces to a $(G_0, \mathfrak{g}_i)$-Higgs pair.
- A result of **Simpson** (1992) states that the $\mathbb{C}^*$-**fixed points** in the moduli space of G-Higgs bundles (under the action of rescaling the Higgs field) are Hodge bundles for some $\mathbb{Z}$-grading.
- Via de non-abelian Hodge correspondence, Hodge bundles correspond to holonomies of **complex variations of Hodge structure**.

Hodge bundles and the Toledo invariant

- Let (E,φ) be a $(G_0,\mathfrak{g}_1)$-Higgs pair and $\chi_T:\mathfrak{g}_0\to\mathbb{C}$ be the **Toledo character** associated to $(G_0,\mathfrak{g}_1)$.
  For a **rational number** q sufficiently large $q\chi_T$ lifts to a character $\widetilde{\chi}_T:G_0\to\mathbb{C}^*$.
- The **Toledo invariant** $\tau(E,\varphi)$ is defined by

$$\tau(E,\varphi)=\frac{1}{q}\deg_{\widetilde{\chi_T}}(E).$$

- **Arakelov–Milnor inequality** ([BCGT], 2021): If (E,φ) is semistable, then $-\operatorname{rk}_T(\varphi)(2g-2)\leq\tau(E,\varphi)\leq 0$, where $\operatorname{rk}_T(\varphi)=\operatorname{rk}_T(\varphi(x))$ for a generic $x\in X$.
  In particular,

$$-\operatorname{rk}_T(G_0,\mathfrak{g}_1)(2g-2)\leq\tau(E,\varphi)\leq 0.$$

Hodge bundles and cyclic Higgs bundles

- Let $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i$ be a $\mathbb{Z}$-**grading** with grading element $\zeta \in \mathfrak{g}_0$, and let $G_0 < G$ be the centralizer of ζ.
- let $m > 1$ be the greatest integer in the for which $\mathfrak{g}_{m-1} \neq 0$ and $\mathfrak{g}_i = 0$ for every $i \geq m$. One has the $\mathbb{Z}/m$-**grading**

$$\mathfrak{g} = \bigoplus_{\underline{i}\in\mathbb{Z}/m} \mathfrak{g}_{\underline{i}}, \quad \text{with} \quad \mathfrak{g}_{\underline{i}} = \mathfrak{g}_i \oplus \mathfrak{g}_{i-m}.$$

  Assume that the element of $\mathrm{Aut}(\mathfrak{g})$ giving this $\mathbb{Z}/m$-grading lifts to an automorphism θ of G, and that $G^\theta = G_0$.
- We want to consider the **Vinberg θ-pair** $(G_0, \mathfrak{g}_{\underline{1}})$ and study $(G_0, \mathfrak{g}_{\underline{1}})$-**Higgs pairs** over X. These correspond to particular type of cyclic Higgs bundles related to Hodge bundles.

Hodge bundles and cyclic Higgs bundles

- Let (E, φ) be a $(G_0, \mathfrak{g}_{\underline{1}})$-Higgs pair Over X. We have that $E(\mathfrak{g}_{\underline{1}}) = E(\mathfrak{g}_1) \oplus E(\mathfrak{g}_{1-m})$, and hence $\varphi = \varphi^+ + \varphi^-$ with

$$\varphi^+ \in H^0(X, E(\mathfrak{g}_1) \otimes K) \text{ and } \varphi^- \in H^0(X, E(\mathfrak{g}_{1-m}) \otimes K).$$

- By means of the Toledo characters of $(G_0, \mathfrak{g}_1)$ and $(G_0, \mathfrak{g}_{1-m})$ one has corresponding Toledo invariants τ^+ and τ^- for $(E, \varphi^+, \varphi^-)$, and we have

$$\tau^+ \geq -\operatorname{rk}_T(\varphi^+)(2g-2) \text{ if } \varphi^- = 0$$

and

$$\tau^- \geq -\operatorname{rk}_T(\varphi^-)(2g-2) \text{ if } \varphi^+ = 0.$$

- **Question**: Do we have these inequalities when both $\varphi^+ \neq 0$ and $\varphi^- \neq 0$?

- One has
$$\tau^- = (1-m)\frac{B(\gamma^-,\gamma^-)}{B(\gamma^+,\gamma^+)}\tau^+$$
where B is the Killing form of $\mathfrak{g}$ and γ^+ and γ^- are the longest roots in $\mathfrak{g}_1$ and $\mathfrak{g}_{1-m}$ respectively. Then if $\tau := \tau^+$ we are asking if the semistability of $(E, \varphi^+, \varphi^-)$ implies
$$-\operatorname{rk}_T(\varphi^+)(2g-2) \leq \tau \leq \frac{1}{m-1}\frac{B(\gamma^+,\gamma^+)}{B(\gamma^-,\gamma^-)}\operatorname{rk}_T(\varphi^-)(2g-2).$$
- This is indeed the case in the **Hermitian situation** where the $\mathbb{Z}$-grading is
$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \quad \text{and} \quad \mathfrak{g}_{\underline{1}} = \mathfrak{g}_1 \oplus \mathfrak{g}_{-1}$$
where one has the **Milnor–Wood inequality**
$$-\operatorname{rk}_T(\varphi^+)(2g-2) \leq \tau \leq \operatorname{rk}_T(\varphi^-)(2g-2).$$
proved by **Biquard-G-Rubio** (2017)

Back to chains and cyclic quivers

- $G = \mathrm{SL}(n, \mathbb{C})$ and consider the cyclic K-twisted quiver bundle

$$
\begin{tikzcd}
E_0 \arrow[r, "\varphi_0"] & E_1 \arrow[r, "\varphi_1"] & \cdots \arrow[r, "\varphi_{m-2}"] & E_{m-1}. \arrow[lll, bend left, "\varphi_{m-1}"]
\end{tikzcd}
$$

  Here $\mathfrak{g}_1$ of the $\mathbb{Z}$-grading is represented by the chain and $\mathfrak{g}_{\underline{1}}$ of the correspongin $\mathbb{Z}/m$-grading is represented by the cyclic quiver, and $\varphi^+ = (\varphi_0, \cdots, \varphi_{m-2})$ and $\varphi^- = \varphi_{m-1}$,
- Joint with **Miguel González** we have shown a positive answer to our question when $\underline{\mathrm{rk}\, E_i = k \text{ for every } i}$ and hence $n = mk$. In this situation we have proved that the semistability of the quiver bundle with both $\varphi^+ \neq 0$ and $\varphi^- \neq 0$ implies

$$\tau^+ \geq -\mathrm{rk}_T(G_0, \mathfrak{g}_1)(2g-2) = (2g-2)\frac{km(m-1)(m+1)}{3}.$$

- Let $\theta \in \mathrm{Aut}(G)$ be of order m. Consider the $\mathbb{Z}/m$-grading defined by θ:
$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}/m} \mathfrak{g}_i,$$
and the Vinberg θ-pair $(G^\theta, \mathfrak{g}_1)$.
- Consider the moduli space $\mathcal{M}(G^\theta, \mathfrak{g}_1)$ of $(G^\theta, \mathfrak{g}_1)$-Higgs pairs over X.
- Recall
$$\mathbb{C}[\mathfrak{g}_1]^{G^\theta} \to \mathbb{C}[\mathfrak{a}]^{W(\mathfrak{a})} = \mathbb{C}[f_1, \cdots, f_r],$$
where $r = \dim \mathfrak{a} = \mathrm{rk}(G^\theta, \mathfrak{g}_1)$. Let $d_i = \deg f_i$
- Evaluating the polynomials f_i on the Higgs field we have the **Hitchin map**:
$$h : \mathcal{M}(G^\theta, \mathfrak{g}_i) \to B(G^\theta, \mathfrak{g}_1) \cong \bigoplus_{i=1}^{r} H^0(X, K^{d_i}),$$

$m = 1$ **(Adjoint representation)**.

- h is the usual Hitchin map (Hitchin, 1987):

$$\mathcal{M}(G) \to B(G) \cong \bigoplus_{i=1}^{r} H^0(X, K^{d_i})$$

  Here $r = \mathrm{rk}\, G$ and $\{d_1, \cdots, d_r\}$ are the **exponents** of G.
- **Hitchin** (1987): **Spectral curve description** of the generic fibres for the **classical groups** (Jacobian/Prym varieties): **Hitchin integrable system**.
- **Donagi–Gaitsgory** (2001): **Cameral curve description** for **general** G as a **gerbe** with generic abelian fibres.
- The Donagi–Gaistgory approach was reformulated by **Ngô** (2010) in his proof of the **Fundamental Lemma**.
- **Hitchin** (1992) constructed a **section** of the Hitchin map which can be identified with a connected component of the character variety for a **split real form** of G: **Hitchin component** (instance of **higher Teichmüller space**).

$m = 2$ **(symmetric pairs)**.

- We get the Hitchin map for the moduli space of Higgs bundles for the real form G^σ $(\sigma = \tau\theta)$
- **Schaposnik** (2013): **Spectral curve approach** for classical real forms.
- **Peón-Nieto** (2013): **Cameral curve approach** for arbitrary real forms.
- From both points of view one can see that the generic fibres are **abelian if and only if the real form is quasi-split**. **Non-abelianization phenomenon** very nicely illustrated for certain real forms by **Hitchin–Schaposnik** (2014).
- A construction of the **gerbe in the quasi-split** case following the Donagi–Gaitsgory approach given by **G–Peón-Nieto** (2021).
- A section of the Hitchin map in this case was constructed by **G–Peón-Nieto–Ramanan** (2018): **Hitchin–Kostant–Rallis** section.

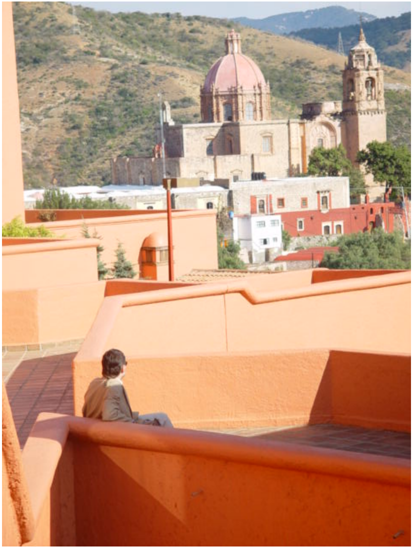
$m > 2$ **(general Vinberg θ-pairs)**. Joint work in progress with **Miguel González**. At the moment there are mostly **questions**:

- When do we get **abelianization**? Generalization of the notion of **quasi-split**. We proposed that $C_{\mathfrak{g}_0}(\mathfrak{a})$ be abelian. Two days after a discussion with **Alastair** in Madrid last September on the particular case of cyclic quiver bundles

$$\begin{tikzcd} E_0 \arrow[r, "\varphi_0"] & E_1 \arrow[r, "\varphi_1"] & \cdots \arrow[r, "\varphi_{m-2}"] & E_{m-1}, \arrow[lll, bend left, "\varphi_{m-1}"] \end{tikzcd}$$

  Alastair suggested that if k is the minimal rank of the E_i, then the ranks should be at most $k+1$. Indeed this satisfied our condition!
- In the case when all the ranks are equal we have a spectral description of the generic fibres generalizing the one given by Schaposnik (2015) in the involutive case for $\mathrm{U}(k,k)$.

HAPPY BIRTHDAY, ALASTAIR!



Guanajuato, 2006 (photo by Leticia Brambila)