Wall-crossing for Calabi–Yau fourfolds and applications

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Structure of the talk.

1. Holomorphic Donaldson invariants and orientations

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   3.1 The general statement of the theorem.
   3.2 Which stability conditions do I use?

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2. Construction of families of vertex algebras
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4. Application to surface counting invariants
   4.1 Wall-crossing formula for surface counting invariants.
   4.2 Simplifying assumptions.
   4.3 Example in the case of an elliptic fibration.

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2. Construction of families of vertex algebras
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5. Further applications soon.

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3. This introduces a notion of orientation torsor given by a $\mathbb{Z}_2$-choice of an isomorphism

$$o : \det \mathrm{Ext}^2(E,E) \xrightarrow{\sim} \mathbb{C}$$

   such that $o^2 = \det(B)$. Gluing these choices as E varies over M leads to the orientation bundle $O \to M$.

Virtual fundamental classes

1. The orientation bundles have been shown to be trivializable for (compactly supported) sheaves [1] by Cao–Gross–Joyce (19') in the compact case and in B.(20') in full generality.

[1] More generally compactly supported perfect complexes.

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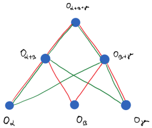
3. As is clear from Dominic's talk, wall-crossing is expressed in terms of taking direct sums of sheaves. So, comparing orientations under direct sums is needed.



Here α, β, γ is some topological data and comparison gives signs $\epsilon_{\alpha,\beta}$.

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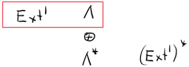
Splitting

1. Suppose that our CY4 obstruction theory splits into two parts



then essentially everything reduces to the wall-crossing Dominic presented[3] for the standard Behrend–Fantechi obstruction theories. We obviously want to do more as this is rather restrictive.

[3] See Liu (22') in the case of CY3.

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3. and a formal u-family of *state-field correspondences* which is a degree zero linear map
$$Y_u : V_\bullet \longrightarrow \mathrm{End}(V_\bullet)[\![z, z^{-1}]\!][\![u]\!] \,,$$
for $\deg(u) = 0, \deg(z) = -2$ extending to a (u)-adic continuous $\mathbb{Q}[\![u]\!]$-linear map
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$$Y_u\colon V_\bullet[\![u]\!] \longrightarrow \mathrm{End}(V_\bullet)[\![z, z^{-1}]\!][\![u]\!]\,.$$

It must additionally induce

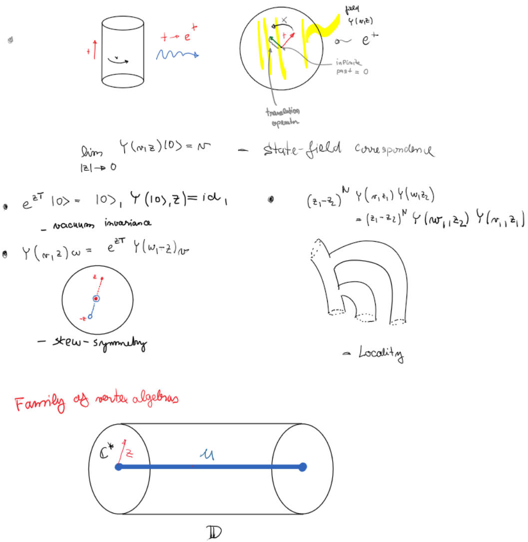
$$Y_u(v, z) = \sum_{n\in\mathbb{Z}} v_{u,n} z^{-n-1} : V_\bullet \to V_\bullet(\!(z)\!)[\![u]\!]\,,$$

for each $v \in V_\bullet \subset V_\bullet[\![u]\!]$ and

$$v_{u,n} : V_\bullet \to V_\bullet[\![u]\!]$$

linear for each $n \in \mathbb{Z}$.

Families of Vertex algebras in a picture



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[4] Should the pole along $z = 0$ be constant for all u or should it vary?

Axioms of families of vertex algebras.

They are required to satisfy the following set of conditions (see Li (02') for a different formulation):

1. *(vacuum)* $T\,|0\rangle = 0\,,\ Y_u(|0\rangle, z) = \mathrm{id}\,,\ Y_u(v,z)|0\rangle \in v + zV_\bullet[\![u,z]\!]\,,$

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3. *(locality)* for any $v, w \in V_\bullet$ and $k \geq 0$, there is an $N \gg 0$ such that the *truncated fields*

$$Y_{\leq k}(v,z) := \sum_{n=0}^{k} u^n\,[t^n]\{Y_t(v,z)\}\,. \tag{1}$$

satisfy

$$(z_1 - z_2)^N[Y_{\leq k}(v,z_1), Y_{\leq k}(w,z_2)] = 0\,,$$

where the supercommutator is defined on $\mathrm{End}(V_\bullet)[\![u]\!]$ by

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The reason for introducing the truncation and $Y_u(v,z)$ mapping to $V_\bullet(\!(z)\!)[\![u]\!]$ rather than $V_\bullet[\![u]\!](\!(z)\!)$ will become apparent from their geometric construction.

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Geometric construction I

The underlying vector space for the geometric construction of the vertex algebras is

$$V_\bullet = H_{\bullet+\mathrm{vdim}_{\mathbb{C}}}(\mathcal{M}_X)$$

where $\mathcal{M}_X$ is the stack of sheaves[6].

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2. There is a K-theory class Θ on $\mathcal{M}_X \times \mathcal{M}_X$ given by the dual of

$$\mathrm{Ext}^\bullet(E,F) \quad \text{at} \quad (E,F) \in \mathcal{M}_X \times \mathcal{M}_X\,.$$

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   It is clearly *additive* with respect to taking direct sums and *multiplicative* with respect to ρ.
3. (New) Consider the trivial $\mathbb{C}^*$ action on $\mathcal{M}_X \times \mathcal{M}_X$ and e^u the weight one line bundle. Take an equivariant K-theory class Ω_u on $\mathcal{M}_X \times \mathcal{M}_X$. It must satisfy the same additivity and scaling property that Ext did. I introduce

$$\Theta_u = \Theta + \Omega_u^\vee + \sigma^*\Omega_u\,.$$

4. Letting $\mathcal{M}_\alpha$ denote a connected component associated to an $\alpha \in K^0_{\mathrm{sst}}(X)$, we also define

$$\kappa(\alpha, \beta) = \mathrm{rk}(\Omega_u|_{\mathcal{M}_\alpha \times \mathcal{M}_\beta})\,, \quad \chi_\Omega(\alpha, \beta) = \chi(\alpha, \beta) + \kappa(\alpha, \beta) + \kappa(\beta, \alpha)\,.$$

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Geometric construction II

Definition

Construct the formal family of vertex algebras on $V_\bullet$ by setting

$$Y_u(v,z)v' = (-1)^{\kappa(\alpha,\beta)+a\chi_\Omega(\beta,\beta)}\epsilon_{\alpha,\beta} z^{\chi_\Omega(\alpha,\beta)} \Sigma_*\Big[(e^{zT}\otimes \mathrm{id})\big(v\boxtimes v'\cap c_{z^{-1}}(\Theta_u)\big)\Big],$$

where Σ is the direct sum map $\mathcal{M}_X\times\mathcal{M}_X\to\mathcal{M}_X$.

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Consider example

$$\Omega_u|_{\mathcal{M}_\alpha\times\mathcal{M}_\beta} = e^u\,\mathcal{O}^{\oplus\chi(\alpha,\beta)},$$

then

$$c_{z^{-1}}\left(\Omega_u|_{\mathcal{M}_\alpha\times\mathcal{M}_\beta}\right) = (1+z^{-1}u)^{\chi(\alpha,\beta)}.$$

It becomes clear that we need to put restrictions on powers of u for any kind of vertex algebra axioms to be satisfied, because for $\chi(\alpha,\beta)<0$, we have an infinite power-series in z^{-1}.

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It becomes clear that we need to put restrictions on powers of u for any kind of vertex algebra axioms to be satisfied, because for $\chi(\alpha,\beta)<0$, we have an infinite power-series in z^{-1}. Hence our choice of axioms leads to the following result.

Theorem (B.(??) extending Joyce (17')[7])

The data $(V_\bullet, Y_u, T, |0\rangle)$ defines a formal u-family of vertex algebras.

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Families of Lie algebras

To understand how vertex algebras are applied to wall-crossing, first, construct a u-family of Lie algebras by

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Starting from a u-family of vertex algebras $(V_\bullet, Y_u, T, |0\rangle)$, define a u-family of Lie algebras $(Q_\bullet, [-,-]_u)$ for

$$Q_\bullet = V_{\bullet+2}/TV_\bullet$$

by

$$[\overline{v}, \overline{w}]_u = \overline{v_{u,0}w}\,, \quad \forall v, w \in V_\bullet[\![u]\!]$$

and $\overline{(-)}$ denotes the associated class in the quotient. [8]

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Outside of the 0 component, we have $Q_\bullet = H_{\bullet+\mathrm{vdim}_{\mathbb{C}}}(\mathcal{M}_X^{\mathrm{rig}})$, where $\mathcal{M}_X^{\mathrm{rig}}$ is the quotient by the action of $[*/\mathbb{G}_m]$ and we use a non-standard symmetric obstruction theory on it.

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2. This construction is not specific to CY4, I already have applications in mind for surfaces too.

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What is the main statement?

Assumption

Fix two stability conditions σ_0, σ_1 and assume that a long list of assumptions holds. One of them is that all the moduli spaces of semistable objects appearing in the proof are projective (or at least proper) for all stability conditions interpolating between σ_0 and σ_1[9].

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1. Then there exist classes $\langle \mathcal{M}_\alpha^\sigma \rangle_u \in Q_\bullet[u]$ counting σ-semistables in class α such that

$$\langle \mathcal{M}_\alpha^\sigma \rangle_u = [M_\alpha^\sigma]_{\mathrm{vir}} \cap c_{\mathrm{rk}}(\Delta^* \Omega_u)$$

when there are no strictly semistables. Here, the class $[M_\alpha^\sigma]_{\mathrm{vir}}$ is the pushforward along the open embedding $M_\alpha^\sigma \hookrightarrow \mathcal{M}_X^{\mathrm{rig}}$ of $[M_\alpha^\sigma]^{\mathrm{vir}}$.

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Let σ_i be two stability conditions for $i = 0, 1$, then for each $\alpha \in K^0(X)$ the classes $\langle M_\alpha^{\sigma_i} \rangle_u$ satisfy

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whenever the Assumptions hold.

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whenever the Assumptions hold.

2. To understand some applications of this statement, let me first describe the stability conditions I use.

[9] I am fiddling with the assumptions right now to make them as easy to check and as general as possible. And that is difficult.

What stability conditions?

1. *Stair polynomial stability condition* is a generalization of *standard polynomial stability* condition of Bayer with an application to surface counting in Calabi–Yau fourfolds.

[10] A subcategory of sheaves E with $p\big(\dim \operatorname{supp}(E)\big) = i$ and no torsion in dimensions d with $p(d) > i$.

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2. Recall that for constructing the latter, one uses a perversity function

$$p : \{0, 1, 2, 3, 4\} \to \mathbb{Z}$$

which is non-increasing and changes in steps at most 1.

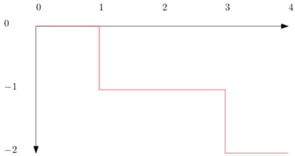

Figure: Perversity function.

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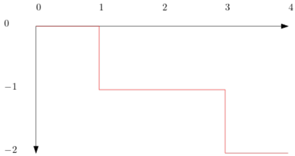

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3. Assign to it a heart (of a bounded t-structure) $\mathcal{A}^p$ consisting of complexes $E^\bullet$ with

$$H^i(E^\bullet) \in \mathrm{Coh}_{p^{-1}(i)}(X)^{10}\,.$$

E.g. if $p : \{0, 1\} \to \{0\}$ and $\{2, 3, 4\} \to \{-1\}$, then we recover the tilted category associated to the torsion pair

$$\langle \mathrm{Coh}_{\leq 1}(X), \mathrm{Coh}_{\geq 2}(X) \rangle\,.$$

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Standard polynomial Bridgeland stability

1. The standard stability condition is given on $\mathcal{A}^p$ by the *central charge*

$$Z_t(E) = \sum_{i=0}^{4} (-1)^{p(i)} \rho_i t^i \int_X \mathrm{ch}(E) H^i \cdot U$$

where H is an ample divisor, U a unipotent transformation, $(-1)^{p(i)}\rho_i \in \mathbb{H}$ and the phases of ρ_i move in the clockwise direction as dimension of support increases,

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2. Instead of comparing phases $\phi_t(E) = \arg_{(0,\pi]} Z_t(E)$ for $t \to \infty$, which would recover the usual Bridgeland stability, one compares their germs at $t = \infty$.

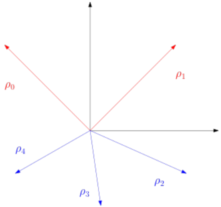

Figure: Standard polynomial stability

Stair polynomial Bridgeland stability

1. Define a stair function $s : \{0, 1, 2, 3, 4\} \to \mathbb{Z}$ satisfying

$$s(i-1) \leq s(i) \leq s(i-1) + 1$$

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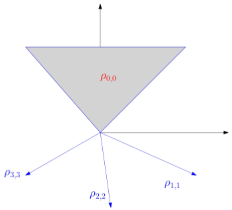

An important example/motivation

1. Take

$$s([0,1]) = 0 \quad s(i) = i-1 \quad \text{for} \quad i \geq 2\,,$$
$$\tilde{p}(\{0\}) = 0\,, \quad \tilde{p}(\{1,2,3\}) = -1\,,$$

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$$\mathcal{A}^p = \langle \mathrm{Coh}_{\geq 2}(X), \mathrm{Coh}_{\leq 1}(X)[-1] \rangle\,.$$

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2. Adapting the work of Pandharipande–Thomas(07') to Calabi–Yau fourfolds, Bae–Kool–Park(22') defined $PT^{(i)}$ stable pairs

$$\mathcal{O}_X \xrightarrow{s} F$$

 for $i = -1, 0, 1$, where F has 2 dimensional support and

$$\dim(\mathrm{coker}\,(s)) \leq i\,, \qquad \mathrm{Hom}(Q, F) = 0 \quad \text{for} \quad \dim(Q) \leq i\,.$$

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3. The next family of stability conditions interpolates between $PT^{(0)}$ and $PT^{(i)}$[11]

Definition

Using $\rho_{i,i} = \rho_i$, the P^t-stability is defined on $\mathcal{A}^p$ for each $t \in \mathbb{R}$ by

$$\begin{aligned}
\rho_0 &: H^0(X) \oplus H^2(X) \to \mathbb{H}\,, & (\beta, n) &\mapsto -n + i(\beta \cdot H)\,,\\
-\rho_1 &: H^4(X) \to \mathbb{H}\,, & \gamma &\mapsto -\gamma \cdot H^2\,,\\
-\rho_3 &: H^8(X) \to \mathbb{H}\,, & r &\mapsto r(-t+i)\,.
\end{aligned}$$

12

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Example in picture

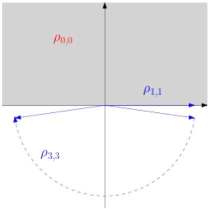

Figure: The grey region represents ≤ 1-dimensional sheaves which are spread out across the upper half-plane. The wall-crossing happens whenever $-\rho_3$ crosses a ray of the phase $\arctan n/\beta \cdot \omega + \pi/2$ for some $(\beta, n) \in N_{\leq 1}(X)$.

An example of an application

1. If additional assumptions are checked, the last example of stability conditions gives us the wall-crossing formula

$$[\mathrm{PT}^{(0)}_{(\gamma,\delta)}]_{\mathrm{vir}} = \sum_{\substack{\underline{\delta}\vdash\delta \\ l(\underline{\delta})=k}} \frac{(-1)^k}{k!}\Big[\cdots\Big[[\mathrm{PT}^{(1)}_{(\gamma,\delta_0)}]_{\mathrm{vir}}, \big[M_{\delta_1}\big]^{\mathrm{in}}\Big], \cdots, \big[M_{\delta_k}\big]^{\mathrm{in}}\Big],$$

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2. Next fix a line bundle L on X and construct a family of vertex algebras for the class Ω_u where

$$\Omega_u|_{\{E\},\{F\}} = e^u \mathrm{Ext}^\bullet(E, F\otimes L) + \text{correction terms}\,.$$

   Its restriction $\iota^*\circ\Delta^*_{\mathcal{M}_X}(\Omega_u)$ to $\mathrm{PT}^{(i)}_{(\gamma,\delta)}$ is given by

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3. This leads to

$$\big\langle\mathrm{PT}^{(0)}_{(\gamma,\delta)}\big\rangle_u = \sum_{\substack{\underline{\delta}\vdash\delta \\ \mathrm{length}(\underline{\delta})=k}} \frac{(-1)^k}{k!}\Big[\cdots\Big[\big\langle\mathrm{PT}^{(1)}_{(\gamma,\delta_0)}\big\rangle_u, [M_{\delta_1}]^{\mathrm{in}}\Big]_u, \cdots, [M_{\delta_k}]^{\mathrm{in}}\Big]_u\,.$$

Taking coefficients

1. From the definition of invariants, conclude

$$\left\langle \mathsf{PT}^{(i)}_{\gamma,\delta} \right\rangle_u = \left[\mathsf{PT}^{(i)}_{\gamma,\delta}\right]^{\mathrm{vir}} \cap c_{\mathrm{rk}}\left(e^u L^{[\gamma,\delta]}\right)$$
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3. Notice that the expression depends only on γ if the orthogonality assumption $\delta c_1(L) = 0$ holds. This motivates the following

Assumption

In the $\mathrm{PT}^{(0)}/\mathrm{PT}^{(1)}$ wall-crossing formula, assume that

$$(\delta - \delta_0) \cdot c_1(L)$$

always holds.

Conjecture of Bae–Kool–Park

1. Under this assumption, taking $\left[u^{\frac{\gamma}{2}(c_1(L)^2+\gamma)}\right]\left\{ - \right\}$ in the wall-crossing formula for $\mathrm{PT}^{(0)}/\mathrm{PT}^{(1)}$ leads to

$$\langle \mathrm{PT}^{(0)}_{\gamma,\delta}\rangle^L = \sum_{\substack{\vec{\delta}\vdash\delta \\ k\geq 0}} \langle \mathrm{PT}^{(1)}_{\gamma,\delta_0}\rangle^L \frac{(-1)^k}{k!}\left[\cdots\left[e^{(-1,0,\gamma,\delta_0)},[M^{\mathrm{ss}}_{\delta_1}]^{\mathrm{in}}\right]^L,\cdots,[M^{\mathrm{ss}}_{\delta_k}]^{\mathrm{in}}\right]^L$$

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2. Let us now take an elliptic fibration $\pi : X \to B$ with a base B[13] and set $L = \mathcal{O}_X$ which always satisfies the orthogonality assumption.

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2. Let us now take an elliptic fibration $\pi : X \to B$ with a base B[13] and set $L = \mathcal{O}_X$ which always satisfies the orthogonality assumption.
3. For the class $(\gamma,\delta) = \pi^*(\beta,n)$ for $(\beta,n)\in H^{\geq 4}(B)$ Bae–Kool–Park define

$$\langle\langle \mathrm{PT}^{(0)}_{\gamma,\delta}\rangle\rangle^{\mathcal{O}_X} = \sum_{d\geq 0}\langle \mathrm{PT}^{(0)}_{\gamma,\delta+dE}\rangle^{\mathcal{O}_X} q^d$$

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Here E is the Poincare dual of a fiber class and PT stands for the usual PT stable pairs which are just $\mathrm{PT}^{(0)}$ stable pairs with $\gamma = 0$.

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Proving BKP conjecture and more

1. Up to a simple structural assumption on $\left[M_{dE,n}\right]^{\text{in}}$[14] that I expect to be able to prove once I return to this point, I can show that

Conjecture (Bae–Kool–Park)

The $\mathrm{PT}^{(0)}/\mathrm{PT}^{(1)}$ *correspondence*

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holds.

2. This is because in the wall-crossing formula twisted by $\mathcal{O}_X$ only the classes $[M_{dE,n}]^{\text{in}}$ contribute. Any bracket with $[M_{dE,n}]^{\text{in}}$ for $n \neq 0$ up to a small additional term is almost trivially zero.

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5. Further applications: Proves my computations of $[\mathrm{Hilb}^n(X)]^{\text{vir}}$, $[\mathrm{Quot}_X(E,n)]^{\text{vir}}$ and expresses them in the framework of families of vertex algebras, proves DT/PT and DT/PT$^{(0)}$ correspondences using families of vertex algebras, many more...

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