Embedding superconformal vertex algebras from Killing spinors and (0,2) mirror symmetry

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Joint with Andoni De Arriba De La Hera & Mario Garcia-Fernandez:
`arXiv:2012.01851 [math.DG]` (to appear in *IMRN*),
and further work in progress.

Subject of this talk

Interaction between

– **Algebra: vertex algebras**

– **Geometry: Killing spinors**

Algebra: vertex-algebra embeddings

The Virasoro algebra

Punctured disc: $\mathring{\mathbb{D}}^1 = \mathrm{Spec}\, \mathbb{C}[t^{\pm 1}]$
Lie algebra of vector fields:

$$\mathrm{Vect}(\mathring{\mathbb{D}}^1) = \mathrm{Der}_{\mathbb{C}}(\mathbb{C}[t^{\pm 1}]) = \mathbb{C}[t^{\pm 1}]\frac{\mathrm{d}}{\mathrm{d}t}$$

Nice basis of vector fields:

$$\mathtt{L}_n := -t^{n+1}\frac{\mathrm{d}}{\mathrm{d}t}, \qquad \text{for } n \in \mathbb{Z}$$

Lie bracket: $[\mathtt{L}_m, \mathtt{L}_n] = (m-n)\mathtt{L}_{m+n}$

Virasoro algebra: the central extension

$$0 \to \mathbb{C}\mathtt{C} \longrightarrow \mathtt{Vir} \longrightarrow \mathrm{Vect}(\mathring{\mathbb{D}}) \to 0,$$

i.e., $\mathtt{Vir} = \mathrm{Vect}(\mathring{\mathbb{D}}) \oplus \mathbb{C}\mathtt{C}$ as a vector space, with commutators

$$[\mathtt{L}_m, \mathtt{L}_n] = (m-n)\mathtt{L}_{m+n} + \delta_{m+n,0}(m^3 - m)\frac{\mathtt{C}}{12},$$
$$[\mathtt{C}, -] = 0.$$

Define $\mathrm{V}^c(\mathtt{Vir}) = \mathrm{U}(\mathtt{Vir})/(\mathtt{C} - c1)$ (localize the universal enveloping algebra at central charge $c \in \mathbb{C}$).

Affinization of quadratic Lie algebras

Finite-dimensional quadratic Lie $\mathbb{C}$-algebra: $(\mathfrak{g}, \langle -, - \rangle)$

Loop algebra: $\mathsf{L}\mathfrak{g} = \mathsf{Map}(\mathring{\mathbb{D}}^1, \mathfrak{g}) = \mathfrak{g}[t^{\pm 1}]$ (with $\mathring{\mathbb{D}}^1 = \mathsf{Spec}\,\mathbb{C}[t^{\pm 1}]$)

Affinization or current algebra: central Lie-algebra extension

$$0 \to \mathbb{C}\mathtt{K} \longrightarrow \widehat{\mathfrak{g}} \longrightarrow \mathsf{L}\mathfrak{g} \to 0,$$

i.e., $\widehat{\mathfrak{g}} = \mathsf{L}\mathfrak{g} \oplus \mathbb{C}\mathtt{K}$ as a vector space, with commutators (for $\mathtt{a}, \mathtt{b} \in \mathfrak{g}$):

$$[\mathtt{a}t^m, \mathtt{b}t^n] = [\mathtt{a}, \mathtt{b}]t^{m+n} + m\delta_{m+n,0}\langle \mathtt{a}, \mathtt{b} \rangle \mathtt{K},$$
$$[\mathtt{K}, \mathtt{a}t^m] = 0.$$

Affinization and the Sugawara construction

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$$[\mathtt{K}, \mathtt{a}t^m] = 0.$$

Define
- $\mathrm{V}^k(\mathfrak{g}) = \mathrm{U}(\widehat{\mathfrak{g}})/(k - \mathtt{K})$ for 'level' $k \in \mathbb{C}$,
- Ω=(Casimir operator of $(\mathfrak{g}, \langle -, - \rangle) \in \mathrm{U}(\mathfrak{g})$ (UEA of $\mathfrak{g}$),
- $2h^\vee \dim \mathfrak{g} = \mathrm{Tr}(\mathrm{ad}(\Omega) \colon \mathfrak{g} \to \mathfrak{g})$ ($h^\vee$:='dual Coxeter number').

Theorem (Sugawara 1968)

If the Lie algebra $\mathfrak{g}$ is simple or abelian and $k + h^\vee \neq 0$, then $\exists$ a canonical embedding $\mathrm{V}^c(\mathtt{Vir}) \subset \mathrm{V}^k(\mathfrak{g})$ of central charge $c = \frac{k \dim \mathfrak{g}}{k + h^\vee}$.

Superconformal algebras (Fattori–Kac, 2002)

Fix N odd variables $\theta^1, \ldots, \theta^N$.

Punctured superdisc: $\mathring{\mathbb{D}}^{1|N} = \mathbb{S}\mathrm{pec}\, \mathbb{C}[t^{\pm 1}, \theta^1, \ldots, \theta^N]$ (super Spec)

Define the supercontact 1-form on $\mathring{\mathbb{D}}^{1|N}$:

$$\alpha := \mathrm{d}t + \sum \theta^i \mathrm{d}\theta^i \in \Omega^1(\mathring{\mathbb{D}}^{1|N})$$

and Lie superalgebra of supercontact vector fields

$$K(1|N) := \{v \mid \exists \text{ function } f_v \text{ s.t. } L_v\alpha = f_v\alpha\} \subset \mathrm{Vect}\left(\mathring{\mathbb{D}}^{1|N}\right).$$

- For $N \leq 3$, $K(1|N)$ admits one non-trivial central extension $\mathcal{K}(1|N) = K(1|N) \oplus \mathbb{C}\mathtt{c}$ (as a Lie conformal algebra).
- For $N = 4$, there are *two* non-trivial central extensions (for the derived Lie algebra $K(1|N)' \subset K(1|N)$), i.e., two central charges.
- For $N \geq 5$, there are no non-trivial central extensions.

$N = 1$: The Neveu–Schwarz superconformal algebra

Neveu–Schwarz superconformal algebra $\mathcal{K}(1|1)$:
Generated by $\mathtt{L}_m, \mathtt{G}_n, \mathtt{C}$ ($m \in \mathbb{Z}, n \in \frac{1}{2} + \mathbb{Z}$), with commutation relations

$$[\mathtt{L}_m, \mathtt{L}_n] = (m - n)\mathtt{L}_{m+n} + \delta_{m+n,0}(m^3 - m)\frac{\mathtt{C}}{12},$$

$$[\mathtt{L}_m, \mathtt{G}_n] = \left(\frac{m}{2} - n\right)\mathtt{G}_{m+n}, \quad [\mathtt{C}, -] = 0,$$

$$[\mathtt{G}_m, \mathtt{G}_n] = 2\mathtt{L}_{m+n} + \delta_{m+n,0}\left(m^2 - \frac{1}{4}\right)\frac{\mathtt{C}}{3}.$$

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Finite-dimensional quadratic Lie superalgebra: $(\mathfrak{g}, \langle -, - \rangle)$

$N = 1$ super loop algebra: $\mathsf{L}^{1|1}\mathfrak{g} = \mathrm{Map}(\mathring{\mathbb{D}}^{1|1}, \mathfrak{g}) = \mathfrak{g}[t^{\pm 1}, \theta]$

Superaffinization or supercurrent algebra: $\widehat{\mathfrak{g}}_{super} = \mathsf{L}^{1|1}\mathfrak{g} \oplus \mathbb{C}\mathtt{K}$ as a vector space, with commutators (for $\mathtt{a}, \mathtt{b} \in \mathfrak{g}$):

$$[\mathtt{a}t^m, \mathtt{b}t^n] = [\mathtt{a}, \mathtt{b}]t^{m+n} + m\delta_{m+n,0}\langle \mathtt{a}, \mathtt{b}\rangle \mathtt{K}, \quad [\mathtt{K}, -] = 0,$$

$$[\mathtt{a}t^m, \mathtt{b}t^n\theta] = [\mathtt{a}, \mathtt{b}]t^{m+n}\theta, \quad [\mathtt{a}t^m\theta, \mathtt{b}t^n\theta] = \delta_{m+n,-1}\langle \mathtt{b}, \mathtt{a}\rangle \mathtt{K}.$$

The Kac–Todorov construction

Define $\mathrm{V}^c(\mathcal{K}(1|1)) = \mathrm{U}(\mathcal{K}(1|1))/(\mathtt{C} - c1)$ (localize universal enveloping algebra at central charge $c \in \mathbb{C}$)

Theorem (Kac–Todorov 1985)

If $k + h^\vee \neq 0$, then $\exists$ a canonical embedding $\mathrm{V}^c(\mathcal{K}(1|1)) \subset \mathrm{V}^{k+h^\vee}(\mathfrak{g}_{super})$ of central charge $c = \frac{k \dim \mathfrak{g}}{k+h^\vee} + \frac{\dim \mathfrak{g}}{2}$.

Kac–Todorov and aim of the talk

Define $\mathrm{V}^c(\mathcal{K}(1|1)) = \mathrm{U}(\mathcal{K}(1|1))/(\mathtt{C} - c1)$ (localize universal enveloping algebra at central charge $c \in \mathbb{C}$)

Theorem (Kac–Todorov 1985)

If $k + h^\vee \neq 0$, then $\exists$ a canonical embedding $\mathrm{V}^c(\mathcal{K}(1|1)) \subset \mathrm{V}^{k+h^\vee}(\mathfrak{g}_{super})$ of central charge $c = \frac{k \dim \mathfrak{g}}{k+h^\vee} + \frac{\dim \mathfrak{g}}{2}$.

Problem: obtain an N=2 SUSY version, i.e., an embedding of the N=2 superconformal algebra

$$\mathbf{V}^{\boldsymbol{c}}(\mathcal{K}(1|2)) \subset \mathbf{V}^{\boldsymbol{k}}(\mathfrak{g}_{super}),$$

where now $\mathrm{V}^c(\mathcal{K}(1|2)) = \mathrm{U}(\mathcal{K}(1|2))/(\mathtt{C} - c1)$.

The most natural framework to construct any of these embeddings is the theory of **vertex algebras**.

Vertex algebras: quantization on the punctured disc

Basic principles to quantize fields on the punctured disc:

- View $\mathbb{C}[z^{\pm 1}]$ as the space of **test functions**.
- Interpret **quantum fields** as operator-valued 'distributions' on $\mathring{\mathbb{D}}^1$, i.e., continuous linear maps into a topological algebra $\mathcal{U}$ of operators:
$$a\colon \mathbb{C}[z^{\pm}] \longrightarrow \mathcal{U}.$$
- Find a suitable set of distributions that are '**local to each other**', so they can be 'multiplied'.

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We write distributions $\mathbb{C}[z^{\pm}] \to \mathcal{U}$ as $\mathcal{U}$-valued formal Laurent series

$$a(z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1} \in \mathcal{U}[\![z^{\pm 1}]\!]$$

($a_{(n)}$='Fourier modes') via the pairing

$$\mathbb{C}[z^{\pm 1}] \otimes \mathcal{U}[\![z^{\pm 1}]\!] \to \mathcal{U}, \; \varphi \otimes a \mapsto \operatorname*{Res}_{z=0}(\varphi(z)a(z)).$$

Vertex algebras: Operator Product Expansion

Two $\mathcal{U}$-valued formal distributions

$$a(z) = \sum_{m\in\mathbb{Z}} a_{(m)} z^{-m-1} \in \mathcal{U}[\![z^{\pm 1}]\!], \quad b(w) = \sum_{n\in\mathbb{Z}} b_{(n)} w^{-n-1} \in \mathcal{U}[\![w^{\pm 1}]\!],$$

are local to each other iff $\exists$ sequence $\{c^j(w)\}_{j=1}^N \subset \mathcal{U}[\![w^{\pm 1}]\!]$ such that

$$[a(z), b(w)] \overset{\text{def}}{=} \sum_{m,n\in\mathbb{Z}} [a_{(m)}, b_{(n)}] z^{-m-1} w^{-n-1} = \sum_{j=0}^{N} \frac{1}{j!} \partial_w^j \delta(z-w)\, c^j(w).$$

In this case, one usually writes an **OPE**

$$a(z)b(w) \sim \sum_{j=0}^{N} \frac{c^j(w)}{(z-w)^{j+1}}$$

and defines the **λ-bracket**

$$[a_\lambda b] = \sum_{j=0}^{N} \frac{\lambda^j}{j!} c^j \in \left(\mathcal{U}[\![w^{\pm 1}]\!]\right)[\lambda]$$

SUSY Lie conformal algebras

We use **superfields** to simplify calculations.

The most basic piece of a SUSY vertex algebra is the Λ-bracket:
Define graded associative algebras (with variables S, χ odd and T, λ even)

$$\mathcal{H} = \frac{\mathbb{C}\langle S, T\rangle}{(S^2 = T)}, \qquad \mathcal{L} = \frac{\mathbb{C}\langle \chi, \lambda\rangle}{(\chi^2 = -\lambda)}.$$

Notation: $\nabla = (T, S)$, $\Lambda = (\lambda, \chi)$ and another copy $\Gamma = (\gamma, \eta)$ of Λ.

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Definition (Heluani–Kac 2007)

A *($N_K = 1$) SUSY Lie conformal algebra* is given by

- an $\mathcal{H}$-module $\mathcal{R}$,
- a parity-reversing Λ-bracket $[\cdot_\Lambda \cdot]\colon \mathcal{R}\otimes\mathcal{R} \longrightarrow \mathcal{L}\otimes\mathcal{R}$,
- axioms ($a, b, c \in \mathcal{R}$):

$$[a_\Lambda b] = (-1)^{|a||b|}[b_{-\Lambda-\nabla} a],$$

$$[a_\Lambda [b_\Gamma c]] = (-1)^{|a|+1}[[a_\Lambda b]_{\Lambda+\Gamma} c] + (-1)^{(|a|+1)(|b|+1)}[b_\Gamma [a_\Lambda c]],$$

$$[Sa_\Lambda b] = \chi[a_\Lambda b], \qquad [a_\Lambda Sb] = -(-1)^{|a|}(S+\chi)[a_\Lambda b].$$

SUSY vertex algebras

Definition (Heluani–Kac 2007; cf. Barron 2000)

A *($N_K = 1$) SUSY vertex algebra* is given by

- a (SUSY) Lie conformal algebra $(V, [\cdot_\Lambda \cdot])$,
- a normally ordered product $V \times V \longrightarrow V, \quad (a, b) \longmapsto {:}ab{:}$,
- a *vacuum vector* $|0\rangle \in V$ that is a right unit for the normally ordered product,
- axioms ($a, b, c \in V$):

$$:ab: - (-1)^{|a||b|} :ba: = \int_{-\nabla}^{0} d\Lambda [a_\Lambda b],$$

$$:(:ab:)c: - :a(:bc:): = :\left(\int_0^\nabla d\Lambda\, a\right)[b_\Lambda c]: + (-1)^{|a||b|} :\left(\int_0^\nabla d\Lambda\, b\right)[a_\Lambda c]:,$$

$$[a_\Lambda :bc:] = :[a_\Lambda b]c: + (-1)^{(|a|+1)|b|} :b[a_\Lambda c]: + \int_0^\Lambda d\Gamma [[a_\Lambda b]_\Gamma c].$$

Remark: A SUSY Lie conformal algebra $(\mathcal{R}, [\cdot_\Lambda \cdot])$ freely generates a 'universal enveloping' SUSY vertex algebra $\mathrm{V}(\mathcal{R})$.

Example 1: The ***N*=2 superconformal vertex algebra** $\mathrm{V}^{\mathtt{C}}(\mathcal{K}(1|2))$ is freely generated by an odd superfield $\mathtt{H}$, an even superfield $\mathtt{J}$ and an (even) central charge $\mathtt{C}$, i.e.,

$$\mathcal{R} = \mathcal{H} \otimes (\mathbb{C}\mathtt{H} \oplus \mathbb{C}\mathtt{J}) \oplus \mathbb{C}\mathtt{C},$$

with non-zero Λ-brackets $[\mathtt{H}_\Lambda \mathtt{H}] = (2T + \chi S + 3\lambda)\,\mathtt{H} + \frac{\chi\lambda^2}{3}\mathtt{C}$,

$$[\mathtt{J}_\Lambda \mathtt{J}] = -\left(\mathtt{H} + \frac{\lambda\chi}{3}\mathtt{C}\right), \quad [\mathtt{H}_\Lambda \mathtt{J}] = (2T + 2\lambda + \chi S)\,\mathtt{J}.$$

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Example 2: Let $(\mathfrak{g}, \langle -, - \rangle)$ be a finite-dimensional quadratic Lie algebra, and $\Pi\mathfrak{g}$ the same vector superspace with reversed parity. The **superaffine vertex algebra** $\mathrm{V}(\mathfrak{g}_{super})$ is freely generated by

$$\mathcal{R} = \mathcal{H} \otimes \Pi\mathfrak{g} \oplus \mathbb{C}\mathtt{K},$$

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Problem: How can we embed the $N = 2$ superconformal vertex algebra?

$$\mathbf{V}^{c}(\mathcal{K}(1|2)) \subset \mathbf{V}^{k}(\mathfrak{g}_{super}).$$

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$$\mathbf{V}^{c}(\mathcal{K}(1\,|\,2)) \subset \mathbf{V}^{k}(\mathfrak{g}_{super}).$$

Our answer: impose conditions on the quadratic Lie algebra $\mathfrak{g}$ inspired by supergravity.

Geometry: Killing spinors

Connections on Courant algebroids

The basic idea to define generalized connections is to replace the tangent bundle by a Courant algebroid in the definition of affine connection.

Fix a differentiable manifold M.

Definition: A **Courant algebroid** is a vector bundle E equipped with a non-degenerate symmetric bilinear form, a bilinear bracket and bundle map,

$$\langle -, - \rangle \colon E \otimes E \longrightarrow \mathbb{R}, \quad [-,-] \colon \Gamma(E) \otimes \Gamma(E) \longrightarrow \Gamma(E), \quad \pi \colon E \longrightarrow TM,$$

with $\Gamma(E) = \{C^\infty \text{ sections of } E\}$, satisfying certain compatibility conditions (Jacobi, Leibniz, compatibility with $\langle -, - \rangle$...).

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Definition: A **generalized metric** on a Courant algebroid is a direct-sum decomposition

$$E = V_+ \oplus V_-,$$

such that $\langle -, - \rangle|_{V_+}$ is non-degenerate and $V_- = V_+^{\perp}$.

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Def: A **generalized connection** on a Courant algebroid is a linear map

$$D\colon \Gamma(E)\longrightarrow \mathrm{End}_{\mathbb{R}}(\Gamma(E)),\quad e\longmapsto \Big(D_e\colon \Gamma(E)\to\Gamma(E)\Big),$$

satisfying the axioms you might guess replacing the tangent bundle by a Courant algebroid in the definition of metric affine connection.

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satisfying some compatibility conditions (Leibniz & compatible with $\langle -,-\rangle$).

Definition: Fix a generalized connection D and a gen. metric $V_+\subset E$.

- **Torsion** $T_D\in\Gamma(\wedge^3 E^*)$ of D: given by
  $T_D(e_1,e_2,e_3) := \langle D_{e_1}e_2 - D_{e_2}e_1 - [e_1,e_2], e_3\rangle + \langle D_{e_3}e_1, e_2\rangle$
- D is **compatible** with the gen. metric $V_+\subset E$ if
  $D_e(\Gamma(V_+))\subset \Gamma(V_+)$ for all $e\in\Gamma(E)$.

Killing spinors on Courant algebroids

We follow (Garcia-Fernandez 2019) to define Killing spinors on Courant algebroids. **This is inspired by supergravity.**

Fix the following data:

- a generalized metric $E = V_+ \oplus V_-$ on a Courant algebroid over M,
- a 'spinor bundle' $S(V_+)$ of (fibrewise) irreducible representations of the bundle of Clifford algebras $\mathrm{Cl}(V_+)$.

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Theorem (Garcia-Fernandez 2019)

- There exists a torsion-free generalized connection D on E that is compatible with generalized metric $V_+ \subset E$, but it is not unique.
- However, the component $D_-^+ \colon \Gamma(V_-) \to \mathrm{End}_{\mathbb{R}}(\Gamma(V_+))$ of such a D is unique.
- The *Dirac operator* $\not{D}^+ \colon \Gamma(S(V_+)) \longrightarrow \Gamma(S(V_+))$ only depends on $V_+ \subset E$ and the *divergence* $\mathrm{div}_D \colon \Gamma(E) \to C^\infty(M)$, $e \longmapsto \mathrm{Tr}(De)$.

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- There exists a torsion-free generalized connection D on E that is compatible with generalized metric $V_+ \subset E$, but it is not unique.
- However, the component $D_-^+ \colon \Gamma(V_-) \to \mathrm{End}_{\mathbb{R}}(\Gamma(V_+))$ of such a D is unique.
- The *Dirac operator* $\not{D}^+ \colon \Gamma(S(V_+)) \longrightarrow \Gamma(S(V_+))$ only depends on $V_+ \subset E$ and the *divergence* $\mathrm{div}_D \colon \Gamma(E) \to C^\infty(M)$, $e \longmapsto \mathrm{Tr}(De)$.

Definition (Garcia-Fernandez 2019)

The *Killing spinor equations* for a non-vanishing spinor $\eta \in \Gamma(S(V_+))$ are

$$D_-^+\eta = 0 \qquad \text{(gravitino equation)},$$

$$\not{D}^+\eta = 0 \qquad \text{(dilatino equation)}.$$

Back to algebra

Killing spinors on quadratic Lie algebras

Goal: apply Garcia-Fernandez's theory of Killing spinors to a (real) **finite-dimensional quadratic Lie algebra, viewed as a Courant algebroid over a point**.

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Fix the following data:

- a (real) finite-dimensional quadratic Lie algebra
$$\mathfrak{g}, \quad \langle -, - \rangle \colon \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathbb{R},$$
- a generalized metric $\mathfrak{g} = V_+ \oplus V_-$,
- a 'divergence' $\varepsilon \in V_+$ (viewed as $\langle \varepsilon, \cdot \rangle \in V_+^*$),
- an irreducible representations $S(V_+)$ of Clifford algebra $\mathrm{Cl}(V_+)$.

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Proposition (AC, De Arriba De La Hera, Garcia-Fernandez)

A solution (V_+, ε, η) of the Killing spinor equations on $(\mathfrak{g}, \langle -,- \rangle)$ with $\dim V_+ = 2n$ even and η pure is equivalent to an isotropic decomposition

$$V_+ \otimes \mathbb{C} = \ell \oplus \bar{\ell}$$

(where $\ell = V_+^{1,0}$ and $\bar{\ell} = V_+^{0,1}$), such that

$$[\ell, \ell] \subset \ell, \quad [\bar{\ell}, \bar{\ell}] \subset \bar{\ell} \qquad \text{(F-term equation)},$$

$$\frac{1}{2} \sum_{i=1}^{n} [\epsilon_i, \epsilon^i] = \varepsilon_{\bar{\ell}} - \varepsilon_{\ell} \qquad \text{(D-term equation)},$$

for basis $\{\epsilon_i\} \subset \ell$, $\{\epsilon^i\} \subset \bar{\ell}$ such that $\langle \epsilon_i, \epsilon^j \rangle = \delta_{ij}$.

Killing spinors on quadratic Lie algebras

- a (real) finite-dimensional quadratic Lie algebra $(\mathfrak{g}, \langle -,-\rangle)$,
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In general, this is not a Manin triple if $V_- \neq 0$.
New 'supergravity' equations for a quadratic Lie algebra?

Main results (Algebra)

Theorem (AC, De Arriba De La Hera, Garcia-Fernandez)

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and furthermore,

$$w := \sum_{i=1}^{n} \left([\epsilon_i, \epsilon^i]_\ell - [\epsilon_i, \epsilon^i]_{\bar{\ell}} \right) \in [\ell, \ell]^\perp \cap [\bar{\ell}, \bar{\ell}]^\perp. \qquad (\star)$$

Then there is an embedding $\mathrm{V}^c(\mathcal{K}(1|2)) \hookrightarrow \mathrm{V}^k(\mathfrak{g}^{\mathbb{C}}_{super})$ with central charge $c = 3n$, where $\mathtt{J}, \mathtt{H} \in \mathrm{V}^c(\mathcal{K}(1|2))$ map into

$$\mathtt{J}_0 = \frac{i}{k} {:}e^j e_j{:}, \qquad \mathtt{H}' = \frac{1}{k} \left({:}e_j \left(S e^j\right){:} + {:}e^j \left(S e_j\right){:} \right) + \frac{1}{k} T(\Pi w)$$
$$+ \frac{1}{k^2} \left({:}e_j \left({:}e^k [e^j, e_k]{:} \right){:} + {:}e^j \left({:}e_k [e_j, e^k]{:} \right){:} \right.$$
$$\left. - {:}e_j \left({:}e_k [e^j, e^k]{:} \right){:} - {:}e^j \left({:}e^k [e_j, e_k]{:} \right){:} \right),$$

where $e_i := \Pi \epsilon_i \in \Pi \ell$ and $e^i := \Pi \epsilon^i \in \Pi \bar{\ell}$.

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Remarks:

1. When $V_+ = \mathfrak{g}$, we recover a classical construction for Manin triples $\mathfrak{g}^{\mathbb{C}} = \ell \oplus \bar{\ell}$ by (Getzler 1995), who required precisely the condition $(\star)$.
2. The condition $(\star)$ is satisfied if w is 'holomorphic', i.e., $[w, \ell] \subset \ell$.

Main results (Algebra)

Theorem (AC, De Arriba De La Hera, Garcia-Fernandez)

A solution $(V_+ \otimes \mathbb{C} = \ell \oplus \bar{\ell}, \varepsilon)$ of the Killing spinor equations on $(\mathfrak{g}, \langle -, - \rangle)$ such that $\dim V_+ = 2n$ is even and $\varepsilon \in V_+$ is holomorphic (i.e., $[\varepsilon, \ell] \subset \ell$), induces an embedding

$$\mathrm{V}^c(\mathcal{K}(1|2)) \hookrightarrow \mathrm{V}^k(\mathfrak{g}^{\mathbb{C}}_{super})$$

with central charge $c = 3\big(n + \frac{4}{k}\langle \varepsilon, \varepsilon \rangle\big)$, where $\mathtt{J}, \mathtt{H} \in \mathrm{V}^c(\mathcal{K}(1|2))$ map into

$$\mathtt{J}^{\mathfrak{g}} = \frac{i}{k} :e^j e_j: - \frac{2}{k} S\left(i\Pi(\varepsilon_\ell - \varepsilon_{\bar{\ell}})\right),$$

$$\mathtt{H}^{\mathfrak{g}} = \frac{1}{k}(:e_j(Se^j): + :e^j(Se_j):) + \frac{1}{k^2}\Big(:e_j\Big(:e^k[e^j, e_k]:\Big): + :e^j\Big(:e_k[e_j, e^k]:\Big): \\ - :e_j\Big(:e_k[e^j, e^k]:\Big): - :e^j\Big(:e^k[e_j, e_k]:\Big):\Big).$$

Geometry:
embeddings in the chiral de Rham complex

The chiral de Rham complex

New problem: find when Killing spinors on Courant algebroids $E \to M$ induce a vertex-algebra embedding:

$$\mathbf{V}^{c}(\mathcal{K}(1\,|\,2)) \longhookrightarrow \Gamma\left(M, \Omega^{\mathrm{ch}}_{E}\right)$$

Here, Ω^{ch}_{E} is the so-called **chiral de Rham complex**, defined by the following result (after Malikov, Schechtman, Vaintrob 1999):

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Theorem (Heluani 2009; cf. Bressler 2007)

To each Courant algebroid E, one can attach functorially a sheaf of $N = 1$ SUSY vertex algebras Ω^{ch}_{E} on M (but only 'localizing' at level $k = 2$).

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We will focus on the so-called 'string' Courant algebroids.

Fix a manifold M, a compact Lie group G with quadratic Lie algebra $(\mathfrak{g}, \langle -, - \rangle)$, a principal G-bundle P with

$$p_1(P) := [\langle F_A \wedge F_A \rangle] = 0 \in H^4_{\mathrm{dR}}(M, \mathbb{R}),$$

for a connection A on P, and $H \in \Omega^3(M)$ satisfying the *Bianchi identity*

$$\mathrm{d}H = \langle F_A \wedge F_A \rangle.$$

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Then there exists a 'string' Courant algebroid $E = E_{H,A}$, given explicitly by

$$\begin{aligned}
E &= TM \oplus \operatorname{ad} P \oplus T^*M,\\
\pi\colon E &\longrightarrow M, \quad X + r + \alpha \longmapsto X,\\
\langle e_1, e_1\rangle &= \alpha(X) + \langle r, r\rangle,\\
[e_1, e_2] &= [X,Y] - F_A(X,Y) + \iota_X \mathrm{d}_A s - \iota_Y \mathrm{d}_A r - [r,s]\\
&\quad + L_X\beta - \iota_Y \mathrm{d}\alpha + \iota_Y\iota_X H + 2\langle \mathrm{d}_A r, s\rangle + 2\langle \iota_X F_A, s\rangle - 2\langle \iota_Y F_A, r\rangle.
\end{aligned}$$

where $e_1 = X + r + \alpha, e_2 = Y + s + \beta \in \Gamma(E)$.

Admissible metrics on string Courant algebroids

Fix a manifold M, a compact Lie group G with quadratic Lie algebra $(\mathfrak{g}, \langle -, - \rangle)$, a principal G-bundle P with

$$p_1(P) := [\langle F_A \wedge F_A \rangle] = 0 \in H^4_{\mathrm{dR}}(M, \mathbb{R}),$$

for a connection A on P, and $H \in \Omega^3(M)$ satisfying the *Bianchi identity*

$$\mathrm{d}H = \langle F_A \wedge F_A \rangle.$$

Then we obtain a 'string' Courant algebroid $E = TM \oplus \mathrm{ad}\, P \oplus T^*M$.

A generalized metric $E = V_+ \oplus V_-$ is called *admissible* if the anchor restricts to an isomorphism

$$\pi|_{V_+} \colon V_+ \xrightarrow{\cong} TM, \quad \text{and} \quad \langle -, - \rangle|_{V_+} > 0.$$

In this case, the generalized metric $E = V_+ \oplus V_-$ determines a Riemannian metric g on TM and an isotropic splitting $\sigma \colon TM \to E$. Conversely,

$$V_+ = \{X + g(X) \mid X \in TM\},$$
$$V_- = \{X + r - g(X) \mid X \in TM,\ r \in \mathrm{ad}\, P\}.$$

The twisted Hull–Strominger system

This system describes a subset of solutions of the Killing spinor equations.

Suppose $X = (M, J)$ is a complex manifold X of dim. n with trivial $K_X = \Lambda^{n,0} T^* X$ (as C^∞ bundle).

Definition: An $\mathrm{SU}(n)$-structure on X is a pair (Ψ, ω) such that

- Ψ is a C^∞ complex $(n, 0)$-form on X,
- $\|\Psi\|_g = 1$ (pointwise norm),
- $\omega \in \Omega^{1,1}(X)$ positive $(1, 1)$-form, with metric $g = \omega(\cdot, J\cdot)$.

Lee form: $\theta_\omega = Jd^*\omega$.

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Twisted Hull–Strominger system (Garcia-Fernandez, Rubio, Shahbazi & Tipler) for an $\mathrm{SU}(n)$-structure (Ψ, ω) and a connection A on P:

$$
\begin{aligned}
F_A^{0,2} = 0, \quad F_A \wedge \omega^{n-1} &= 0, \\
\mathrm{d}\Psi - \theta_\omega \wedge \Psi &= 0, \\
\mathrm{d}\theta_\omega &= 0, \\
\mathrm{d}\mathrm{d}^c\omega + \langle F_A \wedge F_A \rangle &= 0.
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Proposition (cf. Hull; Strominger; Garcia-Fernandez, Rubio & Tipler...)

A solution (Ψ, ω, A) of the twisted Hull–Strominger system induces a solution of the Killing spinor equations on the string Courant algebroid $E = E_{-\mathrm{d}^c\omega, A}$ for an admissible generalized metric and a pure spinor η.

Admissible local frames and the torsion bivector

A solution (Ψ, ω, A) of the twisted Hull–Strominger system determines a unique maximal holomorphic atlas $\{\mathbf{z}_\alpha \colon U_\alpha \to \mathbb{C}^n\}_{\alpha \in I}$ such that the transition maps $\mathbf{z}_\alpha \circ \mathbf{z}_\beta^{-1}$ have constant Jacobian determinants:

$$\det\left(\partial z_\alpha^i / \partial z_\beta^j\right) = \text{const.}$$

Using these local coordinates $\mathbf{z}_\alpha$, we define an '**admissible local frame**'

$$\epsilon_i := \frac{\partial}{\partial \bar{z}^i} + g\frac{\partial}{\partial \bar{z}^i}, \quad \epsilon^i := g^{-1}\mathrm{d}\bar{z}^i + \mathrm{d}\bar{z}^i \in \Gamma(U_\alpha, E^{\mathbb{C}}),$$

so $V_+^{\mathbb{C}} = \ell \oplus \bar{\ell}$, with $\ell = \mathrm{span}\{\epsilon_i\}$, $\bar{\ell} = \mathrm{span}\{\epsilon^i\}$, and $\langle \epsilon_i, \epsilon^j \rangle = \delta_{ij}$.

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The *torsion bivector* $\sigma_\omega \in \Gamma\left(\wedge^{0,2} TX\right)$ of a hermitian metric $g = \omega(\cdot, J\cdot)$ is defined in local holomorphic coordinates $(z^1, \ldots, z^n)$ by the formula

$$\sigma_\omega = \sum \left[g^{-1}\left(\mathrm{d}\bar{z}^i\right), \left(g^{-1} \otimes g^{-1}\right)\left(\iota_{\partial/\partial \bar{z}^i}(\partial\omega)\right)\right]^{0,2},$$

where $[-,-]$ is the Schouten bracket of multivector fields.

Main result (Geometry)

Theorem (AC, De Arriba De La Hera, Garcia-Fernandez)

Suppose (Ψ, ω, A) is a solution of the twisted Hull–Strominger system.

Let • $E = E_{-\mathrm{d}^c\omega, A} = V_+ \oplus V_-$ be the corresponding generalized metric,

- $\{\epsilon_j, \epsilon^j\}$ the admissible local frames of $V_+^{\mathbb{C}} = \ell \oplus \overline{\ell}$,
- $u = -\Pi\theta_\omega \in \Gamma(\Pi E) \subset \Gamma(X, \Omega^{\mathrm{ch}}_{E^{\mathbb{C}}})$.

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Then the following formulas are independent of the choice of admissible local frames and so define global sections of $\Omega^{\mathrm{ch}}_{E^{\mathbb{C}}}$ (with $e_i = \Pi\epsilon_i$, $e^i = \Pi\epsilon^i$):

$$\mathtt{J}^E = \frac{i}{2} {:}e^j e_j{:} - Siu,$$

$$\mathtt{H}^E = \frac{1}{2}\left({:}e_j(Se^j){:} + {:}e^j(Se_j){:}\right) + \frac{1}{4}\Big({:}e_j\Big({:}e^k[e^j, e_k]{:}\Big){:} + {:}e^j\Big({:}e_k[e_j, e^k]{:}\Big){:}$$
$$- {:}e_j\Big({:}e_k[e^j, e^k]{:}\Big){:} - {:}e^j\Big({:}e^k[e_j, e_k]{:}\Big){:}\Big) + T(\pi_{\overline{\ell}} u)$$

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Furthermore, if $\sigma_\omega = 0$, then they induce a vertex-algebra embedding with central charge $c = 3\dim_{\mathbb{C}} X$:

$$\mathrm{V}^c(\mathcal{K}(1|2)) \longrightarrow \Gamma(X, \Omega^{\mathrm{ch}}_{E^{\mathbb{C}}}),$$
$$\mathtt{J} \longmapsto \mathtt{J}^E$$
$$\mathtt{H} \longmapsto \mathtt{H}^E$$

Open problem (in progress): relation to previously known embeddings?:

- Calabi–Yau (Malikov, Schechtman & Vaintrob 1999)
- Kähler manifolds (Ben-Zvi, Heluani, Szczesny 2008; Heluani 2009)
- generalized Calabi-Yau manifolds (Heluani & Zabzine 2011)

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- generalized Calabi-Yau manifolds (Heluani & Zabzine 2011)

Examples:

The following examples satisfy the conditions of the above theorem and so provide $N = 2$ embeddings:

- All solutions (Ψ, ω, A) of the twisted Hull–Strominger system on *compact* complex surfaces.
- Specific families of solutions on the Iwasawa manifold ($\dim_{\mathbb{C}} X = 3$)

$$X = \left\{ \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \middle| z_1, z_2, z_3 \in \mathbb{C} \right\} \Big/ \left\{ \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \middle| z_1, z_2, z_3 \in \mathbb{Z}[\mathrm{i}] \right\}.$$

- Specific families of solutions on Sébastien Picard's Calabi–Yau 3-fold $X = \mathbb{C}^3/\mathbb{Z}[\mathrm{i}]^3$, where now the action of $(a, b, c) \in \mathbb{Z}[\mathrm{i}]^3$ is given by

$$\mathbb{C}^3 \longrightarrow \mathbb{C}^3, \quad (x, y, z) \longmapsto (x + a, y + c, z + \bar{a}y + b).$$

Geometric applications of the algebraic results

Twisted Calabi–Yaus

Definition (Garcia-Fernandez, Rubio, Shahbazi & Tipler 2018)

The *twisted Calabi–Yau system* is the twisted Hull–Strominger system with structure group $G = \{1\}$ (i.e, without gauge fields):

$$\mathrm{d}\Psi = \theta_\omega \wedge \Psi, \qquad \mathrm{d}\theta_\omega = 0, \qquad \mathrm{d}\mathrm{d}^c\omega = 0.$$

A solution (Ψ, ω) of the twisted Calabi–Yau system has pluriclosed metric $(\mathrm{d}\mathrm{d}^c\omega = 0) \Longrightarrow$ positive Aeppli class $[\omega] \in H_A^{1,1}(X, \mathbb{R})$.

Here, $H_A^{\bullet,\bullet}(X) = \dfrac{\ker \partial\bar\partial}{\mathrm{Im}\,\partial + \mathrm{Im}\,\bar\partial}$.

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Here, $H^{\bullet,\bullet}_A(X) = \dfrac{\ker \partial\bar{\partial}}{\mathrm{Im}\,\partial + \mathrm{Im}\,\bar{\partial}}$.

Proposition (AC, De Arriba De La Hera, Garcia-Fernandez)

Let K be an even-dimensional compact Lie group. Then a left-invariant solution (Ψ, ω) of the twisted Calabi–Yau equation on K induces a solution of the Killing spinor equations (with $\varepsilon = 2\theta_\omega|_{V_+}$) on the quadratic Lie algebra (with $H = -d^c\omega$)

$$(\mathfrak{g} = \Gamma(TK \oplus T^*K)^K, [\cdot,\cdot]_H, \langle\cdot,\cdot\rangle).$$

Proposition (AC, De Arriba De La Hera, Garcia-Fernandez)

Let K be an even-dimensional compact Lie group that carries a left-invariant solution (Ψ, ω) of the twisted Calabi–Yau equation. Consider the quadratic Lie algebra (with $H = -d^c\omega$)

$$(\mathfrak{g} = \Gamma(TK \oplus T^*K)^K, [\cdot,\cdot]_H, \langle\cdot,\cdot\rangle).$$

If $\theta_\omega^\sharp$ is holomorphic, then the pair (Ψ, ω) induces a vertex algebra embedding

$$\begin{array}{ccccc} \mathrm{V}^c(\mathcal{K}(1|2)) & \hookrightarrow & \mathrm{V}^2(\mathfrak{g}^{\mathbb{C}}_{super}) & \hookrightarrow & \Gamma(K, \Omega_K^{\mathrm{ch},H}), \\ \mathtt{J} & \longmapsto & \mathtt{J}^{\mathfrak{g}} & \longmapsto & \mathtt{J}(\Psi,\omega) \\ \mathtt{H} & \longmapsto & \mathtt{H}^{\mathfrak{g}} & \longmapsto & \mathtt{H}(\Psi,\omega) \end{array}$$

with central charge $c = 3(n + 2\langle\varepsilon, \varepsilon\rangle)$, in the space of global sections of $\Omega_K^{\mathrm{ch},H}$ with $H = -d^c\omega$.

Here, the *H-twisted standard chiral de Rham complex* is

$$\Omega_M^{\mathrm{ch},H} := \Omega_E^{\mathrm{ch}}, \quad \text{with } E = (TM \oplus T^*M, [\cdot,\cdot]_H).$$

Example: $S^3 \times S^1$ viewed as a Lie group $K = SU(2) \times U(1)$, with Lie alg.

$$\mathfrak{k} = \langle v_1, v_2, v_3, v_4 \rangle$$

where $[v_2, v_3] = -v_1, \quad [v_3, v_1] = -v_2, \quad [v_1, v_2] = -v_3, \quad [v_4, \cdot] = 0.$

$$v_1 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},\ v_1 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ v_3 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \in \mathfrak{su}(2),\ v_4 = 1 \in \mathfrak{u}(1).$$

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Fix $\ell \in \mathbb{R}_{>0}$.
For any $x \in \mathbb{R}_{>0}$, $\exists$ left-invariant solution (Ψ_x, ω_x) of the twisted Calabi–Yau equations

$$\mathrm{d}\Psi = \theta_\omega \wedge \Psi, \qquad \mathrm{d}\theta_\omega = 0, \qquad \mathrm{d}\mathrm{d}^c\omega = 0,$$

given in terms of the dual left-invariant differential forms v^1, v^2, v^3, v^4 by

$$\omega_x = \ell x v^{41} + \ell v^{23},$$
$$\Psi_x = \tfrac{\ell}{2}(iv^1 + xv^4) \wedge (v^2 + iv^3),$$

with

- Lee form $\theta_x = -xv^4$,
- complex structure $I_x \cong x \in \mathbb{R}_{>0}$ given by $I_x v_4 = xv_1$, $I_x v_2 = v_3$,
- $H_\ell := -\mathrm{d}^c\omega_x = \ell v^{123}$,
- Aeppli class on $X_x = (K, I_x)$: $[\omega_x] \cong a := \ell x \in H^{1,1}_A(X_x, \mathbb{R}) \cong \mathbb{R}$.

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In complex coordinates, $X_x \cong (\mathbb{C}^2 \backslash \{0\})/\{(z_1, z_2) \sim (e^x z_1, e^x z_2)\}$ (Hopf surface).

Theorem (AC, De Arriba De La Hera, Garcia-Fernandez)

Let $\ell \in \mathbb{R}_{>0}$. The family of solutions $(V_+^x, \varepsilon_+^x = -xv_+^4, l_x)$ of the Killing spinor equations on $\mathfrak{g}_\ell = (\Gamma(TK \oplus T^*K)^K, [-,-]_{H_\ell})$, for $x \in \mathbb{R}_{>0}$, where $K = SU(2) \times U(1)$, induce vertex-algebra embeddings

$$\begin{array}{ccccc} \mathrm{V}^c(\mathcal{K}(1|2)) & \hookrightarrow & \mathrm{V}^2(\mathfrak{g}^{\mathbb{C}}_{\ell,super}) & \hookrightarrow & \Gamma\left(K, \Omega_K^{\mathrm{ch},H_\ell}\right) \\ \mathtt{J} & \longmapsto & \mathtt{J}^{\mathfrak{g}} & \longmapsto & \mathtt{J}_x := \mathtt{J}(V_+^x, \varepsilon_+^x, l_x) \\ \mathtt{H} & \longmapsto & \mathtt{H}^{\mathfrak{g}} & \longmapsto & \mathtt{H}_x := \mathtt{H}(V_+^x, \varepsilon_+^x, l_x) \end{array}$$

with central charge $c = 6 + 6/\ell$. Furthermore, the solution $(V_+^x, \varepsilon_+^x, l_x)$ for any $x \in \mathbb{R}_{>0}$ is $(0,2)$-mirror to the solution $(V_+^{\hat{x}}, \varepsilon_+^{\hat{x}}, -l_{\hat{x}})$, where

$$\hat{x} = \frac{1}{\ell x}.$$

More precisely, there is a vertex algebra automorphism

$$\psi \in \mathrm{Aut}\left((\Omega_K^{\mathrm{ch},H_\ell})^{\mathfrak{u}(1)}\right),$$

such that

$$\psi(\mathtt{J}_x) = -\mathtt{J}_{\hat{x}}, \quad \psi(\mathtt{H}_x) = \mathtt{H}_{\hat{x}}.$$

Remark: Exchange of complex structure and Aeppli class:

$$\left(a = \ell x \in H_A^{1,1}(X, \mathbb{R}),\ I_x\right) \longleftrightarrow \left(\hat{a} = \ell \hat{x} = x^{-1} \in H_A^{1,1}(X, \mathbb{R}),\ -I_{\hat{x}} = -I_{a^{-1}}\right).$$

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One can use the the above $(0,2)$-mirror symmetry to obtain a canonical isomorphism of sheaves of SUSY vertex algebras

$$\left(\Omega^{\mathrm{ch},H_\ell}_{X_x}\right)^{\mathfrak{u}(1)} \cong \left(\Omega^{\mathrm{ch},H_\ell}_{X_{\hat{x}}}\right)^{\mathfrak{u}(1)}.$$

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Long-term programme: (0,2) mirror symmetry for non-Kähler manifolds.

Thank you!

Happy Birthday Alastair!