Equivariant birational geometry

Equivariant geometry

Solved problem

What are the finite subgroups

$$G \subset \mathrm{PGL}_3 = \mathrm{Aut}(\mathbb{P}^2),$$

up to conjugation?

Blichfeldt 1905.

Equivariant geometry

Open problem

What are the embeddings of finite groups

$$G \hookrightarrow \mathrm{PGL}_3,$$

up to conjugation in

$$\mathrm{Cr}_2 := \mathrm{BirAut}(\mathbb{P}^2)?$$

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- $H_p(x) := \max(1, |x|_p)$, $x \in \mathbb{Q}_p$

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$$\int_{\mathbb{Q}_p} H_p(x_p)^{-s} \mathrm{d}x_p = \int_{U(0)} H_p(x_p)^{-s} \mathrm{d}x_p + \sum_{j \geq 1} \int_{U(j)} H_p(x_p)^{-s} \mathrm{d}x_p$$

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$$= 1 + \sum_{j \geq 1} p^{-js} \mathrm{vol}(U(j))$$

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We interpret this as a volume with respect to a natural measure.

Toric varieties

$X = X_\Sigma$ - projective equivariant compactification of $T = \mathbb{G}_m^d$.

- $N \simeq \mathbb{Z}^d$, $M = \mathrm{Hom}(N, \mathbb{Z})$, $\Sigma = \{\sigma\}$ - fan
- $e_1, \ldots, e_n$ - 1-dimensional cones in Σ
- $$0 \to M \to \mathrm{PL}(\Sigma) \to \mathrm{Pic}(X_\Sigma) \to 0,$$

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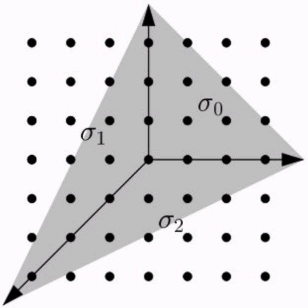
- $N \simeq \mathbb{Z}^d$, $M = \mathrm{Hom}(N, \mathbb{Z})$, $\Sigma = \{\sigma\}$ - fan
- $e_1, \ldots, e_n$ - 1-dimensional cones in Σ
- $$0 \to M \to \mathrm{PL}(\Sigma) \to \mathrm{Pic}(X_\Sigma) \to 0,$$
  $\varphi = \varphi_{\mathbf{s}} \in \mathrm{PL}(\Sigma)$ is defined by its values on e_j: $\varphi_{\mathbf{s}}(e_j) = s_j \in \mathbb{C}$
- $T(\mathbb{Q}_p)/T(\mathbb{Z}_p) = N$

Height integrals



Height integrals

- Local heights:

$$H_p(\varphi_{\mathbf{s}}, t_p) := p^{\varphi_{\mathbf{s}}(\bar{t}_p)}$$

-

$$\int_{T(\mathbb{Q}_p)} H_p(\varphi_{\mathbf{s}}, t_p)^{-1}\, \mathrm{d}t_p = \left(\sum_{r=1}^{d} \sum_{\sigma \in \Sigma(r)} (-1)^r \left(\sum_{n \in \sigma \cap N} p^{-\varphi_{\mathbf{s}}(n)} \right) \right)$$

$$= \sum_{r=1}^{d} \sum_{\sigma \in \Sigma(r)} (-1)^r \prod_{e_j \in \sigma} \frac{1}{1 - p^{-s_j}}$$

Tamagawa numbers / Peyre (1995)

- X – smooth projective Fano variety, $\dim(X) = d$, over a number field F
- $-K_X$ is equipped with an adelic metrization.

For $x \in X(F_v)$ choose local analytic coordinates $x_1, \ldots, x_d$, in a neighborhood U_x. In U_x, a section of the canonical line bundle has the form $\mathsf{s} := \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_d$. Put

$$\tau_v = \tau_{X,v} := \|\mathsf{s}\|_v \mathrm{d}x_1 \cdots \mathrm{d}x_d,$$

where $\mathrm{d}x_1 \cdots \mathrm{d}x_d$ is the standard normalized Haar measure on F_v^d. It globalizes to $X(F_v)$.

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where $\mathrm{d}x_1 \cdots \mathrm{d}x_d$ is the standard normalized Haar measure on F_v^d. It globalizes to $X(F_v)$. For almost all v, and Zariski open $U \subset X$,

$$\int_{U(F_v)} \tau_v = \int_{X(F_v)} \tau_v = \int_{X(\mathfrak{o}_v)} \tau_v = \sum_{\tilde{x} \in X(\mathbb{F}_q)} \int_{\pi^{-1}(\tilde{x})} \tau_v = \frac{\#X(\mathbb{F}_q)}{q^d}.$$

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- $$\int_{X(F_v)} \tau_v = \frac{\#X(\mathbb{F}_q)}{q^n}, \quad \forall' v$$
- If $X \supset U \subset Y$, then

$$\frac{\#X(\mathbb{F}_q)}{q^n} = \int_{X(F_v)} \tau_v = \int_{U(F_v)} \tau_v = \int_{Y(F_v)} \tau_v = \frac{\#Y(\mathbb{F}_q)}{q^n}, \quad \forall' v$$

Igusa integrals: local theory

Let $U := X \setminus D$, with

$$D = \cup_{\alpha \in \mathcal{A}} D_\alpha, \quad -K_X = \sum \rho_\alpha D_\alpha,$$

where D_α are geometrically irreducible, smooth, and intersecting transversally.

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$D_A \subset X$ is smooth, of codimension $\#A$ (or empty).

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$$Z_v(\mathbf{s}) := \int_{U(F_v)} \prod_{\alpha \in \mathcal{A}} H_\alpha(x)^{-s_\alpha} \mathrm{d}\tau_v$$

Local computations

In charts, via partition of unity: in a neighborhood of $x \in D_A^\circ(F)$ it takes the form

$$\int \prod_{\alpha \in A} |x_\alpha|_v^{s_\alpha - \rho_\alpha} \, \mathrm{d}\tau_v$$

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Essentially, this is a product of integrals of the form

$$\int_{|x|_v \leq 1} |x|_v^{s-1} \mathrm{d}x_v.$$

Denef's formula

For almost all v one has:

$$Z_v(\mathbf{s}) = \sum_{A} \frac{\#D_A^\circ(\mathbb{F}_q)}{q^{\dim(X)}} \prod_{\alpha \in A} \frac{q-1}{q^{s_\alpha - \rho_\alpha + 1} - 1}.$$

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Specialize to $s_\alpha = \rho_\alpha$, for all $\alpha \in \mathcal{A}$:

$$Z_v(\rho) = \sum_A \frac{\#D_A^\circ(\mathbb{F}_q)}{q^{\dim(X)}} = \frac{\#X(\mathbb{F}_q)}{q^{\dim(X)}}.$$

Applications

The integral

- is an invariant under blowups,
- encodes information about singularities of X,
- plays a central role in analytic/spectral approches to Manin's conjectures, volume asymptotics, etc.

Basic questions

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- How much geometry can be read off from arithmetic?

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These properties are completely understood in dimensions ≤ 2, and coincide, over $\mathbb{C}$. Stable rationality over nonclosed ground fields k is still an open problem.

Specialization of (stable) rationality

- Voisin (2013): integral decomposition of Δ (Bloch-Srinivas)

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- settle the problem of stable rationality for threefolds, with the exception of threefolds birational to a cubic threefold (Hassett-Kresch-T. 2015)
- find many examples of irrational hypersurfaces of low degree (Schreieder 2017)
- ...

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- **Kontsevich–T. (2017):** Same formula for

$$\mathrm{Burn}(K) \to \mathrm{Burn}(k),$$

  the free abelian group spanned by classes of varieties over the corresponding field, modulo rationality.

Specialization (Kontsevich-T. 2017)

- Let $\mathfrak{o} \simeq k[[t]]$, $K \simeq k((t))$, $\mathrm{char}(k) = 0$.
- Let X/K be a smooth proper (or projective) variety, with function field $L = K(X)$.
- Choose a regular model

$$\pi : \mathcal{X} \to \mathrm{Spec}(\mathfrak{o}),$$

such that π is proper and the special fiber $\mathcal{X}_0$ over $\mathrm{Spec}(k)$ is a simple normal crossings (snc) divisor:

$$\mathcal{X}_0 = \cup_{\alpha \in \mathcal{A}} \, d_\alpha D_\alpha, \quad d_\alpha \in \mathbb{Z}_{\geq 1}.$$

- Put

$$\rho([L/K]) := \sum_{\emptyset \neq A \subseteq \mathcal{A}} (-1)^{\#A-1} [D_A \times \mathbb{A}^{\#A-1}/k] \in \mathrm{Burn}(k),$$

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This motivated the search for ways to "integrate in presence of group actions".

Equivariant bir. types (Kontsevich-T. 2019)

- G - finite abelian group, $A = G^\vee$ its group of characters,
- X - smooth projective, of dimension n, with regular G-action,
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- β_α - (equivalence class of) representation of G, acting in the tangent space $\mathcal{T}_{X,x_\alpha}$, for $x_\alpha \in F_\alpha$, i.e.,

$$\beta_\alpha := [a_{1,\alpha}, \ldots, a_{n,\alpha}],$$

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  $$\beta_\alpha := [a_{1,\alpha}, \dots, a_{n,\alpha}],$$

  an unordered n-tuple of characters $a_i \in A$,
- $$\beta : X \mapsto \sum_\alpha \beta_\alpha.$$

First examples: $\mathbb{P}^2$

Consider an action of $\mathbb{Z}/N\mathbb{Z}$ on $X = \mathbb{P}^2$ given by

$$(x : y : z) \mapsto (\zeta^a x : \zeta^b y : z),$$

$$\zeta = \zeta_N, \quad a, b \in \mathbb{Z}/N\mathbb{Z}, \quad \gcd(a, b, N) = 1, \quad a \neq b.$$

Fixed points are

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Then

$$\beta(X) = [a, b] + [a - b, -b] + [b - a, -a].$$

First examples: $\mathbb{P}^2$

All such actions are equivalent. Declare $\beta(X) = 0$, i.e.,

$$[a, b] = -[b - a, -a] - [a - b, -b]$$

Allowing

$$[a, b] = -[a, -b]$$

we find

$$[a, b] = [a, b - a] + [a - b, b].$$

Birational types $\mathcal{B}_2(\mathbb{Z}/N\mathbb{Z})$

Generators: $[a, b]$, $a, b \in \mathbb{Z}/N\mathbb{Z}$, $\gcd(a, b, N) = 1$

Relations:

- $[a, b] = [b, a]$
- $[a, b] = [a, b - a] + [a - b, b]$ if $a \neq b$
- $[a, a] = [a, 0]$

Birational types

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$$\mathrm{rk}_{\mathbb{Q}}(\mathcal{B}_2(G)) = \frac{p^2+23}{24} = \frac{p^2-1}{24} + 1$$

Birational types $\mathcal{B}_n(G)$

Consider the $\mathbb{Z}$-module

$$\mathcal{B}_n(G)$$

generated by unordered tupels $[a_1, \dots, a_n]$, $\quad a_i \in A$, such that

(G) $\sum_i \mathbb{Z}a_i = A$, and

(B) for all $a_1, a_2, b_1, \dots, b_{n-2} \in A$ we have

$$[a_1, a_2, b_1, \dots b_{n-2}] =$$

$$[a_1 - a_2, a_2, b_1, \dots, b_{n-2}] + [a_1, a_2 - a_1, b_1, \dots, b_{n-2}] \text{ if } a_1 \neq a_2,$$

$$[a_1, 0, b_1, \dots, b_{n-2}] \qquad \text{if } a_1 = a_2.$$

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Jumps at

$$p = 43, 59, 67, 83, \ldots$$

Birational types

Consider $X^G = \sqcup F_\alpha$ and record eigenvalues of G

$$[a_{1,\alpha}, \ldots, a_{n,\alpha}]$$

in the tangent space $\mathcal{T}_{x_\alpha} X$, at some $x_\alpha \in F_\alpha$. Put

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Kontsevich-T. 2019

The class

$$\beta(X) \in \mathcal{B}_n(G)$$

is a well-defined G-equivariant birational invariant.

Birational types

Variant: introduce the quotient

$$\mu^- : \mathcal{B}_n(G) \to \mathcal{B}_n^-(G)$$

by an additional relation

$$[a_1, a_2, \ldots, a_n] = -[-a_1, a_2, \ldots, a_n].$$

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The class of $\mathbb{P}^n$, $n \geq 2$, with linear action of $G := \mathbb{Z}/N\mathbb{Z}$ is

- torsion in $\mathcal{B}_n(G)$ and
- trivial in $\mathcal{B}_n^-(G)$.

Connections to arithmetic groups

$$\mathcal{B}_n^-(G) \otimes \mathbb{Q} \simeq \mathrm{H}^{\frac{n(n-1)}{2}}(\Gamma(G,n), \mathrm{or}_n^{\otimes n}) = \mathrm{H}_0(\Gamma(G,n), \mathrm{St}_n \otimes \mathrm{or}_n)$$

where

- $$\Gamma(G,n) \subset \mathrm{GL}_n(\mathbb{Z})$$

  is a congruence subgroup,
- or is the orientation (the sign of the determinant), and
- St_n is the Steinberg representation.

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In particular, the groups $\mathcal{B}_n(G)$ and $\mathcal{B}_n^-(G)$ carry Hecke operators.

Connections to arithmetic groups

$n = 2$:

$$\mathcal{B}_2(\mathbb{Z}/p) \otimes \mathbb{Q} \simeq \mathrm{H}^1(X_1(p), \mathbb{Q}).$$

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Manin symbols.

Equivariant Burnside group (Kresch-T. 2020)

Let G be a finite group. Let

$$\mathrm{Burn}_n(G)$$

be the quotient of the free abelian group generated by symbols

$$(H, N \circlearrowright K, \beta),$$

where

- $H \subseteq G$ is an abelian subgroup, $N = N_G(H)/H$,
- K is an N-Galois algebra over a field of transcendence degree $d \leq n$ over k, up to isomorphism, and β is a faithful $(n-d)$-dimensional representation of H,

modulo somewhat complicated blowup relations.

Equivariant Burnside group: relations

The symbols are subject to **conjugation** and **blowup** relations:

(C): $(H, N \circlearrowright K, \beta) = (H', N' \circlearrowright K, \beta')$, when

$$H' = gHg^{-1}, \quad N' = N_G(H')/H', \qquad \text{with } g \in G,$$

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(B1): $(H, N \circlearrowright K, \beta) = 0$ when $b_1 + b_2 = 0$.

Equivariant Burnside group: relations

(B2): $(H, N \circlearrowright K, \beta) = \Theta_1 + \Theta_2$, where

$$\Theta_1 = \begin{cases} 0, & \text{if } b_1 = b_2, \\ (H, N \circlearrowright K, \beta_1) + (H, N \circlearrowright K, \beta_2), & \text{otherwise,} \end{cases}$$

with

$$\beta_1 := (b_1, b_2 - b_1, b_3, \ldots, b_{n-d}), \quad \beta_2 := (b_1 - b_2, b_2, b_3, \ldots, b_{n-d}),$$

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$$\Theta_2 = \begin{cases} 0, & \text{if } b_i \in \langle b_1 - b_2 \rangle \text{ for some } i, \\ (\overline{H}, \overline{N} \circlearrowright \overline{K}, \bar{\beta}), & \text{otherwise,} \end{cases}$$

with

$$\overline{H}^\vee := H^\vee / \langle b_1 - b_2 \rangle, \quad \bar{\beta} := (\bar{b}_2, \bar{b}_3, \dots, \bar{b}_{n-d}), \quad \bar{b}_i \in \overline{H}^\vee.$$

Equivariant Burnside group: relations

Model case: Blowing up an isolated point (with abelian stabilizer) on a surface.

It will explain the action of $\overline{N}$ on $\overline{K}$.

Equivariant Burnside group

The class of a G-variety is computed on a standard model X:

- X is smooth projective,
- there exists a Zariski open $U \subset X$ such that G acts freely on U,
- the complement $X \setminus U$ is a normal crossings divisor,
- for every $g \in G$ and every irreducible component D of $X \setminus U$, either $g(D) = D$ or $g(D) \cap D = \emptyset$.

Equivariant Burnside group

Passing to a standard model X, define:

$$[X \circlearrowleft G] := \sum_H \sum_F (H, N \circlearrowright k(F), \beta_F(X)) \in \mathrm{Burn}_n(G),$$

where

- the sum is over all (orbits of) strata $F \subset X$ with generic stabilizer (orbits of) H,
- the symbols record the eigenvalues of H in the tangent space at $x \in F$, as before, as well as the N-action on the function field of F, respectively the orbit of F.

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This is a G-birational invariant.

Equivariant birational geometry

We work over an algebraically closed field k, of characteristic zero. Let X be a smooth projective G-variety (regular, generically free action).

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Basic problems:

- Linearizability: $X \sim_G \mathbb{P}(V)$, where V is a faithful linear representation of G

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- Linearizability: $X \sim_G \mathbb{P}(V)$, where V is a faithful linear representation of G
- Stable Linearizability: $X \times \mathbb{P}^m \sim_G \mathbb{P}(V)$, with trivial action on the second factor

Linear actions

Problem

What are the finite subgroups of PGL_3, up to conjugation?

Linear actions

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What are the finite subgroups of PGL_3, up to conjugation? And PGL_4?

Blichfeldt 1905.

Equivariant geometry

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- The actions are realized as regular actions on minimal rational surfaces X,
- By MMP, X is either a Del Pezzo surface or a conic bundle,
- If the (anticanonical) degree is small, the action is rigid, and visible via the induced action on the Picard group $\mathrm{Pic}(X)$, i.e., through the Weyl group of the associated root lattice.

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- Equivariant birational rigidity (analysis of singularities, ...)
- Cohomology, e.g., $\mathrm{H}^1(G, \mathrm{Pic}(X))$
- G-equivariant intermediate Jacobians (building on work of Clemens-Griffiths, Benoist-Wittenberg, ...), $\ldots$

Basic facts

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- If X is rational and G is cyclic, then $X^G \neq \emptyset$.
- If $Y \dashrightarrow X$ is a G-birational map between smooth projective G-varieties, and G is abelian, then

$$Y^G \neq \emptyset \Leftrightarrow X^G \neq \emptyset.$$

Basic facts

More precisely, let X be smooth projective of dimension n, G abelian, and let $\mathfrak{p} \in X^G$. Let $\{a_1, \ldots, a_n\}$ be the characters (weights) of G in the tangent space to X at $\mathfrak{p}$.

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Let $Y \to X$ be a G-equivariant blowup. Then Y contains a point $\mathfrak{q} \in Y^G$ (in the preimage of $\mathfrak{p}$) with weights $\{b_1, \ldots, b_n\}$ in the tangent space, and such that

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Thus, $\mathbb{Z}/N\mathbb{Z}$-actions on $\mathbb{P}^n$, with $n \geq 2$ are equivariantly birational.

Abelian actions on surfaces

- If there is no curve of genus ≥ 1 in the fixed locus X^G, then all actions are linear, with the exception of one fixed-point free action of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$.

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In particular, $\mathcal{B}_2(G)$ does not give anything new in dimension 2. However, it enters as coefficient group in higher dimensions, and can contribute nontrivially.

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The following examples focus on dimension 4, where we currently do not know how to systematically factor birational maps, and in particular, do not understand the (failure of) rationality of cubic fourfolds.

Cubic fourfolds

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Here are $N > 1$, with $\mathbb{Z}/N\mathbb{Z}$ acting on a smooth cubic fourfold:

$$N = 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 15, 16, 18, 21, 24, 30, 32, 33, 36, 48.$$

Note that

$$d_{\mathbb{Q}} := \dim \mathcal{B}_4(\mathbb{Z}/N\mathbb{Z}) \otimes \mathbb{Q} = 0, \qquad \text{for all } N < 27,\ N = 30, 32,$$

but

N	33	36	48
$d_{\mathbb{Q}}$	2	3	7

One can also work with finite coefficients. Let

$$d_p = d_p(N) := \dim \mathcal{B}_4(\mathbb{Z}/N\mathbb{Z}) \otimes \mathbb{F}_p.$$

We have $d_2, d_3 = 0$, for all $N \leq 15$, and $N = 18, 21$.

N	16	24	30	32	33	36	48
d_2	1	5	10	12	3	19	50
d_3	0	0	0	0	2	3	7

Birational types: using $\mathcal{B}_n(G)$

Consider the cubic fourfold $X \subset \mathbb{P}^5$ given by

$$x_1^2x_2 + x_2^2x_3 + x_3^2x_1 + x_4^2x_5 + x_5^2x_0 + x_0^3 = 0.$$

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$$[4, 24, 31, 22] + [28, 24, 19, 10] + [24, 12, 7, 34] + [9, 5, 17, 29] + [14, 26, 2, 9]$$

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(Solving a system of 443557 linear equations in 82251 variables,...)

$$\beta(X) \neq \beta(\mathbb{P}^4) = 0 \in \mathcal{B}_4(\mathbb{Z}/36\mathbb{Z}) \otimes \mathbb{F}_2 = \mathbb{F}_2^{19}$$

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Thus X is not G-equivariantly birational to $\mathbb{P}^4$ (with linear action).

Birational types: using $\mathrm{Burn}_n(G)$

Consider the cubic fourfold $X \subset \mathbb{P}^5$, given by

$$x_0x_1^2 + x_0^2x_2 - x_0x_2^2 - 4x_0x_4^2 + x_1^2x_2 + x_3^2x_5 - x_2x_4^2 - x_5^3 = 0.$$

$G = \mathbb{Z}/6\mathbb{Z}$ acts with weights $(0,0,0,1,3,4)$. This X is rational, since it contains the disjoint planes

$$x_0 = x_1 - x_4 = x_3 - x_5 = 0 \quad \text{and} \quad x_2 = x_1 - 2x_4 = x_3 + x_5 = 0,$$

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but not G-equivariantly birational to $\mathbb{P}^4$ with linear action.
There is a cubic surface $S \subset X$, with $\mathbb{Z}/3\mathbb{Z}$-stabilizer, $\mathbb{Z}/2\mathbb{Z}$ fixes an elliptic curve, and this S is not stably $\mathbb{Z}/2\mathbb{Z}$-equivariantly rational; the corresponding symbol

$$[\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/2\mathbb{Z} \circlearrowright k(S), \beta] \neq 0 \in \mathrm{Burn}_4(\mathbb{Z}/6\mathbb{Z}),$$

does not interact with any other symbols in $[X \circlearrowleft G]$.

Nonabelian actions on surfaces

Consider the action of $G = C_2 \times \mathfrak{S}_3 = W(\mathsf{G}_2)$ on the corresponding torus T and its Lie algebra $\mathfrak{t}$.

- These are stably equivariantly birational (Lemire-Popov-Reichstein 2005)

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- They are not equivariantly birational (Iskovskikh 2005)

These actions can be realized via:

- the action on $y_1y_2y_3 = 1$ via permutation of variables and taking inverses, with model DP6

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- the action on $y_1 y_2 y_3 = 1$ via permutation of variables and taking inverses, with model DP6
- the action on $x_1 + x_2 + x_3$ via permutation and reversing signs, with model $\mathbb{P}^2$

Nonabelian actions on surfaces

The action on $\mathbb{P}^2 = \mathbb{P}(\mathrm{I} \oplus V)$, with coordinates $(u_0 : u_1 : u_2)$ is given by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{pmatrix}, \quad \iota := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

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There is one fixed point, $(1:0:0)$; after blowing up, the exceptional curve is stabilized by the central involution ι, and comes with a nontrivial $\mathfrak{S}_3$-action, contributing the symbol

$$(C_2, \mathfrak{S}_3 \circlearrowright k(\mathbb{P}^1), (1)) \in [X \circlearrowleft G].$$

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Additionally, the line $\ell_0 := \{u_0 = 0\}$ has as stabilizer the central C_2, contributing the same symbol. ... There are also other terms.

A better model for the second action is the quadric

$$v_0v_1 + v_1v_2 + v_2v_0 = 3w^2,$$

where $\mathfrak{S}_3$ permutes the coordinates $(v_0 : v_1 : v_2)$ and the central involution exchanges the sign on w. There are no G-fixed points, but a conic $R_0 := \{w = 0\}$ with stabilizer the central C_2 and a nontrivial action of $\mathfrak{S}_3$, A better model for the second action is the quadric

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Nonabelian actions on surfaces

The crucial difference is that the summand

$$(C_2, \mathfrak{S}_3 \circlearrowright k(\mathbb{P}^1), (1))$$

appears twice in the $\mathbb{P}^2$ model, and only once in the quadric model.

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appears twice in the $\mathbb{P}^2$ model, and only once in the quadric model. No relations can eliminate this symbol.

This $\mathbb{P}^1$, with $\mathfrak{S}_3$-action, should be viewed as an analog of a curve of genus ≥ 1 in the fixed locus – it will appear on every equivariantly birational model.

Stable linearizability

New examples of nonlinearizable but stably linearizable actions:

Theorem (Hassett-T. 2022)

The following actions are not linearizable but stably linearizable:

- *(generically free) G-actions on quadric surfaces, where G is an extension*

$$1 \to \mathfrak{D}_{2n} \to G \to \mathbb{Z}/2 \to 1$$

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The following actions are not linearizable but stably linearizable:

- *(generically free) G-actions on quadric surfaces, where G is an extension*

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is nontrivial. Moreover, if $\chi \neq \pm\chi'$ then the corresponding classes are distinct.

Equivariant Burnside group

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Unpleasant combinatorial formulas (Kresch-T. 2021); implemented in `magma` by my students Kaiqi Yang and Zhijia Zhang.

Consider $G = \mathfrak{A}_5$ and let V be a faithful 3-dimensional representation of G. Then the class

$$[\mathbb{P}(V) \circlearrowleft G] \in \mathrm{Burn}_2(G)$$

is given by

$$\begin{aligned}
&= (C_1, \mathfrak{A}_5 \circlearrowright k(t,u), ()) + 2(C_2, C_2 \circlearrowright k(t), (1)) \\
&+ 2(C_3, 1 \circlearrowright k, (1,1)) + (C_3, 1 \circlearrowright k, (2,2)) \\
&+ 2(C_5, 1 \circlearrowright k, (1,3)) + (C_5, 1 \circlearrowright k, (4,4)) \\
&+ (C_2^2, 1 \circlearrowright k, ((0,1),(1,0))) + (C_2^2, 1 \circlearrowright k, ((0,1),(1,1))) \\
&= (C_1, \mathfrak{A}_5 \circlearrowright k(t,u), ()) + (C_3, 1 \circlearrowright k, (1,1)) + (C_5, 1 \circlearrowright k, (4,4))
\end{aligned}$$

Burnside groups: incompressibles

Simplifications arise when we focus on geometric properties of strata.

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$$\mathrm{Burn}_n^{\mathrm{inc}}(G) \subset \mathrm{Burn}_n(G),$$

generated by incompressible divisor symbols, i.e.,

$$\mathfrak{s} = (H, Z \circlearrowright K, \beta), \quad \mathrm{trdeg}(K) = n - 1,$$

H a nontrivial cyclic group, and such that $\mathfrak{s}$ cannot arise from Θ_2 in relation (**B2**).

Burnside groups: incompressibles

The subgroup

$$\mathrm{Burn}_n^{\mathrm{inc}}(G) \subseteq \mathrm{Burn}_n(G),$$

is a direct summand, freely generated by incompressible divisor symbols (modulo conjugation).

Burnside groups: incompressibles

$n = 1$ Every divisor symbol in incompressible, hence

$$\mathrm{Burn}_1(G) = \mathrm{Burn}_1^{\mathrm{inc}}(G) \oplus \mathrm{Burn}_1^{\mathbf{H}_{\mathrm{triv}}}(G).$$

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$n = 2$ A divisor symbol

$$(H, Y \circlearrowright K, \beta), \quad \beta = (b),$$

is compressible if and only if Y is cyclic and $K = k(t)$.

Burnside groups: fibrations

Kresch–T. 2021

Let X be a smooth projective G-variety of dimension $n-r$ over k with a G-linearized line bundles $L_0, \ldots, L_r$. Then

$$[\mathbb{P}(L_0 \oplus \cdots \oplus L_r) \circlearrowleft G] = \sum \ldots \in \mathrm{Burn}_n(G),$$

a (rather unpleasant) combinatorial formula.

Burnside groups: fibrations

Let $G := C_5 \times \mathfrak{S}_3$, acting on $X := \mathbb{P}^1$ via an irreducible 2-dimensional representation of $\mathfrak{S}_3$.

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Let $G := C_5 \times \mathfrak{S}_3$, acting on $X := \mathbb{P}^1$ via an irreducible 2-dimensional representation of $\mathfrak{S}_3$. Let L_0 be trivial line bundle and L_1 the twist of $\mathcal{O}_{\mathbb{P}^1}(1)$ by a nontrivial character $\chi : C_5 \to k^\times$.

We would like to compute the class

$$[\mathbb{P}(L_0 \oplus L_1) \circlearrowleft G] \in \mathrm{Burn}_2(G).$$

The outcome of the fibration formula is

$$
\begin{aligned}
[\mathbb{P}(L_0 \oplus L_1) \circlearrowleft G] &= (\mathrm{triv}, G \circlearrowright \mathbb{C}(\mathbb{P}^1)(t), \emptyset) + (\langle(1,2)\rangle, C_5 \overset{\chi}{\circlearrowright} \mathbb{C}(t), 1) \\
&+ (C_5, \mathfrak{S}_3 \circlearrowright \mathbb{C}(\mathbb{P}^1), \chi) + (C_5 \times \langle(1,2)\rangle, \mathrm{triv} \circlearrowright \mathbb{C}, ((0,1),(\chi,0))) \\
&+ (C_5 \times \langle(1,2)\rangle, \mathrm{triv} \circlearrowright \mathbb{C}, ((0,1),(\chi,1))) \\
&+ (C_5 \times \mathfrak{A}_3, \mathfrak{S}_3/\mathfrak{A}_3 \circlearrowright \mathbb{C} \times \mathbb{C}, ((0,1),(\chi,1))) \\
&+ (C_5, \mathfrak{S}_3 \circlearrowright \mathbb{C}(\mathbb{P}^1), -\chi) + (C_5 \times \langle(1,2)\rangle, \mathrm{triv} \circlearrowright \mathbb{C}, ((0,1),(-\chi,0))) \\
&+ (C_5 \times \langle(1,2)\rangle, \mathrm{triv} \circlearrowright \mathbb{C}, ((0,1),(-\chi,1))) \\
&+ (C_5 \times \mathfrak{A}_3, \mathfrak{S}_3/\mathfrak{A}_3 \circlearrowright \mathbb{C} \times \mathbb{C}, ((0,1),(-\chi,1))).
\end{aligned}
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Burnside groups: incompressibles

We have

$$(C_5, \mathfrak{S}_3 \circlearrowright \mathbb{C}(\mathbb{P}^1), \chi) + (C_5, \mathfrak{S}_3 \circlearrowright \mathbb{C}(\mathbb{P}^1), -\chi) \in \mathrm{Burn}_2^{\mathrm{inc}}(G).$$

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These classes are different for $\chi \in \{\pm 1\}$ as compared to $\chi \in \{\pm 2\}$.

Basic terminology: a (faithful) representations $G \to \mathrm{GL}(V)$ is called:

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Linear actions

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- intransitive: if it is reducible, transitive if it is irreducible;
- imprimitive if it is transitive but contains an intransitive normal subgroup G'; in this case G/G' permutes the G' representations;
- primitive if it is neither intransitive, nor imprimitive.

$\mathbb{P}^2$: INTRANSITIVE

$$G \subset k^\times \times \mathrm{GL}_2(k);$$

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finite subgroups of GL_2 arise as binary extensions of subgroups of PGL_2, which in turn are:

$$C_n, \mathfrak{D}_{2n}, \mathfrak{A}_4, \mathfrak{S}_4, \mathfrak{A}_5.$$

$\mathbb{P}^2$: TRANSITIVE AND IMPRIMITIVE

(1) extension of C_3 by $(\mathbb{Z}/n\mathbb{Z})^2$, with the action

$$(\zeta_n x_0, x_1, x_2), \quad (x_0, \zeta_n x_1, x_2), \quad (x_2, x_0, x_1),$$

(2) extension of $\mathfrak{S}_3$ by $(\mathbb{Z}/n\mathbb{Z})^2$, $\mathfrak{S}_3$ permutes the coordinates, the abelian subgroup acts as above,

(3) extension of C_3 by $(\mathbb{Z}/n\mathbb{Z}) \oplus (\mathbb{Z}/m\mathbb{Z})$, $m = n/r$, with $r > 1$, $r \mid n$, $s^2 - s + 1 = 0 \pmod{r}$, and with the action

$$(\zeta_m x_0, x_1, x_2), \quad (\zeta_n^s x_0, \zeta_n x_1, x_2), \quad (x_2, x_0, x_1),$$

(4) extension of $\mathfrak{S}_3$ by $(\mathbb{Z}/n\mathbb{Z}) \oplus (\mathbb{Z}/m\mathbb{Z})$, $m = n/3$, $3 \mid n$, $\mathfrak{S}_3$ permutes the coordinates, the abelian subgroup acts by

$$(\zeta_m x_0, x_1, x_2), \quad (\zeta_n^2, \zeta_n x_1, x_2).$$

$\mathbb{P}^2$: PRIMITIVE

- $\mathfrak{A}_5$
- $3^2 : \mathrm{SL}_2(\mathbb{F}_3)$, and two of its subgroups
- $\mathrm{PSL}_2(\mathbb{F}_7)$ (has dual 3-dimensional representations),
- $\mathfrak{A}_6$

$\mathbb{P}^2$: birational rigidity

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Sakovics (2019)

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This settles the primitive actions. The Burnside invariants allow to distinguish many intransitive and imprimitive actions.

$\mathbb{P}^3$

Kresch-T. 2021

Let G' be

$$\mathfrak{S}_4, \mathfrak{A}_5, \mathrm{PSL}_2(\mathbb{F}_7), \text{ or } 2.\mathfrak{A}_6.$$

Let C_p be the cyclic group of prime order $p > 7$, with and $G := C_n \times G'$. Then there exist embeddings $G \hookrightarrow \mathrm{PGL}_4$ which are not conjugated in Cr_3.

- Construction of new invariants in equivariant birational geometry

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