EQUIVARIANT BIRATIONAL GEOMETRY
Equivariant geometry

Solved problem

What are the finite subgroups

\[ G \subset \text{PGL}_3 = \text{Aut}(\mathbb{P}^2), \]

up to conjugation?

Blichfeldt 1905.
Equivariant geometry

Open problem

What are the embeddings of finite groups

\[ G \hookrightarrow \text{PGL}_3, \]

up to conjugation in

\[ \text{Cr}_2 := \text{BirAut}(\mathbb{P}^2)? \]
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- \( U(j) := \{x \mid |x|_p = p^j\}, \ \text{vol}(U(j)) = p^j \left(1 - \frac{1}{p}\right) \)
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\[
\int_{\mathbb{Q}_p} H_p(x_p)^{-s} \, dx_p = \int_{U(0)} H_p(x_p)^{-s} \, dx_p + \sum_{j \geq 1} \int_{U(j)} H_p(x_p)^{-s} \, dx_p
\]
Calc II

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\]

\[
= 1 + \sum_{j \geq 1} p^{-js} \text{vol}(U(j))
\]
Leading constant

\[ \int_{Q_p} H_p(x_p)^{-s} \, dx_p = \frac{1 - p^{-s}}{1 - p^{-(s-1)}} \]
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\[ \int \ldots = (1 + \frac{1}{p}) = \frac{\# \mathbb{P}^1(\mathbb{F}_p)}{p} \]
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We interpret this as a volume with respect to a natural measure.
Toric varieties

\[ X = X_\Sigma \text{ - projective equivariant compactification of } T = \mathbb{G}_m^d. \]

- \( N \cong \mathbb{Z}^d, \ M = \text{Hom}(N, \mathbb{Z}), \ \Sigma = \{\sigma\} \text{ - fan} \)
- \( e_1, \ldots, e_n \text{ - 1-dimensional cones in } \Sigma \)

\[ 0 \rightarrow M \rightarrow \text{PL}(\Sigma) \rightarrow \text{Pic}(X_\Sigma) \rightarrow 0, \]
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\[
0 \to M \to \text{PL}(\Sigma) \to \text{Pic}(X_\Sigma) \to 0, \]

\( \varphi = \varphi_s \in \text{PL}(\Sigma) \) is defined by its values on \( e_j \): \( \varphi_s(e_j) = s_j \in \mathbb{C} \)

- \( T(\mathbb{Q}_p)/T(\mathbb{Z}_p) = N \)
Height integrals
**Height integrals**

- Local heights:
  \[ H_p(\varphi_s, t_p) := p^{\varphi_s(t_p)} \]

\[
\int_{T(Q_p)} H_p(\varphi_s, t_p)^{-1} \, dt_p = \left( \sum_{r=1}^{d} \sum_{\sigma \in \Sigma(r)} (-1)^r \left( \sum_{n \in \sigma \cap N} p^{-\varphi_s(n)} \right) \right)
\]

\[
= \sum_{r=1}^{d} \sum_{\sigma \in \Sigma(r)} (-1)^r \prod_{e_j \in \sigma} \frac{1}{1 - p^{-s_j}}
\]
- $X$ is a smooth projective Fano variety, $\dim(X) = d$, over a number field $F$.
- $-K_X$ is equipped with an adelic metrization.

For $x \in X(F_v)$ choose local analytic coordinates $x_1, \ldots, x_d$, in a neighborhood $U_x$. In $U_x$, a section of the canonical line bundle has the form $s := dx_1 \wedge \ldots \wedge dx_d$. Put

$$\tau_v = \tau_{X,v} := \|s\|_v dx_1 \cdots dx_d,$$

where $dx_1 \cdots dx_d$ is the standard normalized Haar measure on $F_v^d$. It globalizes to $X(F_v)$. 

**Introduction**
Tamagawa numbers / Peyre (1995)

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where $dx_1 \cdots dx_d$ is the standard normalized Haar measure on $F_v^d$. It globalizes to $X(F_v)$. For almost all $v$, and Zariski open $U \subset X$,

$$\int_U \tau_v = \int_{X(F_v)} \tau_v = \int_{X(\mathcal{O}_v)} \tau_v = \sum_{\tilde{x} \in X(\mathbb{F}_q)} \int_{\pi^{-1}(\tilde{x})} \tau_v = \frac{\#X(\mathbb{F}_q)}{q^d}.$$
Birational Calabi-Yau (Batyrev 1997)

- $X$, $Y$ birational Calabi-Yau of dimension $n$
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- A canonical bundle of a Calabi-Yau variety has a canonical metrization

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Birational Calabi-Yau (Batyrev 1997)

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- If $X \supset U \subset Y$, then

\[ \frac{\# X(\mathbb{F}_q)}{q^n} = \int_{X(F_v)} \tau_v = \int_{U(F_v)} \tau_v = \int_{Y(F_v)} \tau_v = \frac{\# Y(\mathbb{F}_q)}{q^n}, \quad \forall' v \]
Let $U := X \setminus D$, with

$$D = \bigcup_{\alpha \in A} D_\alpha, \quad -K_X = \sum \rho_\alpha D_\alpha,$$

where $D_\alpha$ are geometrically irreducible, smooth, and intersecting transversally.
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$$D_A := \bigcap_{\alpha \in A} D_\alpha, \quad D_A^\circ = D_A \setminus \bigcup_{A' \supset A} D_{A'}.$$
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$D_A \subset X$ is smooth, of codimension $\#A$ (or empty).
Let

$$H_\alpha : U(F_v) \to \mathbb{R}_{\geq 0}$$

be the $v$-adic distance to the boundary component $D_\alpha$. 
**Local heights and height integrals**

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\[ Z_v(s) := \int_{U(F_v)} \prod_{\alpha \in \mathcal{A}} H_\alpha(x)^{-s_\alpha} d\tau_v \]
In charts, via partition of unity: in a neighborhood of $x \in D_A^o(F)$ it takes the form

$$\int \prod_{\alpha \in A} |x_{\alpha}|^{s_{\alpha} - \rho_{\alpha}} \ d\tau_v$$
Local computations

In charts, via partition of unity: in a neighborhood of \( x \in D_A^o(F) \) it takes the form

\[
\int \prod_{\alpha \in A} |x\alpha|_v^{s\alpha - \rho\alpha} \, d\tau_v
\]

Essentially, this is a product of integrals of the form

\[
\int_{|x|_v \leq 1} |x|_v^{s-1} \, dx_v.
\]
Denef’s formula

For almost all $v$ one has:

$$Z_v(s) = \sum_A \frac{\# D_A^\circ(\mathbb{F}_q)}{q^{\dim(X)}} \prod_{\alpha \in A} \frac{q - 1}{q^{s_\alpha - \rho_\alpha + 1} - 1}.$$

Applications
Denef’s formula

For almost all $v$ one has:

$$Z_v(s) = \sum_A \frac{\#D_A^\circ(\mathbb{F}_q)}{q^{\dim(X)}} \prod_{\alpha \in A} \frac{q - 1}{q^{s_\alpha - \rho_\alpha + 1} - 1}.$$ 

Specialize to $s_\alpha = \rho_\alpha$, for all $\alpha \in A$:

$$Z_v(\rho) = \sum_A \frac{\#D_A^\circ(\mathbb{F}_q)}{q^{\dim(X)}} = \frac{\#X(\mathbb{F}_q)}{q^{\dim(X)}}.$$ 

Applications
The integral

- is an invariant under blowups,
- encodes information about singularities of $X$,
- plays a central role in analytic/spectral approaches to Manin’s conjectures, volume asymptotics, etc.
Basic questions

- How much arithmetic is encoded in geometry?
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- How much geometry can be read off from arithmetic?
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Stable rationality over nonclosed ground fields $k$ is still an open problem.
Rationality

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These properties are completely understood in dimensions $\leq 2$, and coincide, over $\mathbb{C}$.
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Specialization of (stable) rationality

- Voisin (2013): integral decomposition of $\Delta$ (Bloch-Srinivas)
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This allowed to:
- show the existence of smooth families with rational and stably irrational fibers (Hassett–Pirutka–T. 2016)
- settle the problem of stable rationality for threefolds, with the exception of threefolds birational to a cubic threefold (Hassett–Kresch–T. 2015)
- find many examples of irrational hypersurfaces of low degree (Schreieder 2017)

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**Specialization of (stable) rationality**

- **Larsen–Lunts (2003):** $K_0(Var_k)/\mathbb{L}$ is isomorphic to the free abelian group spanned by classes of algebraic varieties over $k$, modulo stable rationality.

- **Nicaise–Shinder (2017):** Motivic reduction – formula for homomorphism $K_0(Var_K)/\mathbb{L} \to K_0(Var_k)/\mathbb{L}$, $K = k((t))$, inspired by motivic integration as in Denef–Loeser, ... 

- **Kontsevich–T. (2017):** Same formula for $\text{Burn}(K) \to \text{Burn}(k)$, the free abelian group spanned by classes of varieties over the corresponding field, modulo rationality.
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- **Kontsevich–T. (2017):** Same formula for

\[ \text{Burn}(K) \to \text{Burn}(k), \]

the free abelian group spanned by classes of varieties over the corresponding field, modulo rationality.
Let $\mathfrak{o} \simeq k[[t]]$, $K \simeq k((t))$, char$(k) = 0$.

Let $X/K$ be a smooth proper (or projective) variety, with function field $L = K(X)$.

Choose a regular model

$$\pi : \mathcal{X} \to \text{Spec}(\mathfrak{o}),$$

such that $\pi$ is proper and the special fiber $\mathcal{X}_0$ over $\text{Spec}(k)$ is a simple normal crossings (snc) divisor:

$$\mathcal{X}_0 = \bigcup_{\alpha \in A} d_\alpha D_\alpha, \quad d_\alpha \in \mathbb{Z}_{\geq 1}.$$

Put

$$\rho([L/K]) := \sum_{\emptyset \neq A \subseteq A} (-1)^{\# A - 1} [D_A \times \mathbb{A}^{\# A - 1} / k] \in \text{Burn}(k),$$
There are close similarities between the study of birational properties of varieties over nonclosed fields and the study of birational group actions on varieties over algebraically closed fields.
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This motivated the search for ways to “integrate in presence of group actions”.
Equivariant bir. types (Kontsevich-T. 2019)

- $G$ - finite abelian group, $A = G^\vee$ its group of characters,
- $X$ - smooth projective, of dimension $n$, with regular $G$-action,
- $X^G = \bigsqcup F_\alpha$, where $F_\alpha$ is the (equivalence class of) representation of $G$ acting in the tangent space $T_{X,x\alpha}$ for $x_\alpha \in F_\alpha$. 

$\beta_\alpha : X \mapsto \sum \beta \alpha \alpha$. 

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$$\beta_\alpha := [a_{1,\alpha}, \ldots, a_{n,\alpha}],$$

an unordered $n$-tuple of characters $a_i \in A$,
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an unordered $n$-tuple of characters $a_i \in A$,

$$\beta : X \mapsto \sum_\alpha \beta_\alpha.$$
Consider an action of $\mathbb{Z}/N\mathbb{Z}$ on $X = \mathbb{P}^2$ given by

$$(x : y : z) \mapsto (\zeta^a x : \zeta^b y : z),$$

$\zeta = \zeta_N, \quad a, b \in \mathbb{Z}/N\mathbb{Z}, \quad \gcd(a, b, N) = 1, \quad a \neq b.$

Fixed points are

$$(0 : 0 : 1), \quad (0 : 1 : 0), \quad (1 : 0 : 0).$$
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Then

\[
\beta(X) = [a, b] + [a - b, -b] + [b - a, -a].
\]
First examples: $\mathbb{P}^2$

All such actions are equivalent. Declare $\beta(X) = 0$, i.e.,

$$[a, b] = -[b - a, -a] - [a - b, -b]$$

Allowing

$$[a, b] = -[a, -b]$$

we find

$$[a, b] = [a, b - a] + [a - b, b].$$
Birational types $\mathcal{B}_2(\mathbb{Z}/N\mathbb{Z})$

**Generators:** $[a, b], a, b \in \mathbb{Z}/N\mathbb{Z}, \gcd(a, b, N) = 1$

**Relations:**
- $[a, b] = [b, a]$
- $[a, b] = [a, b - a] + [a - b, b]$ if $a \neq b$
- $[a, a] = [a, 0]$
Let $N = p$ be a prime. We have $\binom{p}{2}$ linear equations in the same number of variables.
Let $N = p$ be a prime. We have $\left(\frac{p}{2}\right)$ linear equations in the same number of variables.

\[
\text{rk}_\mathbb{Q}(\mathcal{B}_2(G)) = \frac{p^2 + 23}{24} = \frac{p^2 - 1}{24} + 1
\]
Birational types $\mathcal{B}_n(G)$

Consider the $\mathbb{Z}$-module $\mathcal{B}_n(G)$

generated by unordered tuples $[a_1, \ldots, a_n]$, $a_i \in A$, such that

(G) $\sum_i \mathbb{Z}a_i = A$, and

(B) for all $a_1, a_2, b_1, \ldots, b_{n-2} \in A$ we have

\[
[a_1, a_2, b_1, \ldots, b_{n-2}] =
\]

\[
[a_1 - a_2, a_2, b_1, \ldots, b_{n-2}] + [a_1, a_2 - a_1, b_1, \ldots, b_{n-2}] \text{ if } a_1 \neq a_2,
\]

\[
[a_1, 0, b_1, \ldots, b_{n-2}] \quad \text{ if } a_1 = a_2.
\]
For $n \geq 3$ the systems of equations are highly overdetermined.
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Jumps at

$$p = 43, 59, 67, 83, \ldots$$
BIRATIONAL TYPES

Consider $X^G = \sqcup F_\alpha$ and record eigenvalues of $G$

$$[a_{1,\alpha}, \ldots, a_{n,\alpha}]$$

in the tangent space $\mathcal{T}_{x_\alpha} X$, at some $x_\alpha \in F_\alpha$. Put

$$\beta(X) := \sum_{\alpha} [a_{1,\alpha}, \ldots, a_{n,\alpha}]$$
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**Kontsevich-T. 2019**

The class

$$\beta(X) \in \mathcal{B}_n(G)$$

is a well-defined $G$-equivariant birational invariant.
Birational types

Variant: introduce the quotient

\[ \mu^- : B_n(G') \to B_{n^-}(G') \]

by an additional relation

\[ [a_1, a_2, \ldots, a_n] = -[-a_1, a_2, \ldots, a_n]. \]
Birational types

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The class of \( \mathbb{P}^n, n \geq 2 \), with linear action of \( G := \mathbb{Z}/N\mathbb{Z} \) is

- torsion in \( B_n(G) \) and
- trivial in \( B_n^{-}(G) \).
Connections to arithmetic groups

\[ \mathcal{B}_n^-(G) \otimes \mathbb{Q} \cong H^{\frac{n(n-1)}{2}}(\Gamma(G,n), \text{or}_n) = H_0(\Gamma(G,n), \text{St}_n \otimes \text{or}_n) \]

where

\[ \Gamma(G, n) \subset \text{GL}_n(\mathbb{Z}) \]

is a congruence subgroup,

or is the orientation (the sign of the determinant), and

\[ \text{St}_n \] is the Steinberg representation.
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- or is the orientation (the sign of the determinant), and

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In particular, the groups \( B_n(G) \) and \( B_n^-(G) \) carry Hecke operators.
$n = 2$: 

$$B_2(\mathbb{Z}/p) \otimes \mathbb{Q} \simeq H^1(X_1(p), \mathbb{Q}).$$
Connections to arithmetic groups

$n = 2$:

\[ \mathcal{B}_2(\mathbb{Z}/p) \otimes \mathbb{Q} \cong H^1(X_1(p), \mathbb{Q}). \]

Manin symbols.
Let $G$ be a finite group. Let

$$\text{Burn}_n(G)$$

be the quotient of the free abelian group generated by symbols

$$(H, N \subseteq K, \beta),$$

where

- $H \subseteq G$ is an abelian subgroup, $N = N_G(H)/H$,
- $K$ is an $N$-Galois algebra over a field of transcendence degree $d \leq n$ over $k$, up to isomorphism, and $\beta$ is a faithful $(n - d)$-dimensional representation of $H$,

modulo somewhat complicated blowup relations.
Equivariant Burnside group: relations

The symbols are subject to conjugation and blowup relations:

(C): \((H, N \subset K, \beta) = (H', N' \subset K, \beta')\), when

\[ H' = gHg^{-1}, \quad N' = N_G(H')/H', \quad \text{with } g \in G, \]

and \(\beta\) and \(\beta'\) are related by conjugation by \(g\).
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(B1): $(H, N \lhd K, \beta) = 0$ when $b_1 + b_2 = 0$. 
Equivariant Burnside group: relations

(B2): \((H, N \lhd K, \beta) = \Theta_1 + \Theta_2\), where

\[
\Theta_1 = \begin{cases} 
0, & \text{if } b_1 = b_2, \\
(H, N \lhd K, \beta_1) + (H, N \lhd K, \beta_2), & \text{otherwise},
\end{cases}
\]

with

\[
\beta_1 := (b_1, b_2 - b_1, b_3, \ldots, b_{n-d}), \quad \beta_2 := (b_1 - b_2, b_2, b_3, \ldots, b_{n-d}),
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\]

and

\[
\Theta_2 = \begin{cases} 
0, & \text{if } b_i \in \langle b_1 - b_2 \rangle \text{ for some } i, \\
(H, N \subset \overline{K}, \overline{\beta}), & \text{otherwise,}
\end{cases}
\]

with

\[
\overline{H}^\vee := H^\vee / \langle b_1 - b_2 \rangle, \quad \overline{\beta} := (\overline{b}_2, \overline{b}_3, \ldots, \overline{b}_{n-d}), \quad \overline{b}_i \in \overline{H}^\vee.
\]
**Equivariant Burnside group: relations**

**Model case:** Blowing up an isolated point (with abelian stabilizer) on a surface.

It will explain the action of $\overline{N}$ on $\overline{K}$. 
The class of a $G$-variety is computed on a standard model $X$:

- $X$ is smooth projective,
- there exists a Zariski open $U \subset X$ such that $G$ acts freely on $U$,
- the complement $X \setminus U$ is a normal crossings divisor,
- for every $g \in G$ and every irreducible component $D$ of $X \setminus U$, either $g(D) = D$ or $g(D) \cap D = \emptyset$. 
Equivariant Burnside group

Passing to a standard model $X$, define:

$$[X \lhd G] := \sum_H \sum_F (H, N \lhd k(F), \beta_F(X)) \in \text{Burn}_n(G),$$

where

- the sum is over all (orbits of) strata $F \subset X$ with generic stabilizer (orbits of) $H$,
- the symbols record the eigenvalues of $H$ in the tangent space at $x \in F$, as before, as well as the $N$-action on the function field of $F$, respectively the orbit of $F$. 

Equivariant birational types
Equivariant Burnside group

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This is a $G$-birational invariant.
Equivariant birational geometry

We work over an algebraically closed field $k$, of characteristic zero. Let $X$ be a smooth projective $G$-variety (regular, generically free action).
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Basic problems:

- Linearizability: $X \sim_G \mathbb{P}(V)$, where $V$ is a faithful linear representation of $G$
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- Stable Linearizability: $X \times \mathbb{P}^m \sim_G \mathbb{P}(V)$, with trivial action on the second factor
Problem
What are the finite subgroups of $\text{PGL}_3$, up to conjugation?
Linear actions

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What are the finite subgroups of $\text{PGL}_3$, up to conjugation? And $\text{PGL}_4$?

Blichfeldt 1905.
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What are the finite subgroups of the Cremona group

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Basic strategy:

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- By MMP, $X$ is either a Del Pezzo surface or a conic bundle.
- If the (anticanonical) degree is small, the action is rigid, and visible via the induced action on the Picard group $\text{Pic}(X)$, i.e., through the Weyl group of the associated root lattice.
In higher dimensions:

- Existence of fixed points upon restrictions to abelian subgroups
Equivariant birational geometry: tools

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If $X$ is rational and $G$ is cyclic, then $X^G \neq \emptyset$. 
Basic facts

- If $X$ is rational and $G$ is cyclic, then $X^G \neq \emptyset$.
- If $Y \to X$ is a $G$-birational map between smooth projective $G$-varieties, and $G$ is abelian, then

$$Y^G \neq \emptyset \iff X^G \neq \emptyset.$$
Basic facts

More precisely, let $X$ be smooth projective of dimension $n$, $G$ abelian, and let $p \in X^G$. Let $\{a_1, \ldots, a_n\}$ be the characters (weights) of $G$ in the tangent space to $X$ at $p$. 

Reichstein-Youssin (2002)

Let $Y \to X$ be a $G$-equivariant blowup. Then $Y$ contains a point $q \in Y^G$ (in the preimage of $p$) with weights $\{b_1, \ldots, b_n\}$ in the tangent space, and such that

$$\det(b_1, \ldots, b_n) = \pm \det(a_1, \ldots, a_n),$$

i.e., this is an equivariant birational invariant.

Thus, $\mathbb{Z}/N\mathbb{Z}$-actions on $\mathbb{P}^n$, with $n \geq 2$ are equivariantly birational.

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Abelian actions on surfaces

- If there is no curve of genus $\geq 1$ in the fixed locus $X^G$, then all actions are linear, with the exception of one fixed-point free action of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$. 

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In particular, $B_2(G)$ does not give anything new in dimension 2. However, it enters as coefficient group in higher dimensions, and can contribute nontrivially.
Abelian actions in dimension 3 are not fully settled, but should be, in principle, accessible.
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The following examples focus on dimension 4, where we currently do not know how to systematically factor birational maps, and in particular, do not understand the (failure of) rationality of cubic fourfolds.
Cubic fourfolds

There is an extensive literature on their automorphisms (and on automorphisms of their variety of lines), e.g., Laza, Zheng, Fu, Mongardi, Mayanskiy, Ouchi, ...
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Here are $N > 1$, with $\mathbb{Z}/N\mathbb{Z}$ acting on a smooth cubic fourfold:

\[ N = 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 15, 16, 18, 21, 24, 30, 32, 33, 36, 48. \]

Note that

\[ d_\mathbb{Q} \coloneqq \dim B_4(\mathbb{Z}/N\mathbb{Z}) \otimes \mathbb{Q} = 0, \quad \text{for all } N < 27, \ N = 30, 32, \]

but

\[
\begin{array}{c|c|c|c}
N & 33 & 36 & 48 \\
\hline
d_\mathbb{Q} & 2 & 3 & 7 \\
\end{array}
\]
Cubic fourfolds

One can also work with finite coefficients. Let

$$d_p = d_p(N) := \dim \mathcal{B}_4(\mathbb{Z}/N\mathbb{Z}) \otimes \mathbb{F}_p.$$ 

We have $d_2, d_3 = 0$, for all $N \leq 15$, and $N = 18, 21$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>16</th>
<th>24</th>
<th>30</th>
<th>32</th>
<th>33</th>
<th>36</th>
<th>48</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_2$</td>
<td>1</td>
<td>5</td>
<td>10</td>
<td>12</td>
<td>3</td>
<td>19</td>
<td>50</td>
</tr>
<tr>
<td>$d_3$</td>
<td>0</td>
<td>0</td>
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</tr>
</tbody>
</table>
Birational types: using $\mathcal{B}_n(G)$

Consider the cubic fourfold $X \subset \mathbb{P}^5$ given by

$$x_1^2x_2 + x_2^2x_3 + x_3^2x_1 + x_4^2x_5 + x_5^2x_0 + x_0^3 = 0.$$
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$G = \mathbb{Z}/36\mathbb{Z}$ acts with weights $(0, 4, 28, 16, 9, 18)$ and isolated fixed points.
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[4, 24, 31, 22] + [28, 24, 19, 10] + [24, 12, 7, 34] + [9, 5, 17, 29] + [14, 26, 2, 9]
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(Solving a system of 443557 linear equations in 82251 variables,...)

\[
\beta(X) \neq \beta(\mathbb{P}^4) = 0 \in \mathcal{B}_4(\mathbb{Z}/36\mathbb{Z}) \otimes \mathbb{F}_2 = \mathbb{F}_2^{19}
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Consider the cubic fourfold $X \subset \mathbb{P}^5$ given by

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$$\beta(X) \neq \beta(\mathbb{P}^4) = 0 \in \mathcal{B}_4(\mathbb{Z}/36\mathbb{Z}) \otimes \mathbb{F}_2 = \mathbb{F}_2^{19}$$

Thus $X$ is not $G$-equivariantly birational to $\mathbb{P}^4$ (with linear action).
Birational types: using $\text{Burn}_n(G)$

Consider the cubic fourfold $X \subset \mathbb{P}^5$, given by

$$x_0x_1^2 + x_0^2x_2 - x_0x_2^2 - 4x_0x_4^2 + x_1^2x_2 + x_3^2x_5 - x_2x_4^2 - x_5^3 = 0.$$  

$G = \mathbb{Z}/6\mathbb{Z}$ acts with weights $(0, 0, 0, 1, 3, 4)$. This $X$ is rational, since it contains the disjoint planes

$$x_0 = x_1 - x_4 = x_3 - x_5 = 0 \quad \text{and} \quad x_2 = x_1 - 2x_4 = x_3 + x_5 = 0,$$

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but not $G$-equivariantly birational to $\mathbb{P}^4$ with linear action. There is a cubic surface $S \subset X$, with $\mathbb{Z}/3\mathbb{Z}$-stabilizer, $\mathbb{Z}/2\mathbb{Z}$ fixes an elliptic curve, and this $S$ is not stably $\mathbb{Z}/2\mathbb{Z}$-equivariantly rational; the corresponding symbol

$$[\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/2\mathbb{Z} \curvearrowright k(S), \beta] \neq 0 \in \text{Burn}_4(\mathbb{Z}/6\mathbb{Z}),$$

does not interact with any other symbols in $[X \curvearrowright G]$. 
Consider the action of \( G = C_2 \times \mathfrak{S}_3 = \mathcal{W}(G_2) \) on the corresponding torus \( T \) and its Lie algebra \( t \).

- These are \textit{stably} equivariantly birational
  (Lemire-Popov-Reichstein 2005)
Consider the action of \( G = C_2 \times S_3 = W(G_2) \) on the corresponding torus \( T \) and its Lie algebra \( \mathfrak{t} \).

- These are stably equivariantly birational (Lemire-Popov-Reichstein 2005)
- They are not equivariantly birational (Iskovskikh 2005)
Nonabelian actions on surfaces

These actions can be realized via:

- the action on $y_1 y_2 y_3 = 1$ via permutation of variables and taking inverses, with model DP6
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- the action on $y_1 y_2 y_3 = 1$ via permutation of variables and taking inverses, with model DP6
- the action on $x_1 + x_2 + x_3$ via permutation and reversing signs, with model $\mathbb{P}^2$
The action on $\mathbb{P}^2 = \mathbb{P}(I \oplus V)$, with coordinates $(u_0 : u_1 : u_2)$ is given by

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0 \\
\end{pmatrix}, \quad
\begin{pmatrix}
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\end{pmatrix}.
\]

There is one fixed point, $(1 : 0 : 0)$; after blowing up, the exceptional curve is stabilized by the central involution $\iota$, and comes with a nontrivial $S_3$-action, contributing the symbol $\left(\mathbb{C}^2, S_3 \right) \in \mathcal{Xr} \left[ G \right]$. Additionally, the line $\ell_0 := \{u_0 = 0\}$ has as stabilizer the central $\mathbb{C}^2$, contributing the same symbol. There are also other terms.

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Nonabelian actions on surfaces

A better model for the second action is the quadric

\[ v_0 v_1 + v_1 v_2 + v_2 v_0 = 3w^2, \]

where \( \mathfrak{S}_3 \) permutes the coordinates \((v_0 : v_1 : v_2)\) and the central involution exchanges the sign on \( w \). There are no \( G \)-fixed points, but a conic \( R_0 := \{w = 0\} \) with stabilizer the central \( C_2 \) and a nontrivial action of \( \mathfrak{S}_3 \),

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The crucial difference is that the summand

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appears twice in the \(\mathbb{P}^2\) model, and only once in the quadric model. No relations can eliminate this symbol.

This \(\mathbb{P}^1\), with \(\mathfrak{S}_3\)-action, should be viewed as an analog of a curve of genus \(\geq 1\) in the fixed locus – it will appear on every equivariantly birational model.
Stable linearizability

New examples of nonlinearizable but stably linearizable actions:

**Theorem (Hassett-T. 2022)**

The following actions are not linearizable but stably linearizable:

- (generically free) $G$-actions on quadric surfaces, where $G$ is an extension

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This gives new examples of failure of the equivariant analog of Zariski's problem.
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**Proof:** via a $G$-equivariant analog of the theory of universal torsors.
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**Kresch-T. 2021**

The class

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Kresch-T. 2021

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is nontrivial. Moreover, if $\chi \neq \pm \chi'$ then the corresponding classes are distinct.
Equivariant Burnside group

How to reach a standard model?

Explicit algorithms for:

- Linear actions on $\mathbb{P}^n$, $G \rtimes \mathbb{PGL}_{n+1}(k)$,
- Toric actions, i.e., $G \rtimes \mathbb{Aut}(X^\ast(T)) = \mathbb{GL}_n(\mathbb{Z})$,

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Unpleasant combinatorial formulas (Kresch-T. 2021); implemented in magma by my students Kaiqi Yang and Zhijia Zhang.
Consider $G = A_5$ and let $V$ be a faithful 3-dimensional representation of $G$. Then the class

$$[\mathbb{P}(V) \simeq G] \in \text{Burn}_2(G)$$

is given by

$$= (C_1, A_5 \simeq k(t, u), ()) + 2(C_2, C_2 \simeq k(t), (1))$$
$$+ 2(C_3, 1 \simeq k, (1, 1)) + (C_3, 1 \simeq k, (2, 2))$$
$$+ 2(C_5, 1 \simeq k, (1, 3)) + (C_5, 1 \simeq k, (4, 4))$$
$$+ (C_2^2, 1 \simeq k, ((0, 1), (1, 0))) + (C_2^2, 1 \simeq k, ((0, 1), (1, 1)))$$
$$= (C_1, A_5 \simeq k(t, u), ()) + (C_3, 1 \simeq k, (1, 1)) + (C_5, 1 \simeq k, (4, 4))$$
Burnside groups: incompressibles

Simplifications arise when we focus on geometric properties of strata.
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\[ \text{Burn}_{n}^{\text{inc}}(G) \subset \text{Burn}_{n}(G), \]

generated by incompressible divisor symbols, i.e.,

\[ s = (H, Z \subseteq K, \beta), \quad \text{trdeg}(K) = n - 1, \]

\( H \) a nontrivial cyclic group, and such that \( s \) cannot arise from \( \Theta_2 \) in relation (B2).
The subgroup

\[ \text{Burn}^\text{inc}_n(G) \subseteq \text{Burn}_n(G), \]

is a direct summand, **freely** generated by incompressible divisor symbols (modulo conjugation).
Burnside groups: incompressibles

\( n = 1 \) Every divisor symbol in incompressible, hence

\[
\text{Burn}_1(G) = \text{Burn}^{\text{inc}}_1(G) \oplus \text{Burn}^{\text{H}_{\text{triv}}}_1(G).
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\[ n = 2 \] A divisor symbol
\[
(H, Y \subseteq K, \beta), \quad \beta = (b),
\]
is compressible if and only if \( Y \) is cyclic and \( K = k(t) \).

Equivariant birational types
Let $X$ be a smooth projective $G$-variety of dimension $n - r$ over $k$ with a $G$-linearized line bundles $L_0, \ldots, L_r$. Then

$$[\mathbb{P}(L_0 \oplus \cdots \oplus L_r) \bowtie G] = \sum \ldots \in \text{Burn}_n(G),$$

a (rather unpleasant) combinatorial formula.
Let $G := C_5 \times \mathfrak{S}_3$, acting on $X := \mathbb{P}^1$ via an irreducible 2-dimensional representation of $\mathfrak{S}_3$. 
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Let $G := C_5 \times S_3$, acting on $X := \mathbb{P}^1$ via an irreducible 2-dimensional representation of $S_3$. Let $L_0$ be trivial line bundle and $L_1$ the twist of $\mathcal{O}_{\mathbb{P}^1}(1)$ by a nontrivial character $\chi : C_5 \to k^\times$.

We would like to compute the class

$$[\mathbb{P}(L_0 \oplus L_1) \cup G] \in \text{Burn}_2(G).$$
The outcome of the fibration formula is

\[ [\mathbb{P}(L_0 \oplus L_1) \curvearrowright G] = (\text{triv}, G \curvearrowright \mathbb{C}(\mathbb{P}^1)(t), \emptyset) + (\langle (1, 2) \rangle, C_5 \overset{\chi}{\curvearrowright} \mathbb{C}(t), 1) \]
\[ + (C_5, S_3 \overset{\chi}{\curvearrowright} \mathbb{C}(\mathbb{P}^1), \chi) + (C_5 \times \langle (1, 2) \rangle, \text{triv} \curvearrowright \mathbb{C}, ((0, 1), (\chi, 0))) \]
\[ + (C_5 \times \langle (1, 2) \rangle, \text{triv} \curvearrowright \mathbb{C}, ((0, 1), (\chi, 1))) \]
\[ + (C_5 \times A_3, S_3/A_3 \overset{\chi}{\curvearrowright} \mathbb{C} \times \mathbb{C}, ((0, 1), (\chi, 1))) \]
\[ + (C_5, S_3 \overset{\chi}{\curvearrowright} \mathbb{C}(\mathbb{P}^1), -\chi) + (C_5 \times \langle (1, 2) \rangle, \text{triv} \curvearrowright \mathbb{C}, ((0, 1), (-\chi, 0))) \]
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Burnside groups: incompressibles

We have

$$(C_5, \mathcal{G}_3 \subset \mathbb{C}(\mathbb{P}^1), \chi) + (C_5, \mathcal{G}_3 \subset \mathbb{C}(\mathbb{P}^1), -\chi) \in \text{Burn}_{2}^{\text{inc}}(G).$$
We have

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These classes are different for $\chi \in \{\pm 1\}$ as compared to $\chi \in \{\pm 2\}$. 
Basic terminology: a (faithful) representations $G \to GL(V)$ is called:

- *intransitive*: if it is reducible, *transitive* if it is irreducible;
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**Linear actions**

**Basic terminology:** a (faithful) representation $G \to \text{GL}(V)$ is called:

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- **primitive** if it is neither intransitive, nor imprimitive.
$\mathbb{P}^2$: INTRANSITIVE

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$\mathbb{P}^2$: INTRANSITIVE

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finite subgroups of $\text{GL}_2$ arise as binary extensions of subgroups of $\text{PGL}_2$, which in turn are:

$C_n, \mathcal{D}_{2n}, \mathcal{A}_4, \mathcal{S}_4, \mathcal{A}_5$. 
(1) extension of $C_3$ by $(\mathbb{Z}/n\mathbb{Z})^2$, with the action

$$(\zeta_n x_0, x_1, x_2), \quad (x_0, \zeta_n x_1, x_2), \quad (x_2, x_0, x_1),$$

(2) extension of $\mathcal{S}_3$ by $(\mathbb{Z}/n\mathbb{Z})^2$, $\mathcal{S}_3$ permutes the coordinates, the abelian subgroup acts as above,

(3) extension of $C_3$ by $(\mathbb{Z}/n\mathbb{Z}) \oplus (\mathbb{Z}/m\mathbb{Z})$, $m = n/r$, with $r > 1$, $r \mid n$, $s^2 - s + 1 = 0 \pmod r$, and with the action

$$(\zeta_m x_0, x_1, x_2), \quad (\zeta_n^s x_0, \zeta_n x_1, x_2), \quad (x_2, x_0, x_1),$$

(4) extension of $\mathcal{S}_3$ by $(\mathbb{Z}/n\mathbb{Z}) \oplus (\mathbb{Z}/m\mathbb{Z})$, $m = n/3$, $3 \mid n$, $\mathcal{S}_3$ permutes the coordinates, the abelian subgroup acts by

$$(\zeta_m x_0, x_1, x_2), \quad (\zeta_n^2, \zeta_n x_1, x_2).$$
\[ P^2: \text{ PRIMITIVE} \]

- \( \mathfrak{A}_5 \)
- \( 3^2 : \text{SL}_2(\mathbb{F}_3) \), and two of its subgroups
- \( \text{PSL}_2(\mathbb{F}_7) \) (has dual 3-dimensional representations),
- \( \mathfrak{A}_6 \)
\(\mathbb{P}^2: \text{BIRATIONAL RIGIDITY}\)

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Let $G \subset \text{PGL}_3$ be a finite group. Then $\mathbb{P}^2$ is $G$-birationally rigid if and only if $G$ is transitive and not isomorphic to $A_4$ or $S_4$. 

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This settles the primitive actions. The Burnside invariants allow to distinguish many intransitive and imprimitive actions.
Let $G'$ be $S_4, A_5, \text{PSL}_2(\mathbb{F}_7)$, or $2.A_6$.

Let $C_p$ be the cyclic group of prime order $p > 7$, with and $G := C_n \times G'$. Then there exist embeddings $G \hookrightarrow \text{PGL}_4$ which are not conjugated in $C_{r_3}$. 

**Applications**
Birational types: summary

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