5D SCFT from $C^3/\Gamma$ (and more*)

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Main results

Quick tour of 5D SCFT

$\mathbb{C}^3/\Gamma$ singularities
   Example (Physics)
   An application: 1-form symmetry

Other $\mathbb{C}^3$ orbifolds

*Bridging 5D and 4D

*A class of TCS $G_2$

*Evidences of the conjectured duality
   BPS spectrum
   $SL(2, \mathbb{Z})$ transformation

*Outlook
Main results

▶ Read physical data of 5D SCFT from the finite algebra $\Gamma \subset SU(3)$ using 3D McKay correspondence
  ▶ Class of theories with exceptional flavor algebra
  ▶ Calculation of 1-form symmetry
  ▶ (Brane-web construction)
▶ *Relate D3-brane moduli with moduli in a class of $G_2$-manifolds
  ▶ Looking at $SL(2, \mathbb{Z})$ transformation in the geometry
  ▶ Singular limit of M-theory (on certain class of $G_2$) leads to SCFT dual to D3-brane on top of 7-branes
1. M-theory on threefold canonical singularity $X$  [Intrilligator, Morrison, Seiberg], [Xie, Yau], [Jefferson, Katz, Kim, Vafa], [Apruzzi, Lawrie, Lin, Schäfer-Nameki, Wang], [Lakshya], [Closset, Schäfer-Nameki, Wang], [Martone, Zafrir], ...

- Coulomb branch: resolution $\tilde{X} \rightarrow X$
- $U(1)$ gauge field on CB: compact 4-cycle $S_i \subset \tilde{X}$
- rank $r = \#$ of $S_i$
- flavor rank $f = \#$ of non-compact 4-cycles $D_\alpha \subset \tilde{X}$
- 5D vector/hypermultiplets: M2 wrapping compact 2-cycles
- 5D prepotential: triple intersections on $\tilde{X}$
- Higgs branch: a bit tricky...
2. Brane web in IIB  [Hanany, Witten], [DeWolfe, Hauer, Iqbal, Barton], ...

▶ CB and HB: [Benini, Benvenuti, Tachikawa], [Kim, Yagi], ...
▶ Map $X$ to brane web: no general method
We will focus on $X = C^3/\Gamma$

$\Gamma$ is a finite subgroup of $SU(3)$: has been fully classified [Miller, Blichfeldt, Dickson], [Yau, Yau, Yu], [Roan], ...
At least one crepant resolution exists for all $\Gamma$.

Why these singularities?

Physics: Rich 5D theories
$\mathbb{C}^3/\Gamma$ singularities

Example 1: $\Gamma = \mathbb{Z}_N \times \mathbb{Z}_N$, $\widehat{\mathbb{C}^3}/\Gamma$ is toric, 5D $T_N$ theory [Eckhard, Schäfer-Nameki, Wang]

Figure: $\frac{\mathbb{C}^3}{\mathbb{Z}_3 \times \mathbb{Z}_3}$
$\mathbb{C}^3/\Gamma$ singularities

Example 2: $\Gamma = \mathbb{Z}_3$, generator is $\frac{1}{3}(1,1,1)$ (we will explain this notation in a moment), $\mathbb{C}^3/\Gamma$ is toric, 5D $E_0$ theory [Closet, Del Zotto, Saxena]

Figure: $\mathbb{C}^3/\mathbb{Z}_3$
C³/Γ singularities

Why these singularities?

Math: McKay correspondence in \( n \)-dims \[\text{[Reid]}\]

Finite subgroup \( \Gamma \subset SL(n) \):

- Conjugacy class \( g_i \) of \( \Gamma \) ↔ cycle \( C_i \) in \( \mathbb{C}^n/\Gamma \) (Recall our dictionary!)
- Irrep \( \rho_i \) of \( \Gamma \) ↔ rank-dim(\( \rho_i \)) vector bundle \( V_i \) over \( \mathbb{C}^n/\Gamma \)
2D McKay correspondence

\[ \Gamma \subset SU(2) \text{ is ADE classified} \]

Non-trivial \( g_i \leftrightarrow \text{exceptional 2-cycle } C_i \subset \mathbb{C}^2/\Gamma \)

\[ \rho_i \otimes \pi = \bigoplus_j a_{ij} \rho_j \leftrightarrow a_{ij} \text{ arrows from } \rho_i \text{ to } \rho_j \text{ in the McKay quiver} \]
We can write:

\[ g \in \mathfrak{g} \rightarrow \]

\[ g \mapsto g_{\text{diag}} = \left( e^{\frac{2\pi i}{k} a}, e^{\frac{2\pi i}{k} b}, e^{\frac{2\pi i}{k} c} \right) := \frac{1}{r}(a, b, c), \quad g^k = 1, \quad \forall g \in \mathfrak{g} \]

\[ \text{age}(g) := \frac{1}{k} (a + b + c) \]

**Theorem 1 [Ito, Reid]**:

\[ \text{age}(g) = 1 \rightarrow D_g \text{ is an exceptional divisor of } \mathbb{C}^3/\Gamma \rightarrow \mathbb{C}^3/\Gamma \]

**Translate to physics**: \( r + f = |\Gamma_1| \)
3D McKay correspondence

Theorem 2 [Ito, Reid]: \(\text{age}(g) = 2 \rightarrow D_g\) is dual to compactly based \(H^2_c\left(\mathbb{C}^3/\Gamma\right)\)

Translate to physics: \(r = |\Gamma_2|\)

Theorem 3 [Ito], [Dixon, Harvey, Vafa, Witten]:
\[
\chi\left(\mathbb{C}^3/\Gamma\right) = \chi\left(\mathbb{C}^3/\Gamma\right) = \#g
\]

Translate to physics: \(\chi\left(\mathbb{C}^3/\Gamma\right) = 1 + |\Gamma_1| + |\Gamma_2| = 1 + f + 2r\)
Example 2110.14441 Warning: Lot of physics ahead

\[ \mathbb{C}^3/\Delta(3n^2) = \frac{\mathbb{C}^3}{\mathbb{Z}_n \times \mathbb{Z}_n}/\mathbb{Z}_3 \]

Generators:

\[ M_1 = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^{-1} \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \]

M-theory on \( \frac{\mathbb{C}^3}{\mathbb{Z}_n \times \mathbb{Z}_n} \rightarrow T_n \)-theory [Gaiotto], [Gaiotto, Maldacena], [Benini, Benvenuti, Tachikawa], [Eckhard, Schäfer-Nameki, Wang], ...

\[ r = (n - 1)(n - 2)/2, \quad G_F = SU(n)^3 \]

(Note: the above physical data can be read off from 3D McKay correspondence as well 2110.15129)
Example

Motivation

Figure: \( T_3 \)-theory

Extra \( \mathbb{Z}_3 \) symmetry permutes three \( SU(n) \)
Example

Convenient to consider CB of $T_n$-theory

Figure: $\mathbb{Z}_3$ symmetric and asymmetric resolutions

Consider only $\mathbb{Z}_3$-invariant CB and gauge the $\mathbb{Z}_3$
Example

Key difference: $3 \mid n$ vs. $3 \nmid n$

$\Rightarrow$ fixed divisor vs. fixed point

$\Rightarrow$ extra $3 \times A_2$ vs. extra $A_2$ [Ito]
Example

From 3D McKay correspondence of $\Delta(3n^2)$:

$$|\Gamma_0| = 1$$

$$\text{rk}_G(T_{\Delta(3n^2)}) = |\Gamma_2| = \begin{cases} 
\frac{1}{6} (n^2 - 3n + 2), & 3 \nmid n \\
\frac{1}{6} (n^2 - 3n + 6), & 3 | n
\end{cases}, \quad (1)$$

$$\text{rk}_F(T_{\Delta(3n^2)}) = |\Gamma_1| - |\Gamma_2| = \begin{cases} 
n + 1, & 3 \nmid n \\
n + 5, & 3 | n
\end{cases}. \quad (2)$$
Example

$\mathbb{C}^3/\Delta(3n^2)$ as hypersurface in $\mathbb{C}^4$:

\[-16w^3z^n + 24wxz^n + 24yz^n + 72z^{2n} + 3w^2x^2 - 3w^4x - 4w^3y + w^6 + 4wxy - x^3 + 8y^2 = 0\]

$SU(n)^3 \xrightarrow{\text{gauge}} SU(n) \times G$

$SU(n)$ is from codim-2 singularity of the hypersurface which is part of $G_F$ [Xie, Yau]

$G \sim SU(3)^3$, $3|n$, rings a bell? $SU(3)^3 \to E_6$

$G \sim SU(3)$, $3 \nmid n$

Blow-up:

$(y, x, w, z) \to (\delta_1^3y, \delta_1^2x, \delta_1w, \delta_1z),$

$(y, x, w, \delta_i) \to (\delta_i^3y, \delta_i^2x, \delta_{i+1}w, \delta_{i+1}\delta_i), \ i = 1, \ldots, k - 1,$

$k = n/3$
Example

The most interesting exceptional divisor is $S_k := \{\delta_k = 0\}$: degree 6 equation in $\mathbb{P}^{3,2,1,1}$ with 4 $A_2$ singularities

$gdP_8$ of the type $4A_2 \hookrightarrow E_8$ [Derenthal] $\Rightarrow$ 5D rank-1 $E_8$ if $n = 3$

What if $n > 3$?

1 $A_2$ is the intersection of $S_k$ and another compact divisor
3 $A_2$’s are the intersections of $S_k$ and non-compact divisors

$\Rightarrow$

- Gauging the compact $A_2$ and couple it to the fields from other compact exceptional divisors
- Left with $E_6$ flavor symmetry
Example

Conclusion: $G_F = SU(n) \times E_6$ when $3 | n$, $G_F = SU(n) \times SU(3)$ when $3 \nmid n$.

Evidences are also collected via brane web construction [2110.14441]
An application: Electric 1-form symmetry

Recipe given in [Morrison, Schäfer-Nameki, Willett] Intersection matrix \( \mathcal{M}_{ij} = S_i \cdot C_j \)

\( \mathcal{M} \) is \( r \times (r + f) \) matrix

\( U(1)^r \) 1-form symmetry in the presence of M2-brane wrapping \( C_i \)'s broken to:

Electric 1-form symmetry: \( \Gamma_e^{(1)} = \mathbb{Z}^r / \mathcal{M} \mathbb{Z}^{r+f} \)

\( \mathcal{M} = S D T \) where \( D = \begin{pmatrix} D' & 0 \\ 0 & 0 \end{pmatrix} \), \( D = \text{diag}(\alpha_1, \alpha_2, \ldots, \alpha_r) \) Smith normal form

\( \Rightarrow \Gamma_e^{(1)} = \bigoplus_{i=1}^{r} \mathbb{Z} / \alpha_i \mathbb{Z} \)
An application: Electric 1-form symmetry

Recall the McKay quiver: $\rho_i \otimes \pi = \oplus_j a_{ji} \rho_j$

$A_{ij} := a_{ji} - a_{ij}$

Smith normal form: $A_{ij} = S D_A T$

$D_A = \text{diag}(\alpha_1, \alpha_1, \alpha_2, \alpha_2, \ldots, \alpha_r, \alpha_r, 0, \ldots, 0)$

$\Rightarrow$ Read off electric 1-form symmetries from McKay quiver
An application: Electric 1-form symmetry

Example: $\Gamma = \mathbb{Z}_5 = \langle \frac{1}{5} (2, 2, 1) \rangle$
An application: Electric 1-form symmetry

\[ A = \begin{pmatrix}
0 & 1 & 2 & -2 & -1 \\
-1 & 0 & 1 & 2 & -2 \\
-2 & -1 & 0 & 1 & 2 \\
2 & -2 & -1 & 0 & 1 \\
1 & 2 & -2 & -1 & 0 \\
\end{pmatrix} \]

\[ \Rightarrow D_A = \text{diag}(5, 5, 1, 1, 0) \]

\[ \Rightarrow \Gamma_e^{(1)} = \mathbb{Z}_5 \]

Can also be checked via toric geometry construction in these Abelian cases
Other $\mathbb{C}^3$ orbifolds

Geometry: hypersurface/complete intersection equations $\rightarrow$ codim-2 ADE singularities $[2110.15129]$

Algebra: 3D McKay correspondence $\rightarrow$ conjugacy classes to exceptional divisors

Extra: brane web

CB is more than just $G$ and $G_F$
Studying CB requires a resolution, which is highly technical and case dependent

HB?

- Brane web is a good tool but is hard to construct for most geometries.
- Deformation theory of non-isolated singularities?
5D to 4D

M-theory on $X \times S^1 \leftrightarrow$ IIA on $X \leftrightarrow$ IIB on $\tilde{X}$

$\uparrow$

D3 probing 7-branes [Hori, Iqbal, Vafa]

$\Rightarrow$

Is there a setup where both M-theory compactification and the physics of 3/7 system play important roles?
So that we could study $SL(2, \mathbb{Z})$ on both sides.
A special class of TCS $G_2$ [Corti, Haskins, Nordström, Pacini], [Braun, Schäfer-Nameki]

M-theory on

The K3 fiber is itself elliptically fibered.
A special class of TCS $G_2$

Conjectured F-theory dual w/o $G_4$-flux

$+ 12$ D3-branes

$\Rightarrow$ D3-brane probing 7-brane background ($\mathcal{N} = 1$)
A special class of TCS $G_2$

\[ Z_+ := -y^2 + x^3 + f_{8,4}(z, \hat{z})x + g_{12,6}(z, \hat{z}) \]

\[ Z_- := -y^2 + x^3 + z_1^4 z_2^4 f_{0,4}(z, \hat{z})x + z_1^5 z_2^5 g_{2,6}(z, \hat{z}) \]

$\Rightarrow E_8 \times E_8$ on $z_1 = 0$ and $z_2 = 0$

Worsen over 12 points on $\widehat{\mathbb{P}^1}$ to $E_8 - I_1$

$\Rightarrow K3$ degenerates to $dP_{9-n} \cup_E dP_{9+n}$ [Kulikov]
Match the moduli

\[ H^3(X, \mathbb{Z}) = \mathbb{Z}[S] \oplus \Gamma^{3,19} / (N_+ + N_-) \oplus (N_- \cap T_+) \oplus (N_+ \cap T_-) \]
\[ \oplus H^3(Z_+) \oplus K_+ \oplus H^3(Z_-) \oplus K_- \]

(3)

(where \( K_\pm := \ker(\rho_\pm)/[S_\pm] \) and \( \rho_\pm : H^2(Z_\pm, \mathbb{Z}) \to H^2(S_\pm, \mathbb{Z}) \))

Conjecture:

<table>
<thead>
<tr>
<th>D3 moduli</th>
<th>Multiplet</th>
<th>Z_ moduli</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \phi_P )</td>
<td>hyper</td>
<td>( H^{1,2}(Z_-) \oplus H^{2,1}(Z_-) )</td>
</tr>
<tr>
<td>( \phi_T )</td>
<td>vector</td>
<td>( K_- )</td>
</tr>
</tbody>
</table>

0th order check 1: \(|K_-| = 12\), matching number of D3's
Physics simplifies in the Kovalev limit: [Guio, Jockers, Klemm, Yeh]

\[ \mathcal{N} = 2 \ \frac{3}{7} \ \text{system} \longleftrightarrow \ \text{M-theory on } Z_-/S_- \times S^1 \sim X \times S^1 \longleftrightarrow \ \text{IIA on } X + \text{KK modes} \]

0\text{th} \text{ order check 2: } K_- \subset H^{1,1}(X), \text{ matching basic IIA fact}

\( \epsilon \text{th} \text{ order check: } 2 \text{ real DOF for each of the 12 } E_8-I_1 \text{ points} \)

\[ \xrightarrow{\text{complexify}} 12+12 \text{ complex DOF } \subset 20+20 \text{ of } H^{1,2}(Z_-) \oplus H^{2,1}(Z_-) \]

\[ \Rightarrow \text{dist}(D3,7\text{-branes}) = \text{vol}(dP_9_{-n}) \]
1\textsuperscript{st} order check: BPS spectrum

Example: \( Z_+ := -y^2 + x^3 + z_1^4 f_{4,4}(z, \hat{z}) x + z_1^5 g_{7,6}(z, \hat{z}) \)

\( D3 \)-brane probing \( E_8 \left( + i_1^{(KK)} \right) \) \( \longleftrightarrow \) M-theory on \( X_{dP_8} \times \mathbb{S}^1 \)
\( \Rightarrow \) Junctions \( \longleftrightarrow \) Cycles

MN \( E_8 \) theory in 5D-4D context [Martone, Zafrir], [Closset, Magureanu]

1\textsuperscript{st} order check: matching electric BPS states [Distler, Martone, Neitzke], [Apruzzi, Lawrie, Lin, Schäfer-Nameki, Wang]

<table>
<thead>
<tr>
<th>Reps</th>
<th>Junctions ( (n_{(p,q)}, g_J) )</th>
<th>2-cycles ( C \left( n_{C.C}, g_C \right) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>248</td>
<td>((1_{(1,0)}, 0))</td>
<td>((−1, 0))</td>
</tr>
<tr>
<td>3875</td>
<td>((2_{(1,0)}, 0))</td>
<td>((0, 0))</td>
</tr>
<tr>
<td>(\ldots)</td>
<td>(\ldots)</td>
<td>(\ldots)</td>
</tr>
</tbody>
</table>

\( C \subset dP_{9-n} \sim dP_8 \)
Higher order check: $SL(2, \mathbb{Z})$ transformation on both sides

([Hauer, Iqbal] for $dP_9$ in particular)

The crucial map, coarse version:

$(p, q)$-cycle (junction) $\leftrightarrow$ 2- and 4-cycles in $dP_n$

3/7 system: $(p, q)$-cycle in $H^1(E)$ experiences monodromy

$\leftrightarrow$

d$P_n$: “Exchange” among 2- and 4-cycles
$SL(2, \mathbb{Z})$ transformation on both sides

The crucial map, finer version:
Junctions $\leftrightarrow$ Sheaves on $dP_n$

$J \in H^2(Y, E, \mathbb{Z})$ \xrightarrow{a(J)} $H^1(E, \mathbb{Z})$

$K_{\text{num}}(dP_n)$

$Y$ is an elliptic fibration over the disk $\Delta$ with 7-branes, $E = \pi^{-1}(p)$ where $p$ is the D3-brane position on $\Delta$

$\Rightarrow$

$SL(2, \mathbb{Z})$ around all 7-branes $\leftrightarrow \bigotimes \omega_{dP_n}$
Outlook

Ultimate goal: What is the incarnation of $SL(2, \mathbb{Z})$ in $G_2$?

- 3/7 system is a nice playground
- “Lift” of $\otimes \omega_{dP_n}$?
- More general $G_2$
- ...
Thank you!