5D SCFT from $\mathbb{C}^3/\Gamma$ (and more*)

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Main results

Quick tour of 5D SCFT

C^3/Γ singularities

- Example (Physics)
- An application: 1-form symmetry

Other $\mathbb{C}^3$ orbifolds

*Bridging 5D and 4D

*A class of TCS G_2

*Evidences of the conjectured duality

- BPS spectrum
- $SL(2,\mathbb{Z})$ transformation

*Outlook

Main results

- Read physical data of 5D SCFT from the finite algebra $\Gamma \subset SU(3)$ using 3D McKay correspondence
  - Class of theories with exceptional flavor algebra
  - Calculation of 1-form symmetry
  - (Brane-web construction)
- *Relate D3-brane moduli with moduli in a class of G_2-manifolds
  - Looking at $SL(2,\mathbb{Z})$ transformation in the geometry
  - Singular limit of M-theory (on certain class of G_2) leads to SCFT dual to D3-brane on top of 7-branes

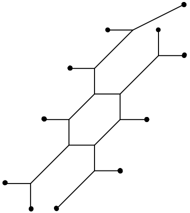
5D SCFT

1. M-theory on threefold canonical singularity X [Intrilligator, Morrison, Seiberg], [Xie, Yau], [Jefferson, Katz, Kim, Vafa], [Apruzzi, Lawrie, Lin, Schäfer-Nameki, Wang], [Lakshya], [Closset, Schäfer-Nameki, Wang], [Martone, Zafrir], ...

- Coulomb branch: resolution $\widetilde{X} \to X$
- $U(1)$ gauge field on CB: compact 4-cycle $S_i \subset \widetilde{X}$
- rank $r = \#$ of S_i
- flavor rank $f = \#$ of non-compact 4-cycles $D_\alpha \subset \widetilde{X}$
- 5D vector/hypermultiplets: M2 wrapping compact 2-cycles
- 5D prepotential: triple intersections on $\widetilde{X}$
- Higgs branch: a bit tricky...

5D SCFT

2. Brane web in IIB [Hanany, Witten], [DeWolfe, Hauer, Iqbal, Barton], ...



- CB and HB: [Benini, Benvenuti, Tachikawa], [Kim, Yagi], ...
- Map X to brane web: no general method

$\mathbb{C}^3/\Gamma$ singularities

We will focus on $X = \mathbb{C}^3/\Gamma$

Γ is a finite subgroup of $SU(3)$: has been fully classified [Miller, Blichfeldt, Dickson], [Yau, Yau, Yu], [Roan], ...
At least one crepant resolution exists for all Γ.

Why these singularities?

Physics: Rich 5D theories

$\mathbb{C}^3/\Gamma$ singularities

Example 1: $\Gamma = \mathbb{Z}_N \times \mathbb{Z}_N$, $\widetilde{\mathbb{C}^3/\Gamma}$ is toric, 5D T_N theory [Eckhard, Schäfer-Nameki, Wang]

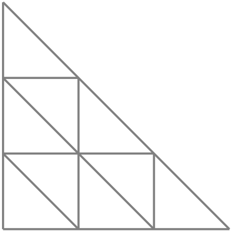


Figure: $\widetilde{\frac{\mathbb{C}^3}{\mathbb{Z}_3 \times \mathbb{Z}_3}}$

C^3/Γ singularities

Example 2: $\Gamma = \mathbb{Z}_3$, generator is $\frac{1}{3}(1,1,1)$ (we will explain this notation in a moment), $\widetilde{\mathbb{C}^3/\Gamma}$ is toric, 5D E_0 theory [Closet, Del Zotto, Saxena]

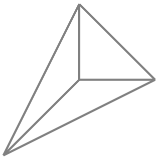


Figure: $\widetilde{\frac{\mathbb{C}^3}{\mathbb{Z}_3}}$

$\mathbb{C}^3/\Gamma$ singularities

Why these singularities?

Math: McKay correspondence in n-dims [Reid]

Finite subgroup $\Gamma \subset SL(n)$:

- Conjugacy class $\mathfrak{g}_i$ of $\Gamma \longleftrightarrow$ cycle C_i in $\widetilde{\mathbb{C}^n/\Gamma}$ (Recall our dictionary!)
- Irrep ρ_i of $\Gamma \longleftrightarrow$ rank-dim(ρ_i) vector bundle V_i over $\widetilde{\mathbb{C}^n/\Gamma}$

2D McKay correspondence

$\Gamma \subset SU(2)$ is ADE classified

Non-trivial $\mathfrak{g}_i \longleftrightarrow$ exceptional 2-cycle $C_i \subset \widetilde{\mathbb{C}^2/\Gamma}$

$\rho_i \otimes \pi = \oplus_j a_{ij} \rho_j \longleftrightarrow a_{ij}$ arrows from ρ_i to ρ_j in the McKay quiver

3D McKay correspondence [Ito], [Ito, Reid], [Ito, Nakajima]

We can write:

$g \in \mathfrak{g} \longrightarrow$

$g \stackrel{\text{diag}}{=} \left(e^{\frac{2\pi i}{k}a}, e^{\frac{2\pi i}{k}b}, e^{\frac{2\pi i}{k}c}\right) := \frac{1}{r}(a, b, c),\ g^k = 1,\ \forall g \in \mathfrak{g}$

$\text{age}(\mathfrak{g}) := \frac{1}{k}(a + b + c)$

Theorem 1 [Ito, Reid]:

$\text{age}(\mathfrak{g}) = 1 \rightarrow D_{\mathfrak{g}}$ is an exceptional divisor of $\widetilde{\mathbb{C}^3/\Gamma} \rightarrow \mathbb{C}^3/\Gamma$

Translate to physics: $r + f = |\Gamma_1|$

3D McKay correspondence

Theorem 2 [Ito, Reid]: $\mathrm{age}(\mathfrak{g}) = 2 \to D_{\mathfrak{g}}$ is dual to compactly based $H_c^2\left(\widetilde{\mathbb{C}^3/\Gamma}\right)$

Translate to physics: $r = |\Gamma_2|$

Theorem 3 [Ito], [Dixon, Harvey, Vafa, Witten]:
$\chi\left(\widetilde{\mathbb{C}^3/\Gamma}\right) = \chi\left(\mathbb{C}^3/\Gamma\right) = \#\mathfrak{g}$

Translate to physics: $\chi\left(\widetilde{\mathbb{C}^3/\Gamma}\right) = 1 + |\Gamma_1| + |\Gamma_2| = 1 + f + 2r$

Example 2110.14441 Warning: Lot of physics ahead

$$\mathbb{C}^3/\Delta(3n^2) = \frac{\mathbb{C}^3}{\mathbb{Z}_n \times \mathbb{Z}_n}/\mathbb{Z}_3$$

Generators:

$$M_1 = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix},\ M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^{-1} \end{pmatrix},\ T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

M-theory on $\frac{\mathbb{C}^3}{\mathbb{Z}_n \times \mathbb{Z}_n} \to T_n$-theory [Gaiotto], [Gaiotto, Maldacena], [Benini, Benvenuti, Tachikawa], [Eckhard, Schäfer-Nameki, Wang], ...

$$r = (n-1)(n-2)/2,\ G_F = SU(n)^3$$

(Note: the above physical data can be read off from 3D McKay correspondence as well 2110.15129)

Example

Motivation

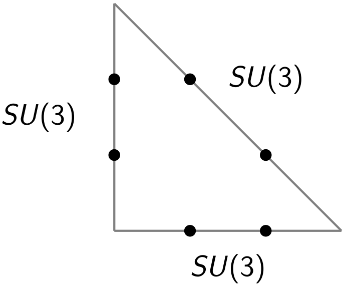


Figure: T_3-theory

Extra $\mathbb{Z}_3$ symmetry permutes three $SU(n)$

Example

Convenient to consider CB of T_n-theory

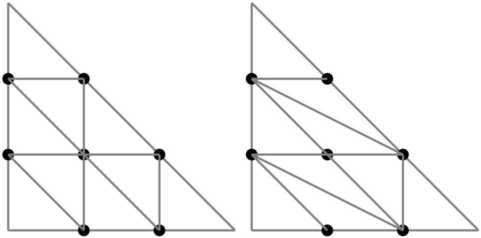
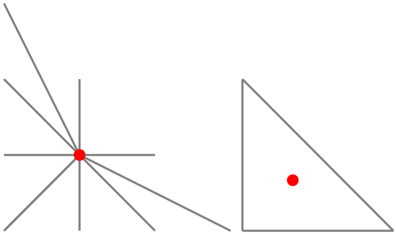


Figure: $\mathbb{Z}_3$ symmetric and asymmetric resolutions

Consider only $\mathbb{Z}_3$-invariant CB and gauge the $\mathbb{Z}_3$

Example

Key difference: $3|n$ vs. $3 \nmid n$



$\Rightarrow$ fixed divisor vs. fixed point

$\Rightarrow$ extra $3 \times A_2$ vs. extra A_2 [Ito]

Example

From 3D McKay correspondence of $\Delta(3n^2)$:

$|\Gamma_0| = 1$

$$\mathrm{rk}_G(T_{\Delta(3n^2)}) = |\Gamma_2| = \begin{cases} \frac{1}{6}\left(n^2 - 3n + 2\right), & 3 \nmid n \\ \frac{1}{6}\left(n^2 - 3n + 6\right), & 3 | n \end{cases}, \qquad (1)$$

$$\mathrm{rk}_F(T_{\Delta(3n^2)}) = |\Gamma_1| - |\Gamma_2| = \begin{cases} n + 1, & 3 \nmid n \\ n + 5, & 3 | n \end{cases}. \qquad (2)$$

Example

$\mathbb{C}^3/\Delta(3n^2)$ as hypersurface in $\mathbb{C}^4$:
$-16w^3z^n + 24wxz^n + 24yz^n + 72z^{2n} + 3w^2x^2 - 3w^4x - 4w^3y + w^6 + 4wxy - x^3 + 8y^2 = 0$

$SU(n)^3 \xrightarrow{\text{gauge}} SU(n) \times G$
$SU(n)$ is from codim-2 singularity of the hypersurface which is part of G_F [Xie, Yau]

$G \sim SU(3)^3,\ 3|n$, rings a bell? $SU(3)^3 \to E_6$
$G \sim SU(3),\ 3 \nmid n$

Blow-up:
$(y, x, w, z) \to (\delta_1^3 y, \delta_1^2 x, \delta_1 w, \delta_1 z)$,
$(y, x, w, \delta_i) \to (\delta_{i+1}^3 y, \delta_{i+1}^2 x, \delta_{i+1} w, \delta_{i+1}\delta_i),\ i = 1, \ldots, k-1$,
$k = n/3$

Example

The most interesting exceptional divisor is $S_k := \{\delta_k = 0\}$: degree 6 equation in $\mathbb{P}^{3,2,1,1}$ with 4 A_2 singualrities

gdP_8 of the type $4A_2 \hookrightarrow E_8$ [Derenthal] $\Rightarrow$ 5D rank-1 E_8 if $n = 3$

What if $n > 3$?

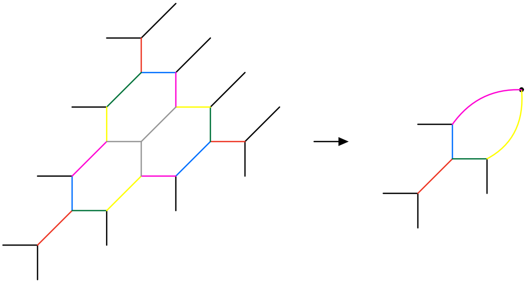
1 A_2 is the intersection of S_k and another compact divisor
3 A_2's are the intersections of S_k and non-compact divisors
$\Rightarrow$

- Gauging the compact A_2 and couple it to the fields from other compact exceptional divisors
- Left with E_6 flavor symmetry

Example

Conclusion: $G_F = SU(n) \times E_6$ when $3|n$, $G_F = SU(n) \times SU(3)$ when $3 \nmid n$.

Evidences are also collected via brane web construction [2110.14441]



An application: Electric 1-form symmetry

Recipe given in [Morrison, Schäfer-Nameki, Willett] Intersection matrix $\mathcal{M}_{ij} = S_i \cdot C_j$

$\mathcal{M}$ is $r \times (r+f)$ matrix

$U(1)^r$ 1-form symmetry in the presence of M2-brane wrapping C_i's broken to:
Electric 1-form symmetry: $\Gamma_e^{(1)} = \mathbb{Z}^r / \mathcal{M}\mathbb{Z}^{r+f}$

$\mathcal{M} = SDT$ where $D = \begin{pmatrix} D' & 0 \\ 0 & 0 \end{pmatrix}$, $D = \mathrm{diag}(\alpha_1, \alpha_2, \ldots, \alpha_r)$ Smith normal form

$\Rightarrow \Gamma_e^{(1)} = \oplus_{i=1}^r \mathbb{Z}/\alpha_i\mathbb{Z}$

An application: Electric 1-form symmetry

Recall the McKay quiver: $\rho_i \otimes \pi = \oplus_j a_{ji} \rho_j$

$A_{ij} := a_{ji} - a_{ij}$

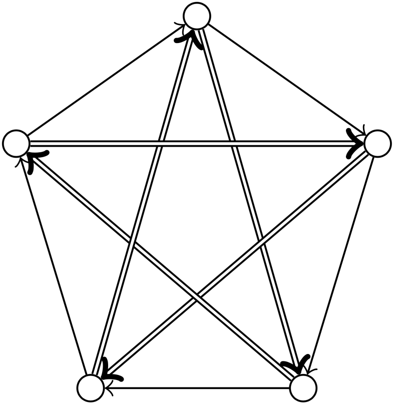
Smith normal form: $A_{ij} = S D_A T$

$D_A = \text{diag}(\alpha_1, \alpha_1, \alpha_2, \alpha_2, \ldots, \alpha_r, \alpha_r, 0, \cdots, 0)$

$\Rightarrow$ Read off electric 1-form symmetries from McKay quiver

An application: Electric 1-form symmetry

Example: $\Gamma = \mathbb{Z}_5 = \langle \frac{1}{5}(2,2,1) \rangle$



An application: Electric 1-form symmetry

$$A = \begin{pmatrix} 0 & 1 & 2 & -2 & -1 \\ -1 & 0 & 1 & 2 & -2 \\ -2 & -1 & 0 & 1 & 2 \\ 2 & -2 & -1 & 0 & 1 \\ 1 & 2 & -2 & -1 & 0 \end{pmatrix}$$

$$\Rightarrow D_A = \mathrm{diag}(5,5,1,1,0)$$

$$\Rightarrow \Gamma_e^{(1)} = \mathbb{Z}_5$$

Can also be checked via toric geometry construction in these Abelian cases

Other $\mathbb{C}^3$ orbifolds

Geometry: hypersurface/complete intersection equations $\rightarrow$ codim-2 ADE singularities [2110.15129]

Algebra: 3D McKay correspondence $\rightarrow$ conjugacy classes to exceptional divisors

Extra: brane web

CB is more that just G and G_F
Studying CB requires a resolution, which is highly technical and case dependent

HB?

- Brane web is a good tool but is hard to construct for most geometries.
- Deformation theory of non-isolated singularities?

5D to 4D

M-theory on $X \times S^1 \leftrightarrow$ IIA on $X \leftrightarrow$ IIB on $\check{X}$

$$\updownarrow$$

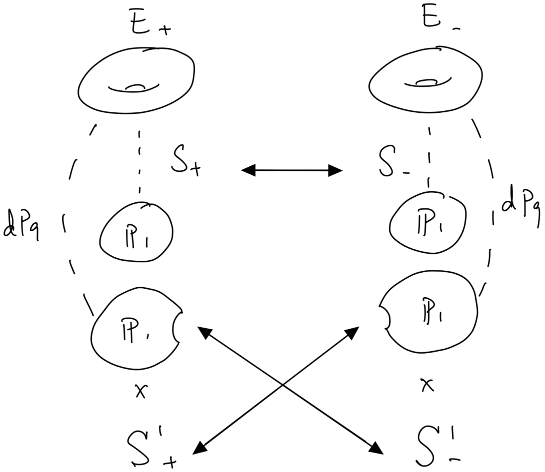
D3 probing 7-branes [Hori, Iqbal, Vafa]

$\Rightarrow$

Is there a setup where both M-theory compactification and the physics of $3/7$ system play important roles?
So that we could study $SL(2,\mathbb{Z})$ on both sides.

A special class of TCS G_2 [Corti, Haskins, Nordstr öm, Pacini], [Braun, Schäfer-Nameki]

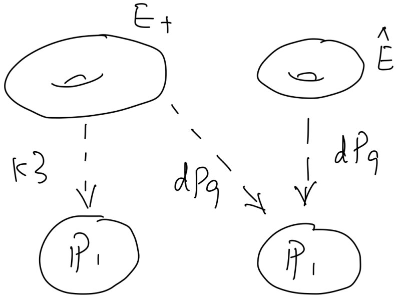
M-theory on



The K3 fiber is itself elliptically fibered.

A special class of TCS G_2

Conjectured F-theory dual w/o G_4-flux



+ 12 D3-branes

$\Rightarrow$ D3-brane probing 7-brane background ($\mathcal{N} = 1$)

A special class of TCS G_2

$$Z_+ := -y^2 + x^3 + f_{8,4}(z,\hat{z})x + g_{12,6}(z,\hat{z})$$

$$Z_- := -y^2 + x^3 + z_1^4 z_2^4 f_{0,4}(z,\hat{z})x + z_1^5 z_2^5 g_{2,6}(z,\hat{z})$$

$\Rightarrow E_8 \times E_8$ on $z_1 = 0$ and $z_2 = 0$

Worsen over 12 points on $\widehat{\mathbb{P}^1}$ to E_8-I_1

$\Rightarrow K3$ degenerates to $dP_{9-n} \cup_E dP_{9+n}$ [Kulikov]

Match the moduli

$$H^3(X,\mathbb{Z}) = \mathbb{Z}[S] \oplus \Gamma^{3,19}/(N_+ + N_-) \oplus (N_- \cap T_+) \oplus (N_+ \cap T_-) \oplus H^3(Z_+) \oplus K_+ \oplus H^3(Z_-) \oplus K_- \quad (3)$$

(where $K_\pm := \ker(\rho_\pm)/[S_\pm]$ and $\rho_\pm : H^2(Z_\pm,\mathbb{Z}) \to H^2(S_\pm,\mathbb{Z})$)

Conjecture:

$D3$ moduli	Multiplet	Z_- moduli
ϕ_P	hyper	$H^{1,2}(Z_-) \oplus H^{2,1}(Z_-)$
ϕ_T	vector	K_-

0^{th} order check 1: $|K_-| = 12$, matching number of $D3$'s

Simplify the physics

Physics simplifies in the Kovalev limit: [Guio, Jockers, Klemm, Yeh]
$\mathcal{N} = 2$ 3/7 system $\longleftrightarrow$ M-theory on $Z_-/S_- \times S^1 \sim X \times S^1$
$\longleftrightarrow$ IIA on X + KK modes

0^{th} order check 2: $K_- \subset H^{1,1}(X)$, matching basic IIA fact

ϵ^{th} order check: 2 real DOF for each of the 12 E_8-I_1 points
$\xrightarrow{complexify}$ 12+12 complex DOF $\subset$ 20+20 of $H^{1,2}(Z_-) \oplus H^{2,1}(Z_-)$

$$\Rightarrow \boxed{\text{dist}(D3,7\text{-branes}) = \text{vol}(dP_{9-n})}$$

1^{st} order check: BPS spectrum

Example: $Z_+ := -y^2 + x^3 + z_1^4 f_{4,4}(z,\hat{z})x + z_1^5 g_{7,6}(z,\hat{z})$

$D3$-brane probing E_8 $(+I_1^{(KK)}) \longleftrightarrow$ M-theory on $X_{dP_8} \times S^1$
$\Rightarrow$ Junctions $\longleftrightarrow$ Cycles

MN E_8 theory in 5D-4D context [Martone, Zafrir], [Closset, Magureanu]

1^{st} order check: matching electric BPS states [Distler, Martone, Neitzke], [Apruzzi, Lawrie, Lin, Schäfer-Nameki, Wang]

Reps	Junctions $(n_{(p,q)}, g_J)$	2-cycles C $(n_{C\cdot C}, g_C)$
248	$(1_{(1,0)}, 0)$	$(-1,0)$
3875	$(2_{(1,0)}, 0)$	$(0,0)$
$\dots$	$\dots$	$\dots$

$C \subset dP_{9-n} \sim dP_8$

Higher order check: $SL(2,\mathbb{Z})$ transformation on both sides

([Hauer, Iqbal] for dP_9 in particular)

The crucial map, coarse version:
(p,q)-cycle (junction) $\longleftrightarrow$ 2- and 4-cycles in dP_n

$$\begin{array}{c}3/7 \text{ system: } (p,q)\text{-cycle in } H^1(E) \text{ experiences monodromy} \\ \longleftrightarrow \\ dP_n\text{: “Exchange” among 2- and 4-cycles}\end{array}$$

$SL(2,\mathbb{Z})$ transformation on both sides

The crucial map, finer version:
Junctions $\longleftrightarrow$ Sheaves on dP_n

$$\begin{array}{ccc}
J \in H^2(Y,E,\mathbb{Z}) & \xrightarrow{\;a(J)\;} & H^1(E,\mathbb{Z}) \\
\text{mirror} \updownarrow & \nearrow & (\deg(i^*\mathcal{F}),\mathrm{rk}(i^*\mathcal{F})) \\
K_{\mathrm{num}}(dP_n) & &
\end{array}$$

Y is an elliptic fibration over the disk Δ with 7-branes,
$E = \pi^{-1}(p)$ where p is the D3-brane position on Δ
$\Rightarrow$

$$SL(2,\mathbb{Z}) \text{ around all 7-branes} \longleftrightarrow \otimes\omega_{dP_n}$$

Outlook

Ultimate goal: What is the incarnation of $SL(2,\mathbb{Z})$ in G_2?

- 3/7 system is a nice playground
- "Lift" of $\otimes\omega_{dP_n}$?
- More general G_2
- ...

Thank you!