The Physics of Joyce-Karigiannis Manifolds

Daniel Baldwin

King's College London and ICTP

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Summary

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i.e. what four dimensional theory do we get when we 'compactify' M-theory onto such orbifolds.

Outline of Talk

- Brief outline of the results and the motivation
- State the Joyce-Karigiannis result
- Introduce a simple first model to illustrate the ideas
- Discuss the geometric viewpoint and then the moduli space of flat connections viewpoint
- Discuss generalisations

Brief Outline of Results

M-theory on a G_2-orbifold of 'J-K type', meaning the singular set is locally

$$\frac{Q \times \frac{\mathbb{C}^2}{\mathbb{Z}_{\mathbb{k}}}}{\mathbb{Z}_2} \tag{1}$$

and Q is a 3-manifold admitting at least one nowhere vanishing $\mathbb{Z}_2$-twisted harmonic 1-form, gives a supersymmetric gauge theory in four dimensions. Generally each family of singularities will have a moduli space with several components.

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Coulomb branch: If $b_1(Q/\mathbb{Z}_2) = 0$ then we have pure $\mathcal{N} = 1$ SU(k) super Yang-Mills theory. If $b_1(Q/\mathbb{Z}_2) > 0$ then we get additional adjoint scalars and a Coulomb branch. This was known (see Acharya '98 [1]).

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Higgs branches: There are also non-identity connected Higgs branches and the moduli parametrising them arise from the $\mathbb{Z}_2$-twisted harmonic 1-forms. Describing these is our main result.

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In the case where $Q = T^3$ this problem was also considered by Rodrigo Barbosa in his PhD thesis (where the lower group is $\mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$). His results agree with ours.

We also extended this to more general ADE singularities and to more general finite groups than $\mathbb{Z}_2$.

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- Since it lives in 11 dimensions $M^{10,1}$, to get 4d real world physics one must 'compactify' the theory on a compact 7-manifold X^7 to get an effective 4d theory, usually we just take $M^{10,1} = \mathbb{R}^{3,1} \times X^7$.

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- Then, if X^7 has a Ricci flat metric, this spacetime will automatically solve the 11d vacuum Einstein equations.

- One finds the massless 4d dimensional fields by Kaluza-Klein analysis of the 11d fields. i.e. write the 11d fields as a product of a field in 4d and a field in 7d and plug it into the equation of motion. e.g. for the 3-form C:

$$C(x^M) = \sum \omega^{(3)}(x^m) \wedge \phi^{(0)}(x^i) + \sum \beta^{(2)}(x^m) \wedge A^{(1)}(x^i) + \dots \tag{2}$$

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- G_2-manifolds are good models for the extra 7 dimensions of M-theory.
- They are 7d, Ricci flat and compactification onto them gives $\mathcal{N} = 1$ supersymmetric theories.
- In order to get phenomenologically desirable things in 4d such as non-Abelian gauge symmetry and chiral fermions, X^7 must have singularities (codim 4 ADE or codim 7 conical respectively).

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- ADE singularities occur along 3-dimensional submanifolds $Q \subset X^7$. Nearby Q, X^7 looks like $Q \times (\mathbb{C}^2/\Gamma_{ADE})$.
- M-theory on $\mathbb{R}^7 \times (\mathbb{C}^2/\Gamma_{ADE})$ gives 7d super Yang-Mills theory with gauge group G_{ADE}. M-theory on $Q \times (\mathbb{C}^2/\Gamma_{ADE})$ gives a 4d supersymmetric gauge theory arising from the reduction of 7d SYM on Q.

Motivation

- More generally, one might consider the space,

$$\frac{Q \times \frac{\mathbb{C}^2}{\Gamma_{ADE}}}{G} \tag{4}$$

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- This was done by Joyce and Karigiannis for the case $\Gamma_{ADE} = \mathbb{Z}_2$ and $G = \mathbb{Z}_2$.

The Joyce-Karigiannis Construction

Joyce and Karigiannis [5] published a new construction of smooth compact G_2-manifolds. One starts by taking a compact Riemannian 7-manifold M with a torsion-free G_2-structure and assumes it to have an involution i which preserves the G_2-structure. Then $M/\langle i \rangle$ is a G_2-orbifold. The fixed point set of i is a 3-dimensional associative submanifold L of M and the singularities locally look like $\mathbb{R}^3 \times (\mathbb{C}^2/\mathbb{Z}_2)$.

Now, if there exists a harmonic (wrt to the induced metric) nowhere vanishing 1-form on L then one can resolve the singularities of $M/\langle i \rangle$ by gluing in Eguchi-Hanson spaces and thus obtain a new smooth and compact G_2-manifold.

This construction can in fact be made slightly more general in two ways. Firstly, everything still goes through if one replaces $M/\langle i \rangle$ by a general G_2-orbifold M' that is not globally a quotient and has orbifold singularities supported on a compact 3-submanifold L that admits a harmonic nowhere vanishing 1-form such that near L, M' looks like $\mathbb{R}^3 \times (\mathbb{C}^2/\mathbb{Z}_2)$. Secondly, the construction still works if one replaces the 1-form with a 1-form defined up to a sign locally in L, we will call this a $\mathbb{Z}_2$-twisted 1-form.

At the end of the paper, the authors suggest that the result could carry through for singularities of the form $\mathbb{R}^3 \times (\mathbb{C}^2/\Gamma)$ for Γ another finite subgroup of SU(2). In this case one would resolve the singularities by gluing in the ALE hyperKähler 4-manifold that is the desingularisation of $\mathbb{C}^2/\Gamma$.

Next, we will look at a first simple example of a G_2-orbifold of J-K type.

We will consider the desingularisation of this orbifold and the different group actions on the arising ALE space and then look at the arising 4d physics via Kaluza-Klein analysis (harmonic analysis).

A Simple First Example

As a first example we take our space to be,

$$\frac{T^3 \times \frac{\mathbb{C}^2}{\mathbb{Z}_k}}{\mathbb{Z}_2} \tag{5}$$

where the lower $\mathbb{Z}_2$ acts freely on the T^3. The action on T^3 is $\mathbb{Z}_2 : (y^1, y^2, y^3) \mapsto (-y^1, -y^2, y^3 + 1/2)$.

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We can define a torsion-free G_2-structure on this space,

$$\varphi = dy^1 dy^2 dy^3 + d\vec{y} \cdot \vec{\omega}, \tag{6}$$

where ω_i are the hyperKähler 2-forms on $\mathbb{C}^2/\mathbb{Z}_k$. Then the metric is,

$$g = d\vec{y}^2 + h \tag{7}$$

where h is the Euclidean metric on $\mathbb{C}^2/\mathbb{Z}_k$. We want to preserve φ so $\mathbb{Z}_2$ must act on $\mathbb{C}^2/\mathbb{Z}_k$ as -1 on two ω_i and 1 on the other.

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Then $T^3/\mathbb{Z}_2$ has one harmonic 1-form dy^3 and two $\mathbb{Z}_2$-twisted harmonic 1-forms $dy^{1,2}$ that are all nowhere vanishing.

The desingularisation of $\mathbb{C}^2/\mathbb{Z}_k$ is the k-centre Gibbons-Hawking space, denoted $M_{GH}^{(k)}$. The action of $\mathbb{Z}_2$ on $\mathbb{C}^2/\mathbb{Z}_k$ only determines the action on $M_{GH}^{(k)}$ 'at infinity'. In Joyce '98 [4] it is explained that generally the data required to determine a K-action on the hyperKähler ALE space resolving $\mathbb{C}^2/\Gamma_{ADE}$ is a group homomorphism,

$$\chi : K \to \mathrm{Aut}(\mathrm{Dynkin}_{ADE}) \ltimes Weyl(ADE) \tag{8}$$

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Thus, the action of $\mathbb{Z}_2$ on $M_{GH}^{(k)}$ will not necessarily be determined by the asymptotic conditions on it and there may be several actions differing by the Weyl group.

Here we review the space $M_{GH}^{(k)}$. This is a 4d hyperKähler ALE space, the metric and hyperKähler forms are:

$$ds^2 = g = V(\vec{x})\, d\vec{x} \cdot d\vec{x} + V(\vec{x})^{-1}(dt + A_i\, dx^i)^2 \tag{9}$$

$$V(\vec{x}) = \sum_{j=0}^{k-1} \frac{1}{|\vec{x} - \vec{a}_j|} \tag{10}$$

$$\vec{\nabla} \times \vec{A} = \vec{\nabla} V(\vec{x}) \quad \text{or equivalently} \quad * \, dA = dV. \tag{11}$$

$$\omega_1 = (dt + A_i\, dx^i) \wedge dx_1 + V(\vec{x})\, dx^2 \wedge dx^3 \tag{12}$$

where $\vec{x} \in \mathbb{R}^3$ and $t \in S^1$. Note, that by adding a 1 to $V(\vec{x})$ one gets ALF geometry.

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where $\vec{x} \in \mathbb{R}^3$ and $t \in S^1$. Note, that by adding a 1 to $V(\vec{x})$ one gets ALF geometry. The hyperKähler moment map is $\vec{\mu} : M_{\mathrm{GH}}^{(k)} \to \mathbb{R}^3$. The preimage of a line segment between two centres $\vec{a}_i$ and $\vec{a}_j$, that meets no other centres, is topologically a 2-sphere, $C_{i,j}$. The set of 2-spheres coming from line segments between consecutive centres, $C_{i,i+1}$, generates the second homology $H_2(M_{GH}^{(k)}, \mathbb{Z}) = \mathbb{Z}^{k-1}$. The intersection matrix of the $[C_{i,i+1}]$ is minus the Cartan matrix for the A_{k-1} Lie algebra.

We want to realise a $\mathbb{Z}_2$-action on the coordinates of $M_{GH}^{(k)}$. Requiring that we also preserve the G_2 3-form on $T^3 \times M_{GH}^{(k)}$ means that the $\mathbb{Z}_2$ action on the coordinates must induce an SO(3) action on the hyperKähler forms ω_i, since this will preserve the hyperKähler structure. Then from the expressions for the ω's, the $\mathbb{Z}_2$ action must act on $\vec{x} \in \mathbb{R}^3$ as an element of SO(3), must leave $V(\vec{x})$ and t invariant and must act on A_i in the inverse way to the action on $\vec{x}$.

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In order to leave invariant:

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Thus, for example, if we arrange the centres in $\mathbb{R}^3$ as the vertices of some regular polygon. Then the $\mathbb{Z}_2$-action on $\vec{x}$ will be some order two element from the dihedral group $D_k \subset \mathrm{SO}(3)$.

This will induce an action on the 2-spheres $C_{i,i+1}$. There will be some number of $\mathbb{Z}_2$-even linear combinations of 2-spheres and some number of $\mathbb{Z}_2$-odd linear combinations.

We can explicitly work out the action on the second homology.

Identify all the order two elements of $D_k \subset SO(3)$, i.e. all the flips and the order two rotation.

The action of these elements on the $\mathbb{R}^3$ coordinates of $M_{GH}^{(k)}$ will induce an action on the centres which will in turn induce an action on the 2-spheres (they arise from the line segments between the centres).

Thus we can view the order two elements of D_k as order two (involutory) matrices acting on $H_2(M_{GH}^{(k)}, \mathbb{Z}) = \mathbb{Z}^{k-1}$.

The even ($\mathbb{Z}_2$-invariant) linear combinations of spheres are then the eigenvectors with eigenvalue 1 whereas the odd linear combinations are those with eigenvalue -1.

One can construct a homomorphism from $\mathbb{Z}_2$ to $\mathrm{Aut}(A_{k-1}) \ltimes \mathrm{Weyl}$, call this homomorphism χ. In the paper (Joyce '98 [4]) it is explained that different choices of χ give different $\mathbb{Z}_2$-actions on $T^3 \times M_{GH}^{(k)}$ which may give rise to topologically distinct desingularisations of $(T^3 \times M_{GH}^{(k)})/\mathbb{Z}_2$. We would expect each topologically distinct desingularisation to give rise to a different connected component of the moduli space of the 4d field theory.

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$$\chi : \mathbb{Z}_2 \to \mathbb{Z}_2 \tag{14}$$

and there are two such possibilities: either the generator of the first $\mathbb{Z}_2$ is mapped to the identity or it is mapped to the generator of the second $\mathbb{Z}_2$ (which is the Weyl group). The generator of the Weyl group reverses the orientation of the 2-sphere that generates $H_2(M_{GH}^{(2)}, \mathbb{Z})$, so it acts as -1 on the second homology. Thus we see that the two different possible $\mathbb{Z}_2$-actions on $T^3 \times M_{GH}^{(2)}$ are topologically distinct and we should get two topologically distinct desingularisations of $(T^3 \times M_{GH}^{(2)})/\mathbb{Z}_2$. The two corresponding branches of the moduli space are the so-called 'Coulomb' and 'Higgs' branches discussed later. This geometric picture carries over to the multi-centre case.

Now, in order to work out what fields we get in four dimensions we must perform a Kaluza-Klein analysis of the fields in the 11d supergravity limit of M-theory.

The fields in 11d are a metric G_{MN}, a 3-form C and a gravitino Ψ.

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In 4d, the massless bosonic fields come from expanding the 3-form C in terms of harmonic forms on the compactification manifold,

$$\frac{T^3 \times M_{GH}^{(k)}}{\mathbb{Z}_2} \tag{15}$$

On this space we have harmonic 3-forms coming from either the harmonic 1-form on $T^3/\mathbb{Z}_2$ wedged with the harmonic 2-form that is the Poincaré dual of one of the invariant 2-spheres in $M_{GH}^{(k)}$ or from the $\mathbb{Z}_2$-twisted harmonic 1-form wedged with the Poincaré dual of the odd 2-spheres.

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Each even harmonic 2-form gives a massless vector (gauge boson) in 4d. Each harmonic 3-form gives a massless scalar in 4d. The massless scalars are complexified since they pair up with massless scalars coming from fluctuations of the metric.

So Far...

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- We have found out what 4d fields arise from Kaluza-Klein analysis of the M-theory fields and seen how the 4d field content depends on the $\mathbb{Z}_2$-action on Gibbons-Hawking space.
- We will now learn more about the 4d physics by considering a different point of view where we look at the moduli space of flat connections on the 3-manifold...

Moduli Space of Flat Connections on the 3-manifold

It is known that one can study the moduli space of vacua arising from compactification onto a G_2-manifold with ADE singularities supported on a 3-manifold L by studying the space $\mathrm{Hom}(\pi_1(L), G^{\mathbb{C}}_{ADE})$ modulo the action of the gauge group on these homomorphisms by conjugation.

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This is because M-theory on $(\mathbb{C}^2/\Gamma) \times \mathbb{R}^7$ gives 7 dimensional super Yang-Mills with ADE gauge group (the bosonic content of this theory is a gauge field A and a triple of adjoint scalars B) and what we are doing here is compactifying this Yang-Mills theory on the 3-manifold L. The conditions of supersymmetry then imply firstly that the B become a 1-form along L and these can be combined with the components of A along L to form a complex Lie-algebra valued 1-form.

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Secondly, they imply that A and B satisfy the dimensional reduction of the Hermitian Yang-Mills equations which amounts to saying that $A + iB$ is a complex flat connection (see [2] and also [6]). The space $\mathrm{Hom}(\pi_1(L), G^{\mathbb{C}}_{ADE})$ (modulo gauge) is equivalent to the set of equivalence classes of principal $G^{\mathbb{C}}_{ADE}$-bundles with flat connections over L [7].

Moduli Space of Flat Connections on the 3-manifold

The gauge group of the theory at a point in the moduli space will be the subgroup of the original gauge group that commutes with the Wilson line solution at that point, this is because the dimensionally reduced Yang-Mills action gives mass terms to the 4d gauge bosons not commuting with the Wilson line background and the corresponding generators of the gauge group will be 'broken'.

Then, instead of transforming in the adjoint, the massless fields will transform in representations of the broken group arising from the decomposition of the adjoint.

Now we see an example of this.

Moduli Space of Flat Connections on the 3-manifold

Example of computation for two centres

Take our space to be $\frac{T^3 \times M_{GH}^{(2)}}{\mathbb{Z}_2}$.

The fundamental group of $T^3/\mathbb{Z}_2$ (where, as before, $\mathbb{Z}_2 : (y^1, y^2, y^3) \mapsto (-y^1, -y^2, y^3 + 1/2)$) is,

$$\langle g_1,\ g_2,\ g_3,\ \beta \,|\, g_i g_j = g_j g_i \quad i = 1, 2, 3, \quad \beta^2 = g_3, \quad \beta g_3 \beta^{-1} = g_3, \\ \beta g_{1,2} \beta^{-1} = g_{1,2}^{-1} \Big\rangle \tag{16}$$

The two classes solutions are the identity connected 'Coulomb branch',

$$g_1 \mapsto \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ g_2 \mapsto \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ g_3 \mapsto \begin{pmatrix} e^{i\theta_3} & 0 \\ 0 & e^{-i\theta_3} \end{pmatrix},\ \beta \mapsto \begin{pmatrix} e^{i\theta_3/2} & 0 \\ 0 & e^{-i\theta_3/2} \end{pmatrix}$$

Note that, at the origin ($\theta_3 = 0$), the centraliser of the solution in SU(2) is the whole group and away from the origin we break down to the maximal torus U(1). Another solution is not identity connected and we call it the 'Higgs branch',

$$g_1 \mapsto \begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{-i\theta_1} \end{pmatrix},\ g_2 \mapsto \begin{pmatrix} e^{i\theta_2} & 0 \\ 0 & e^{-i\theta_2} \end{pmatrix},\ g_3 \mapsto \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},\ \beta \mapsto \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

At the origin of this branch an SO(2) subgroup survives which is completely broken away from the origin. Here it is important to note that β is an element of the Weyl group of SU(2).

The low-energy theory on the Coulomb branch at the origin is $\mathcal{N} = 2$ SU(2) super Yang-Mills, giving a VEV to the adjoint scalar breaks SU(2) to U(1) and leaves a massless complex scalar. This can be identified with θ_1 (which should be complexified), this corresponds to the direction in the torus associated to the $\mathbb{Z}_2$-invariant harmonic 1-form.

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We get a potential for the scalars coming from the reduction of 7d SU(2) SYM on the 3-manifold and generic VEVS for the scalars break SO(2) completely. Before this higgsing we have 4 real bosonic degrees of freedom in the vector multiplet and 8 in the two hypermultiplets. After Higgsing, one hypermultiplet is eaten to create a massive vector multiplet and there are 4 real degrees of freedom left over. The left over degrees of freedom correspond to two massless complex scalars without a potential. These moduli can be identified with θ_1 and θ_2, these correspond to the directions in the torus associated to the two $\mathbb{Z}_2$-twisted 1-forms.

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In conclusion, the moduli space for this theory is $\mathbb{C} \cup \mathbb{C}^2$ where the first factor corresponds to the Coulomb branch and the second to the Higgs branch. This agrees with the results in Rodrigo Barbosas PhD thesis [3] when he computes the moduli space of $SL(2, \mathbb{C})$ connections on $T^3/\mathbb{Z}_2$.

Also, we can easily extend the previous calculation to the case $T^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$. In this case we will have two $\mathbb{Z}_2$ generators in the fundamental group, call them α and β. They act on T^3 as,

$$\alpha : (y^1, y^2, y^3) \mapsto (y^1 + 1/2, -y^2, -y^3) \tag{17}$$

$$\beta : (y^1, y^2, y^3) \mapsto (-y^1, y^2 + 1/2, -y^3 + 1/2) \tag{18}$$

We have no Coulomb branch (since $b_1 = 0$) but we get three Higgs branches coming from the three non-trivial homomorphisms

$$\mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathrm{Aut}(A_1) \ltimes \mathrm{Weyl}(A_1) = \mathbb{Z}_2 \tag{19}$$

i.e. the 2-sphere in the ALE space is either odd under one of the two $\mathbb{Z}_2$ generators and even under the other or odd under both of them. In each case there is one corresponding non vanishing twisted harmonic 1-form (i.e. if the 2-sphere is odd under α and even under β then one can pair the PD with dy^2 to get a harmonic 3-form).

Thus the moduli space is,

$$\mathbb{C} \cup \mathbb{C} \cup \mathbb{C} \tag{20}$$

again in agreement with Barbosa. This can be seen as the intersection of 3-copies of the previous solution.

We can utilise this SU(2) solution to learn about the physics of more general ADE singularities. For example we can embed the SU(2) Higgs branch solution diagonally into an SU($2N$) matrix to learn about the unbroken gauge group at the origin of the Higgs branch for the space,

$$\frac{T^3 \times M_{GH}^{(2N)}}{\mathbb{Z}_2} \tag{21}$$

The centraliser of the matrix with N β's on the diagonal as a subgroup of SU($2N$) is,

$$\mathrm{S}(\mathrm{U}(N) \times \mathrm{U}(N)) \cong (\mathrm{SU}(N) \times \mathrm{SU}(N) \times \mathrm{U}(1))/\mathbb{Z}_N, \tag{22}$$

The fields in the higher dimensional theory (i.e. before compactification onto $T^3/\mathbb{Z}_2$) transform in the adjoint of SU($2N$). So in order to see how the hypermultiplets in the lower dimensional theory transform we must look at the decomposition of this representation,

$$\mathrm{SU}(2N) \rightarrow (\mathrm{SU}(N) \times \mathrm{SU}(N) \times \mathrm{U}(1))/\mathbb{Z}_N$$

$$\mathbf{4N}^2 - \mathbf{1} \rightarrow \underbrace{(\mathbf{N}^2 - \mathbf{1}, \mathbf{1})_0 + (\mathbf{1}, \mathbf{N}^2 - \mathbf{1})_0 + (\mathbf{1}, \mathbf{1})_0}_{\text{Adjoint}} + \underbrace{(\mathbf{N}, \bar{\mathbf{N}})_2 + (\bar{\mathbf{N}}, \mathbf{N})_{-2}}_{\text{bifundamentals}}$$

This representation is the adjoint of the new group plus a bifundamental and an anti-bifundamental. The $\mathcal{N} = 2$ vector multiplet will transform in the adjoint and the matter in the theory must transform in the bifundamentals. There is one $\mathcal{N} = 2$ hypermultiplet transforming in the $(\mathbf{N}, \bar{\mathbf{N}})_2$ and one in the $(\bar{\mathbf{N}}, \mathbf{N})_{-2}$ (so we have $8N^2$ real scalars in total).

Now we can move away from the origin of the moduli space by giving VEVs to the scalars. If we give a VEV to a bifundamentals the group breaks to,

$$(\mathrm{SU}(N) \times \mathrm{SU}(N) \times \mathrm{U}(1))/\mathbb{Z}_N \to \frac{\mathrm{SU}(N)_D}{\mathbb{Z}_N} \tag{23}$$

and the matter representation decomposes into adjoints and singlets and $4N^2$ of the scalars become massive. Giving a VEV to one of the remaining scalars in the adjoint representation then breaks the group to,

$$\frac{\mathrm{SU}(N)_D}{\mathbb{Z}_N} \to \frac{\mathrm{U}(1)^{N-1}}{\mathbb{Z}_N} \tag{24}$$

$4(N^2 - N)$ of the real scalars become massive.

So we are left with gauge group $\mathrm{U}(1)^{N-1}$ and $4N$ massless real scalars. We can't break this group any further.

From the geometric point of view outlined earlier. This physics corresponds to having $N-1$ $\mathbb{Z}_2$-invariant and N $\mathbb{Z}_2$-odd linear combinations of 2-spheres. One can find such an action on $M_{GH}^{(2N)}$ explicitly by arranging all the centres on, e.g., a regular polygon in $\mathbb{R}^3$.

The $\mathbb{Z}_2$-invariant spheres provide the U(1) gauge bosons and the complex scalars that pair up with them to form the (bosonic content of) the $\mathcal{N}=2$ vector multiplet. The $\mathbb{Z}_2$-odd spheres provide two massless complex scalars each.

We could also embed the SU(2) Coulomb branch solution into SU($2N$) to learn about the Coulomb branch in this case, or even a mix of Coulomb and Higgs to get a mixed branch.

We can also consider D-type singularities by embedding our SU(2) solution into Spin($4N$),

$$\mathrm{Spin}(4N) \to \mathrm{Spin}(4)^N \cong \mathrm{SU}(2)^{2N} \tag{25}$$

and play a similar game. In this case, the gauge group should be completely broken at a generic point on the Higgs branch, leaving behind $8N$ massless real scalars.

Also we could do the same for E_8 singularities using the subgroup,

$$E_8 \to \mathrm{SU}(2)^8/\mathbb{Z}_2. \tag{26}$$

and in this case, the gauge group is also completely broken at a generic point and we have 32 massless real scalars.

Again we could also embed the Coulomb branch solution or the mixed branch solutions.

So we see that we can squeeze a lot out of the simple SU(2) solution.

We have considered the physics of G_2 orbifolds of the form,

$$\frac{T^3 \times \frac{\mathbb{C}^2}{\Gamma_{ADE}}}{\mathbb{Z}_2} \tag{27}$$

we could generalise $\mathbb{Z}_2$ to one of $G = \mathbb{Z}_3,\ \mathbb{Z}_4,\ \mathbb{Z}_6,\ \mathbb{Z}_2 \times \mathbb{Z}_2$ so that T^3/G is one of the flat, compact, orientable Riemannian 3-manifolds. We could again carry out the flat connections computation quite explicitly here due to the simple presentations of the fundamental groups. They give $\mathcal{N} = 2$ 4d SUSY apart from the last case which gives $\mathcal{N} = 1$.

But we could also consider a much more general 3-manifold Q/G which may have G-invariant harmonic 1-forms and G-twisted harmonic 1-forms (nowhere vanishing). The G-twisted harmonic 1-forms will play the role of the $\mathbb{Z}_2$-twisted ones from before. So, via KK analysis, the PD of G-invariant linear combinations of 2-spheres in the ALE space will give rise to massless 4d gauge bosons and the PD of G-twisted linear combinations of 2-spheres can pair with an oppositely G-twisted 1-form to give a massless 4d scalar. All the previous analysis should carry through as long as these G-twisted harmonic 1-forms exist.

- We have physically interpreted the type of G_2-orbifolds that can be resolved by the methods of Joyce and Karigiannis and generalisations of such orbifolds (not covered by their method) by studying the 4d field theories arising from M-theory compactifications onto the orbifolds.

- We have two complementary viewpoints:
  - A geometric viewpoint where we consider the PDs of 'G-invariant' and 'G-twisted' 2-spheres in the ALE space and the harmonic 1-forms on the 3-manifold and then do Kaluza-Klein analysis. Here, different branches of the moduli space arise from topologically distinct group actions on the ALE space (i.e. actions which act differently on the second homology).
  - A moduli space of flat connections viewpoint where we consider the field theory on the 3-manifold. This amounts to working out the space of homomorphsims from the fundamental group of the 3-manifold to the gauge group. Here, different branches correspond to different classes of solution.

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Thank you for your attention