Heterotic systems, balanced SU(3)-structures and coclosed G_2-structures in cohomogeneity one manifolds

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Introduction

Heterotic string theory is a ten dimensional supergravity theory that includes a supersymmetric gauge theory at classical level. It is formulated on a ten dimensional spin manifold N.

To study this theory in lower dimensions, we impose the following ansatz:

$$N = Y^D \times M^{10-D},$$

where

- Y^D is a D-dimensional Lorentzian manifold;
- M^{10-D} is a Riemannian spin manifold.

This mechanism is called compactification and reduces the equations the fields must satisfy to a system of coupled differential equations on the smooth manifold M known as the Killing spinor equations in (heterotic) supergravity.

First main result

In the case where $D = 4$, we get the Hull–Strominger system (with three complex dimensions) (Hull, Strominger 1986).

We look for solutions to this system in the cohomogeneity one setting. We consider balanced metrics on complex manifolds with holomorphically trivial canonical bundle, most commonly known as balanced SU(3)-structures. We find a non-existence result.

Theorem (A., Salvatore, '22)

Let M be a six-dimensional simply connected cohomogeneity one manifold under the almost effective action of a connected Lie group G. Then, M admits no G-invariant balanced non-Kähler SU(3)-structures.

Setting of the Hull–Strominger system

- (X, Ω) Calabi-Yau manifold of dimension n with underlying smooth manifold by M and almost complex structure J;
- hermitian metric g on (X, Ω);
- ε holomorphic vector bundle over X;
- hermitian metric h on ε;
- integrable Dolbeault operator $\bar{\partial}_T$ on (TM, J). We denote the holomorphic vector bundle $(TM, J, \bar{\partial}_T)$ by T;
- F_h curvature of the Chern connection on (ε, h);
- R_g curvature of the Chern connection of g, regarded as a hermitian metric on the holomorphic vector bundle T;

Definition of the Hull–Strominger system

Let α be a non-vanishing real constant (proportional to slope parameter in string theory).

Definition

In the hypothesis above, the Hull–Strominger system, for g, h and $\bar{\partial}_T$ is the following system of coupled non-linear differential equations of mixed order:

$$\Lambda_\omega F_h = 0,$$

$$\Lambda_\omega R_g = 0,$$

$$\mathrm{d}^*\omega - \mathrm{d}^c \log ||\Omega||_\omega = 0, \quad (\text{or } \mathrm{d}(||\Omega||_\omega \omega^{n-1}) = 0),$$

$$\mathrm{dd}^c\omega - \alpha(\mathrm{tr} R_g \wedge R_g - \mathrm{tr} F_h \wedge F_h) = 0.$$

G-invariant SU(3)-structures

Let M be a cohomogeneity one manifold of complex dimension 3 for the almost effective action of a compact connected Lie group G, with a G-invariant SU(3)-structure. Let K be the principal isotropy group.

Lemma (Podestà, Spiro '09)

We can identify K with a subgroup of SU(2). Then $\mathfrak{k} = \mathrm{Lie}(K)$ is $\{0\}$, $\mathbb{R}$ or $\mathfrak{su}(2)$.

We can then classify all possible pairs $(\mathfrak{g}, \mathfrak{k})$ which may admit an SU(3)-structure in cohomogenity one, as $\dim(\mathfrak{g}) - \dim(\mathfrak{k}) = 5$.

If we further assume that the manifold M is simply connected, the possible principal orbit types are:

1. $\mathfrak{g} = \mathfrak{su}(3)$, $\mathfrak{k} = \mathfrak{su}(2)$,
2. $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, $\mathfrak{k} = \mathbb{R}$,
3. $\mathfrak{g} = \mathfrak{su}(2) \oplus 2\mathbb{R}$, $\mathfrak{k} = \{0\}$.

Van Kampen theorems

Proposition (Hoelscher, '10)

Let M be the compact cohomogeneity one manifold given by the group diagram $G \supset H_-, H_+ \supset K$ with $H_\pm/K = S^{l_\pm}$ and assume $l_\pm \geq 1$. M is simply connected if and only if the images of $\pi_1(H_\pm/K) = \pi_1(S^{l_\pm})$ generate $\pi_1(G/K)$ under the natural inclusions.

Proposition (A., Salvatore '22)

Let M be the non-compact cohomogeneity one manifold given by the group diagram $G \supset H \supset K$ with $H/K = S^l$, and $l \geq 1$. M is simply connected if and only if the image of $\pi_1(H/K) = \pi_1(S^l)$ generates $\pi_1(G/K)$ under the natural inclusions.

Conditions for (ω, ψ_+)

For each of the remaining cases, we start with a generic pair of G-invariant forms on M^{princ} of degree two and three (ω, ψ_+).

In order for the pair (ω, ψ_+) to define a G-invariant balanced non-Kähler SU(3)-structure on M^{princ}, we have to impose the following conditions:

1. Stability conditions: $\omega^3 \neq 0$, $\lambda := \lambda(\psi_+) < 0$,
2. Compatibility conditions $\psi_\pm \wedge \omega = 0$,
3. Normalization condition: $\psi_+ \wedge \psi_- = \frac{2}{3}\omega^3$,
4. $\mathrm{d}\psi_\pm = 0$,
5. Balanced condition $\mathrm{d}\omega^2 = 0$,
6. Non-Kähler condition $d\omega \neq 0$,
7. Positive-definiteness of the induced symmetric bilinear form $g = \mathrm{d}t^2 + g_t$ on M^{princ}.

Case $\mathfrak{g} = \mathfrak{su}(3)$, $\mathfrak{k} = \mathfrak{su}(2)$

A pair of generic G-invariant forms (ω, ψ_+) on M^{princ} of degree two and three is given respectively by

$$\begin{aligned}\omega =& h_1\, e^{16} + h_2\,(e^{23} + e^{45}) + h_3\,(e^{24} - e^{35}) + h_4(e^{25} + e^{34}),\\ \psi_+ =& p_1\,(e^{123} + e^{145}) + p_2\,(e^{124} - e^{135}) + p_3(e^{246} - e^{356})\\ &+ p_4(e^{236} + e^{456}) + p_5\,(e^{125} + e^{134}) + p_6\,(e^{256} + e^{346}),\end{aligned}$$

where $h_i, p_j \in C^\infty(\mathring{I})$, $i = 1, ..., 4, j = 1, ..., 6$.

The collection of ODE's and algebraic equations for the coefficients (1)-(7) is incompatible.

Hence M admits no such G-invariant SU(3)-structures.

Case $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, $\mathfrak{k} = \mathbb{R}$

We need to divide the discussion depending on the embeddings of $\mathfrak{k} = \mathbb{R}$ in $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$.

- If $\mathfrak{k} = \mathbb{R}$ is not diagonally embedded in $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, M admits no G-invariant stable 3-forms inducing an almost complex structure on M.
- If $\mathfrak{k} = \mathbb{R}$ is diagonally embedded in $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, a pair of generic G-invariant forms (ω, ψ_+) on M^{princ} of degree two and three is given respectively by

$$\begin{aligned}
\omega =& h_1\, e^{12} + h_2\, e^{35} + h_3\, e^{46} + h_4(e^{34} + e^{56}) + h_5(e^{36} + e^{45}),\\
\psi_+ =& p_1\, e^{135} + p_2\, e^{146} + p_3(e^{134} + e^{156}) + p_4(e^{136} + e^{145})\\
&+ p_5\, e^{235} + p_6\, e^{246} + p_7(e^{234} + e^{256}) + p_8(e^{236} + e^{245}),
\end{aligned}$$

  where $h_i, p_j \in C^\infty(\mathring{I})$, $i = 1, \ldots, 5, j = 1, \ldots, 8$. M admits no such G-invariant $SU(3)$-structures.

Case $\mathfrak{g} = \mathfrak{su}(2) \oplus 2\mathbb{R}$, $\mathfrak{k} = \{0\}$

A pair of generic G-invariant forms (ω, ψ_+) on M^{princ} of degree two and three is given respectively by

$$\begin{aligned} \omega \quad &:= h_1 e^{12} + h_2 e^{13} + h_3 e^{14} + h_4 e^{15} + h_5 e^{16} \\ &+ h_6 e^{23} + h_7 e^{24} + h_8 e^{25} + h_9 e^{26} + h_{10} e^{34} \\ &+ h_{11} e^{35} + h_{12} e^{36} + h_{13} e^{45} + h_{14} e^{46} + h_{15} e^{56}, \end{aligned}$$

$$\begin{aligned} \psi_+ \quad &:= p_1 e^{123} + p_2 e^{124} + p_3 e^{125} + p_4 e^{126} + p_5 e^{134} \\ &+ p_6 e^{135} + p_7 e^{136} + p_8 e^{145} + p_9 e^{146} + p_{10} e^{156} \\ &+ p_{11} e^{234} + p_{12} e^{235} + p_{13} e^{236} + p_{14} e^{245} + p_{15} e^{246} \\ &+ p_{16} e^{256} + p_{17} e^{345} + p_{18} e^{346} + p_{19} e^{356} + p_{20} e^{456}, \end{aligned}$$

where $h_i, p_j \in C^\infty(-1,1)$, $i = 1, \ldots, 15, j = 1, \ldots, 20$. The pair

$$\begin{aligned} \omega \quad &= \frac{3}{2} \frac{e^{4t}}{\sqrt{9+3e^{6t}}} e^{12} - \frac{1}{3} \left(-3 + \sqrt{9 + 3e^{6t}} \right) e^{-2t} e^{34} \\ &+ e^{35} + e^{36} - e^{45} + e^{46} + e^{2t} e^{56}, \end{aligned}$$

$$\psi_+ = e^{123} + e^{234} + e^{2t}(e^{136} - e^{145} + e^{235} + e^{246}),$$

defines a G-invariant balanced non-Kähler SU(3)-structure on the corresponding M^{princ}.

Case $\mathfrak{g} = \mathfrak{su}(2) \oplus 2\mathbb{R}$, $\mathfrak{k} = \{0\}$

Theorem (A., Salvatore, '22)

Let M be a six-dimensional simply connected cohomogeneity one manifold under the almost effective action of a connected Lie group G and let K be the principal isotropy group. Then the principal part M^{princ} admits a G-invariant balanced non-Kähler SU(3)-structure (g, J, ψ) if and only if M is compact and $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{su}(2) \oplus 2\mathbb{R}, \{0\})$.

Using the method from Eschenburg-Wang ('00), we find that it is not possible to extend this structures to the singular orbits.

Theorem (A., Salvatore, '22)

Let M be a six-dimensional simply connected cohomogeneity one manifold under the almost effective action of a connected Lie group G. Then, M admits no G-invariant balanced non-Kähler SU(3)-structures.

Key point: the relevant case was

$$M \cong S^3 \times S^3.$$

Balanced metrics on $S^3 \times S^3$

- Fu, Li and Yau ('12) constructed balanced metrics on the connected sum of $k \geq 2$ copies of $S^3 \times S^3$.
- Michelsohn ('82) proved that $S^3 \times S^3$ endowed with the Calabi–Eckmann complex structure does not admit any compatible balanced metric.
- Alexandrov ('01) showed that in a manifold with six real dimensions there is no non-Kähler Hermitian metric which is simultaneously balanced and strong Kähler-with-torsion (SKT).
- Fino and Vezzoni ('15) conjectured that on non-Kähler compact complex manifolds it is never possible to find an SKT metric and also a balanced metric.
- Grantcharov ('08) provided an example of a SKT structure on $S^3 \times S^3$.

Corollary (A., Salvatore, '22)

There is no non-Kähler balanced SU(3)-structure on $S^3 \times S^3$ which is invariant under a cohomogeneity one action.

Geometry of G_2-structures

Definition

A G_2-structure a smooth seven-manifold M is a smooth 3-form φ on M such that, at every $p \in M$, there exists a linear isomorphism $T_pM \cong \mathbb{R}^7$ with respect to which $\varphi_p \in \Lambda^3(T_p^*M)$ corresponds to
$\varphi_0 = e^{123} - e^{167} - e^{527} - e^{563} - e^{415} - e^{426} - e^{437} \in \Lambda^3(\mathbb{R}^7)^*$.
The coassociative form is given by $\psi = *\varphi$.

There exist unique differential forms $\tau_0 \in \Omega^0$, $\tau_1 \in \Omega^1$, $\tau_2 \in \Omega^2_{14}$ and $\tau_3 \in \Omega^3_{27}$, so that the following equations hold:

$$d\varphi = \tau_0\psi + 3\tau_1 \wedge \varphi + *\tau_3,$$

$$d\psi = 4\tau_1 \wedge \psi + \tau_2 \wedge \varphi.$$

- Torsion-free ($\nabla\varphi = 0$): $\tau_0 = \tau_1 = \tau_2 = \tau_3 = 0$;
- Closed ($d\varphi = 0$): $\tau_0 = \tau_1 = \tau_3 = 0$;
- Coclosed ($d\psi = 0$): $\tau_1 = \tau_2 = 0$.

The heterotic G_2 system

The heterotic G_2 system on (M^7, φ) has the following degrees of freedom

- Geometric fields: Scalar field $\lambda \in \mathbb{R}$, Dilaton $\mu \in C^\infty(M)$, Flux $H \in \Omega^3(M)$.
- Gauge fields:
  - $A \in \mathcal{A}(E)$ where $E \to M$ is a vector bundle and A is a G_2-instanton, i.e. $F_A \wedge \psi = 0$.
  - $\theta \in \mathcal{A}(TM)$ such that θ is a G_2-instanton, i.e. $R_\theta \wedge \psi = 0$.

Definition

Let $\alpha' \neq 0$ be a (small) real constant. The heterotic G_2 system consists on the following relations between the geometric fields and the intrinsic torsion forms:

$$\tau_0 = \frac{3}{7}\lambda, \quad \tau_1 = \frac{1}{2}d\mu, \quad \tau_2 = 0, \quad H = \frac{1}{6}\tau_0\varphi - \tau_1 \lrcorner \psi - \tau_3.$$

together with the anomaly free condition or heterotic Bianchi identity that relates the curvatures of the gauge fields:

$$dH = \frac{\alpha'}{4}(\mathrm{tr} F_A \wedge F_A - \mathrm{tr} R_\theta \wedge R_\theta). \tag{1}$$

Cohomogeneity one G_2-structures

Let M be a seven-dimensional simply connected manifold of cohomogeneity one for the almost effective action of a compact connected Lie group G with a G_2-structure φ. We require that the action of G preserves the G_2-structure, so the principal isotropy group K acts of T_pM with $K \subset \mathrm{SU}(3) \subset G_2$.

All the possible principal orbit types at the Lie algebra level $(\mathfrak{g}, \mathfrak{k})$ are:

1. $\mathfrak{k} = \{0\}$, $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$,
2. $\mathfrak{k} = 2\mathbb{R}$, $\mathfrak{g} = \mathfrak{su}(3)$,
3. $\mathfrak{k} = \mathfrak{su}(2) \oplus \mathbb{R}$, $\mathfrak{g} = \mathfrak{sp}(2)$,
4. $\mathfrak{k} = \mathfrak{su}(3)$, $\mathfrak{g} = \mathfrak{g}_2$.

In '01, Cleyton and Swann studied coclosed G_2-structures on cohomogeneity one manifolds where the Lie group acting G was simple.

On account of this, from now on, our focus will be on the first case.
Up to finite quotients, the principal orbits are

$$\frac{G}{K} = \mathrm{SU}(2)^2 \cong S^3 \times S^3.$$

Construction of a G_2-structure from a half flat SU(3)-structure

Consider $M^{\text{princ}} = I \times N^6$. Let $(\omega(t), \Omega_2(t))$ be a 1-parameter family of $SU(3)$-structures on N parameterized by $t \in I$, we can construct a G_2-structure on M^{princ}

$$\begin{aligned} \varphi &= \mathrm{d}t \wedge \omega(t) + \Omega_1(t), \\ \psi &= \frac{\omega^2(t)}{2} - \mathrm{d}t \wedge \Omega_2(t). \end{aligned} \tag{2}$$

The G_2-structure is coclosed if $(\omega(t), \Omega_2(t))$ is a solution of:

$$\omega \wedge \dot{\omega} = -\mathrm{d}\Omega_2. \tag{3}$$

Definition

We say that an $SU(3)$-structure $(\omega, \Omega_1, \Omega_2)$ is half-flat if

$$\mathrm{d}\Omega_1 = 0, \quad \mathrm{d}\omega^2 = 0.$$

We consider G_2-structures constructed from half-flat SU(3)-structures.

G_2-structures on cohomogeneity one manifolds

A generic $SU(2)^2$-invariant half-flat SU(3)-structure on the principal bundle is given by (Madsen and Salamon '13)

$$
\begin{aligned}
\omega &= 4\sum_{i=1}^3 A_i B_i \eta_i^- \wedge \eta_i^+, \\
\Omega_1 &= 8B_1B_2B_3\eta_{123}^- - 4\sum_{i=1}^3 \epsilon_{ijk} A_i A_j B_k \eta_i^+ \wedge \eta_j^+ \wedge \eta_k^-, \\
\Omega_2 &= -8A_1A_2A_3\eta_{123}^+ + 4\sum_{i=1}^3 \epsilon_{ijk} B_i B_j A_k \eta_i^- \wedge \eta_j^- \wedge \eta_k^+,
\end{aligned} \tag{4}
$$

for real-valued functions A_i, B_i of $t \in I$, $i = 1, 2, 3$.
The resulting metric on $I \times N$ compatible with the G_2-structure $\varphi = \mathrm{d}t \wedge \omega + \Omega_1$, determined by this SU(3)-structure is:

$$
g = \mathrm{d}t^2 + \sum_{i=1}^{3} \left((2A_i)^2 \eta_i^+ \otimes \eta_i^+ + (2B_i)^2 \eta_i^- \otimes \eta_i^- \right). \tag{5}
$$

Conditions for extension to the singular orbit(s)

Consider the non-compact cohomogeneity one manifold $M = \mathbb{R}^4 \times S^3$ with group diagram $\mathrm{SU}(2)^2 \supset \Delta\mathrm{SU}(2) \supset \{1\}$. Lotay and Oliveira used Eschenburg-Wang's technique to determine when one of the previous metrics extends smoothly over the singular orbit $Q \cong S^3$ in M.

Lemma (Lotay, Oliveira '18)

The metric g in (5) extends smoothly over the singular orbit $Q = \mathrm{SU}(2)^2/\Delta\mathrm{SU}(2)$ if and only if A_i, B_i are sign definite for $t > 0$ and:

1. the A_i's are odd with $\dot{A}_i(0) = 1/2$;
2. the B_i's are even with $B_1(0) = B_2(0) = B_3(0) \neq 0$ and $\ddot{B}_1(0) = \ddot{B}_2(0) = \ddot{B}_3(0)$.

Now let $M = S^4 \times S^3$ be the cohomogeneity one manifold with group diagram $\mathrm{SU}(2) \times \mathrm{SU}(2) \supset \Delta\mathrm{SU}(2), \Delta\mathrm{SU}(2) \supset \{1\}$. The metric g in (5) extends smoothly over the singular orbits if the conditions for A_i, B_i from the previous lemma are satisfied around both singular orbits.

Coclosed equations

Proposition

Let M be a seven-dimensional simply connected cohomogeneity one manifold under the action of $\mathrm{SU}(2)^2$, with a G_2-structure coming from a half-flat $\mathrm{SU}(3)$-structure. Let the $\mathrm{SU}(3)$-structure $(\omega, \Omega_1, \Omega_2)$ be written as in (4). Then, the equations for the G_2-structure to be coclosed are:

$$\begin{cases} A_2B_2(\dot{A}_3B_3 + A_3\dot{B}_3) + & A_3B_3(\dot{A}_2B_2 + A_2\dot{B}_2) = \\ & A_1A_2A_3 - B_2B_3A_1 + B_1B_3A_2 + B_1B_2A_3, \\ A_1B_1(\dot{A}_3B_3 + A_3\dot{B}_3) + & A_3B_3(\dot{A}_1B_1 + A_1\dot{B}_1) = \\ & A_1A_2A_3 + B_2B_3A_1 - B_1B_3A_2 + B_1B_2A_3, \\ A_1B_1(\dot{A}_2B_2 + A_2\dot{B}_2) + & A_2B_2(\dot{A}_1B_1 + A_1\dot{B}_1) = \\ & A_1A_2A_3 + B_2B_3A_1 + B_1B_3A_2 - B_1B_2A_3. \end{cases} \tag{6}$$

Rewriting the equations

We will assume that for t in an open interval all A_i's are sign definite. If we define

$$\begin{cases} D_1 = A_2 B_2 A_3 B_3, \\ D_2 = A_1 B_1 A_3 B_3, \\ D_3 = A_1 B_1 A_2 B_2, \end{cases}$$

Then we can write the coclosed equations as a system of ODEs for the functions D_1, D_2, D_3, where the coefficients depend on free functions A_1, A_2, A_3.
Writing $D = (D_1, D_2, D_3)^T$ the system of equations now becomes

$$\dot{D} = \frac{1}{A_1 A_2 A_3} \begin{pmatrix} -A_1^2 & A_2^2 & A_3^2 \\ A_1^2 & -A_2^2 & A_3^2 \\ A_1^2 & A_2^2 & -A_3^2 \end{pmatrix} D + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} A_1 A_2 A_3.$$

Example

Take $A_1 = A_2 = A_3 = t/2$. Metrics (on $\mathbb{R}^4 \times S^3$) are $(c > 0)$:

$$g = dt^2 + t^2 \sum_{i=1}^{3} \eta_i^+ \otimes \eta_i^+ + (t^2 + c) \sum_{i=1}^{3} \eta_i^- \otimes \eta_i^- .$$

Existence and uniqueness results

Theorem (Foscolo, Haskins '17)

Consider the singular initial value problem

$$\dot{y} = \frac{1}{t} M_{-1}(y) + M(t, y), \quad y(0) = y_0, \tag{7}$$

where y takes values in $\mathbb{R}^k$, $M_{-1} : \mathbb{R}^k \to \mathbb{R}^k$ is a smooth function of y in a neighbourhood of y_0 and $M : \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}^k$ is smooth in t, y in a neighbourhood of $(0, y_0)$. Assume that

1. $M_{-1}(y_0) = 0$;
2. $h\mathrm{Id} - d_{y_0} M_{-1}$ is invertible for all $h \in \mathbb{N}$, $h \geq 1$.

Then there exists a unique solution $y(t)$ of (7). Furthermore y depends continuously on y_0 satisfying (i) and (ii).

Applying Foscolo-Haskins Theorem

- We perform the change of variable $E(t) = D(t)/t^2$. Let $b_0 \neq 0$. Fix $E_0 = (b_0^2/4, b_0^2/4, b_0^2/4)^T$. The new singular initial value problem for $E(t)$ is:

$$\dot{E} = \frac{1}{t} M'_{-1}(E) + M'(t, E), \quad E(0) = E_0, \tag{8}$$

  for some smooth functions M' and M'_{-1}.
- There exists a unique solution $E(t)$ in a neighbourhood of 0, which depends continuously on E_0. We recover D_1, D_2, D_3 by $D(t) = t^2 E(t)$.
- This result only gives a short-time solution. However, we can further extend the solution using Picard-Lindëlof Theorem.
- We can show that $D_i(t) > 0$ for $t \in (0, L)$, so it is possible to recover the B_i's from the D_i's with:

$$B_i(t) = \mathrm{sign}(b_0) \sqrt{\frac{D_j D_k}{D_i A_i^2}}, t > 0, \quad B_i(0) = b_0. \tag{9}$$

Extension to the singular orbit(s)

Non-compact case: $M = \mathbb{R}^4 \times S^3$

We take functions A_1, A_2, A_3 such that they satisfy the conditions for extension to the singular orbit, and a constant $b_0 \neq 0$. We have found functions B_1, B_2, B_3 such that the A_i's, B_i's solve the coclosed equations. In the previous conditions,

- $B_1(0) = B_2(0) = B_3(0) = b_0 \neq 0$ (by construction);
- the functions B_1, B_2, B_3 are even;
- $\ddot{B}_1(0) = \ddot{B}_2(0) = \ddot{B}_3(0)$.

Compact case: $M = S^4 \times S^3$

We can proceed as before and find solutions in neighbourhoods around both of the singular orbits.

The solutions around both singular orbits patch up well: the previous solution gives a G_2-structure and the corresponding metric can also be extended to the other singular orbit.

Families of coclosed G_2-structures

Corollary 1 (A. '22)

On the cohomogeneity one manifold $M = \mathbb{R}^4 \times S^3$ with group diagram $\mathrm{SU}(2)^2 \supset \Delta\mathrm{SU}(2) \supset \{1\}$, there is a family of coclosed G_2-structures which is given by three smooth functions satisfying certain boundary conditions around the singular orbit and a non-zero parameter. Moreover, any coclosed G_2-structure constructed from a half flat SU(3)-structure is in this family.

Corollary 2 (A. '22)

On the cohomogeneity one manifold $M = S^4 \times S^3$ with group diagram $\mathrm{SU}(2)^2 \supset \Delta\mathrm{SU}(2), \Delta\mathrm{SU}(2) \supset \{1\}$, there is a family of coclosed G_2-structures which is given by three smooth functions satisfying certain boundary conditions around the singular orbits and a non-zero parameter. Moreover, any coclosed G_2-structure constructed from a half flat SU(3)-structure is in this family.

Observations

- There are many more G_2-structures constructed from half-flat SU(3)-structures that are coclosed than that are torsion free:
  - Two-parameter family of torsion-free G_2-structures (Lotay-Oliveira '18) on $M = \mathbb{R}^4 \times S^3$.
  - Family of coclosed G_2-structures which is given by three smooth functions satisfying certain boundary conditions and a non-zero parameter.
- Contrast between 6 and 7 dimensional cases:
  - M^6 cohomogeneity one under a.e. action of G with a G-invariant balanced SU(3)-structure $\Rightarrow M$ Kähler.
  - M^7 cohomogeneity one under a.e. action of G with a G-invariant coclosed G_2-structure $\varphi \not\Rightarrow \varphi$ torsion-free.

Future directions

After having found a class of coclosed G_2-structures, the aim is to find gauge fields $A \in \mathcal{A}(E)$ and $\theta \in \mathcal{A}(TM)$ that, together with the flux given by the family of G_2-structures found, solve the heterotic Bianchi identity in the heterotic G_2-system:

$$dH = \frac{\alpha'}{4}(\mathrm{tr} F_A \wedge F_A - \mathrm{tr} R_\theta \wedge R_\theta).$$

- Finding G_2-instantons over a vector bundle $E \to M$.
- Finding G_2-instantons over the tangent bundle TM.

First, we compute the torsion form τ_0, and find that

$$\tau_0 = 0.$$

We may also compute

$$dH = F_1(t)\eta_{23}^- \wedge \eta_{23}^+ + F_2(t)\eta_{13}^- \wedge \eta_{13}^+ + F_3(t)\eta_{12}^- \wedge \eta_{12}^+.$$

Thank you for your attention.