# $\nu$-invariants of Joyce's $G_2$-manifolds

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# $G_2$-structures

Writing $dx_{12} = dx_1 \wedge dx_2$, etc., define a 3-form $\phi_0$ on $\mathbb{R}^7$ by

$$\phi_0 = dx_{127} + dx_{136} + dx_{145} + dx_{235} - dx_{246} + dx_{347} + dx_{567}$$

The exceptional compact Lie group $G_2 \subset GL(7, \mathbb{R})$ is the stabilizer of $\phi_0$.
dim 14 dim 49

A *$G_2$-structure* on a 7-manifold $M$ is a 3-form $\phi \in \Omega^3(M)$ such that for each $x \in M$ there is an isomorphism $g : T_x M \to \mathbb{R}^7$ with $g^*(\phi_0) = \phi_x$.

Equivalently, a $G_2$-structure of $M$ is a principal $G_2$ bundle which is a sub-bundle of the principal $GL(7, \mathbb{R})$ frame bundle of $M$.

$G_2$ is connected and simply-connected.
Thus manifolds with $G_2$-structures are automatically oriented and spin.

$G_2 \subset SO(7)$. So a $G_2$-structure has a preferred Riemannian metric $g_\phi$.

# Torsion-free $G_2$-structures

Let $(M, \phi)$ be a 7-manifold with $G_2$-structure. Let $\nabla$ be the Levi–Civita connection on $M$ associated to $g_\phi$.

In the case $\nabla\phi = 0$, then the $G_2$-structure is called *torsion-free*, and $(M, \phi, g_\phi)$ is called a $G_2$-manifold. Equivalent condition: $d\phi = d *_{g_\phi} \phi = 0$.

A $G_2$-manifold has Riemannian holonomy group $\mathrm{Hol}(g_\phi) \subset G_2$. Furthermore, if $\mathrm{Hol}(g_\phi)$ is connected, it is isomorphic to one of $\{1\}, SU(2), SU(3), G_2$.

Conversely, if $(M, g)$ is a Riemannian manifold with holonomy $\mathrm{Hol}(g) \subset G_2$, then $M$ is naturally a $G_2$-manifold.

**Observation** (Joyce): If $(M, g)$ is compact and has $\mathrm{Hol}(g) \subset G_2$, then

$$\boxed{\mathrm{Hol}(g) = G_2 \quad \Longleftrightarrow \quad \pi_1(M) \text{ is finite}}$$

In particular, a compact 7-manifold with $\mathrm{Hol} = G_2$ necessarily has $b_1 = 0$.

# Compact $G_2$-manifolds

Using Yau's resolution of the Calabi conjecture, many examples of compact $G_2$-manifolds can be constructed.

**Examples of compact $G_2$-manifolds:**

- $(T^7, g_{flat})$: Hol $= \{1\}$
- $(K3 \times T^3, g_{\text{Yau}} + g_{flat})$: Hol $= SU(2)$
- $(Y \times S^1, g_{\text{Yau}} + g_{S^1})$, $Y$ is a $\pi_1 = 0$ Calabi–Yau 3-fold: Hol $= SU(3)$

None of these could possibly have a metric with Hol $= G_2$, as $b_1 > 0$.

For some time, the existence of compact examples with Hol $= G_2$ was unknown.

**Theorem** (Joyce, '96): There exist (many) compact manifolds with Hol $= G_2$.

**Theorem** (Joyce, '96): There exist (many) compact manifolds with $\mathrm{Hol} = G_2$.

**Rough Idea**: Let $\Gamma$ be a finite group acting on a flat 7-torus $T^7$, such that $\Gamma$ preserves the $G_2$-structure. Then $\mathcal{O} = T^7/\Gamma$ is a flat orbifold.

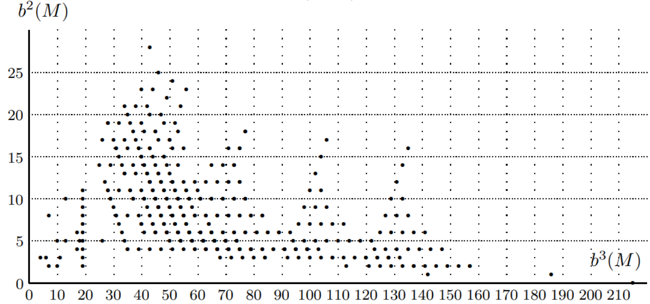
Cut out the singularities of $\mathcal{O}$ and replace them with smooth pieces that admit torsion-free $G_2$-structures. The torsion-free $G_2$ metrics on these resolution pieces are pasted together with the flat metric on the smooth part of $\mathcal{O}$.

With careful analysis, Joyce shows this construction can be done so that there exists a torsion-free $G_2$ metric $g$ close to the cut-and-paste one just described.

Finally, if $\pi_1$ of the resolved 7-manifold is finite, then $\mathrm{Hol}(g) = G_2$.

Joyce constructs several hundred $\pi_1 = 0$ compact $G_2$-manifolds. The following is taken from Joyce, "Constructing compact manifolds with exceptional holonomy":

Figure 1: Betti numbers $(b^2, b^3)$ of compact $G_2$-manifolds




There are now several other constructions of compact 7-manifolds with Hol $= G_2$.

- Twisted connected sum (Kovalev '03, Corti–Haskins–Nordström–Pacini '12)
- Extra-twisted connected sum (Nordström '18)
- Resolution of $G_2$-manifold/involution (Joyce–Karigiannis '17)

# Spaces of $G_2$-structures

Fix a compact oriented spin 7-manifold $M$. Define

$$\mathcal{G}_2(M) = \{\phi \text{ is a } G_2\text{-structure on M}\} / \mathrm{Diff}_0(M)$$

where each $\phi$ is compatible with the orientation and spin structure on $M$, and $\mathrm{Diff}_0(M)$ is the group of spin diffeomorphisms of $M$ isotopic to $\mathrm{id}_M$.

**Q1**: What is the topology of $\mathcal{G}_2(M)$? What about $\mathcal{G}_2(M)/\mathrm{Diff}(M)$?

Let $\mathcal{G}_2^{\mathrm{met}}(M) \subset \mathcal{G}_2(M)$ be the subset of torsion-free $G_2$-structures.

**Q2**: What is the topology of $\mathcal{G}_2^{\mathrm{met}}(M)$? What about $\mathcal{G}_2^{\mathrm{met}}(M)/\mathrm{Diff}(M)$?

**Q1** is topological, while **Q2** is geometric.

These questions are hard to answer in the generality given.

(And **Q2** is much harder than **Q1**.)

**Q1**: What is the topology of $\mathcal{G}_2(M)$? What about $\mathcal{G}_2(M)/\mathrm{Diff}(M)$?

$\mathcal{G}_2(M)$ is an infinite dimensional (singular) manifold.

(Witt, see also Crowley & Nordström): $\pi_0\mathcal{G}_2(M)$ is non-empty, and has a free and transitive action by $H^7(M;\pi_7(S^7)) \cong \mathbb{Z}$.

Crowley & Nordström (2014) introduced a numerical invariant, $\nu(M,\phi)$, defined for a manifold with $G_2$-structure. It induces a map

$$\nu : \pi_0\left(\mathcal{G}_2(M)/\mathrm{Diff}(M)\right) \longrightarrow \mathbb{Z}/48$$

This invariant satisfies $\nu(-M,-\phi) = -\nu(M,\phi)$ and also

$$\nu(M,\phi) \equiv 1 + b_1(M) + b_2(M) + b_3(M) \pmod 2$$

The map $\nu$ surjects onto the 24 values in $\mathbb{Z}/48$ which satisfy this constraint.

Crowley & Nordström also determine circumstances for which $\nu$ is an injection.

**Summary**: **Q1** is well-understood at the level of $\pi_0$.

(For higher $\pi_k$, see Crowley's talk ...)

**Q2**: What is the topology of $\mathcal{G}_2^{\text{met}}(M)$? What about $\mathcal{G}_2^{\text{met}}(M)/\text{Diff}(M)$?

The various constructions of holonomy $G_2$-metrics mentioned earlier allow us to verify cases in which $\mathcal{G}_2^{\text{met}}(M) \neq 0$.

There is also the following structural result, due to Joyce.

**Theorem**: $\mathcal{G}_2^{\text{met}}(M)$ is either empty or a smooth manifold of dim$= b_3(M)$. There is a local diffeomorphism

$$\mathcal{G}_2^{\text{met}}(M) \longrightarrow H^3(M;\mathbb{R})$$

induced by sending a torsion-free $G_2$-structure $\phi$ to the de Rham class $[\phi]$.

There are constructions of $G_2$-metrics in families (e.g. in the work of Joyce).

All constructions of torsion-free $G_2$-structures are "near the boundary of $\mathcal{G}_2^{\text{met}}(M)$" in the sense that the metrics are close to ones which are almost degenerate along a hypersurface, or otherwise close to some singular metric

**Q2**: What is the topology of $\mathcal{G}_2^{\text{met}}(M)$? What about $\mathcal{G}_2^{\text{met}}(M)/\text{Diff}(M)$?

Following the story of $\nu$, Crowley, Goette & Nordström (2015) defined a numerical invariant $\overline{\nu}(M, \phi)$ for manifolds with *torsion-free* $G_2$-structure. It induces a map

$$\overline{\nu} : \pi_0\left(\mathcal{G}_2^{\text{met}}(M)/\text{Diff}(M)\right) \longrightarrow \mathbb{Z}$$

This is a refinement of $\nu$ in the sense that

$$\overline{\nu}(M, \phi) \equiv \nu(M, \phi) - 24(1 + b_1(M)) \pmod{48}$$

Some (open) questions:

- For any given $M$, what is the range of $\overline{\nu}$?
- Is the map $\pi_0\left(\mathcal{G}_2^{\text{met}}(M)/\text{Diff}(M)\right) \to \pi_0\left(\mathcal{G}_2(M)/\text{Diff}(M)\right)$ ever onto?

This leads us to the matter of computing the invariants $\nu$, $\overline{\nu}$.

# Computations of $\nu$, $\overline{\nu}$

Main interest: compute $\nu$, $\overline{\nu}$ for $G_2$-manifolds.

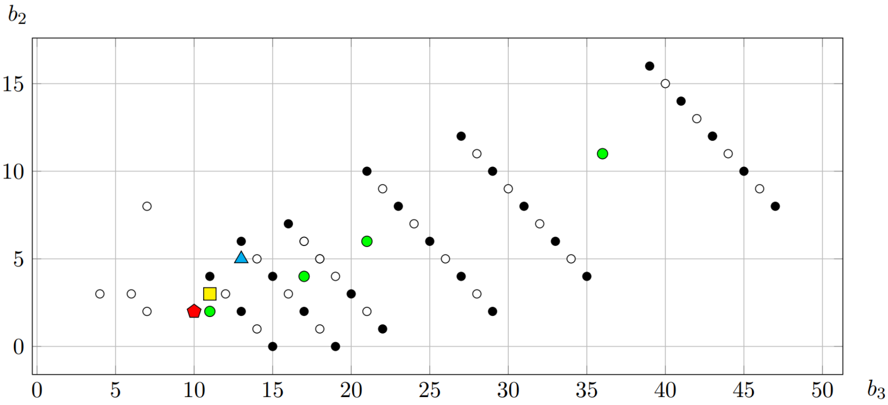
- $\nu \equiv 24 \pmod{48}$ for twisted connected sums (Crowley–Nordström)
- $\overline{\nu} = 0 \in \mathbb{Z}$ for twisted connected sums (Crowley–Goette–Nordström)
- $\overline{\nu} \neq 0$ for many extra-twisted connected sums (Goette–Nordström)

What about Joyce's original $G_2$-manifolds?

**Main goal today**:

Explain how to compute $\nu$ for many of Joyce's $G_2$-manifolds.

Here is a preview of the results that we obtain:




Betti numbers $b_2$, $b_3$ of the compact $G_2$-manifolds from Joyce's 1996 paper.

● : $\nu \equiv 0 \pmod{48}$ ○ : $\nu \equiv 0 \pmod{24}$

green ● : $\nu \equiv 24 \pmod{48}$ blue ▲ : $\nu \equiv 3 \pmod{48}$

yellow ■ : $\nu \equiv 45 \pmod{48}$ red ⬟ : $\nu \equiv 45, 33 \pmod{48}$

Some nodes have more than one manifold. There are two at ⬟ with distinct $\nu$.

# Definitions of $\nu$ and $\overline{\nu}$

Original definition of $\nu$ of Crowley–Nordström:

For $(M, \phi)$ where $\phi$ is a $G_2$-structure, let $W$ be a compact 8-manifold with $\partial W = M$ and which admits a Spin(7) structure restricting to $\phi$ at $\partial W$. Then

$$\nu(M, \phi) := \chi(W) - 3\sigma(W) \pmod{48}$$

Crowley–Goette–Nordström gave a spectral description of $\nu$ as follows. Assume for simplicity that $\phi$ is a torsion-free $G_2$-structure. Then

$$\boxed{\nu(M, \phi) = 3\eta(B_M) - 24\eta(D_M) - 24(1 + b_1(M)) \pmod{48}}$$

where

$$B_M = *d - d* : \Omega^{\text{even}}(M) \longrightarrow \Omega^{\text{even}}(M)$$ odd signature operator

$$D_M : \Gamma(S) \longrightarrow \Gamma(S), \quad \text{spin bundle } S \to M$$ Dirac operator

and $\eta$ is a spectral invariant of an elliptic operator on a compact manifold.

$$\nu(M,\phi) = 3\eta(B_M) - 24\eta(D_M) - 24(1 + b_1(M)) \pmod{48}$$

The $\eta$-invariant of an elliptic operator $D$ at a complex parameter $s$ is

$$\eta(D)(s) = \sum_{\lambda \neq 0} \operatorname{sign}(\lambda) \dim(E_\lambda) |\lambda|^s$$

where $E_\lambda$ is the $\lambda$-eigenspace of $D$. The $\eta$-invariant is then

$$\eta(D) = \eta(D)(0) = \text{``}\#\{\lambda > 0\} - \#\{\lambda < 0\}\text{''}$$

If the spectrum of $D$ is symmetric, then $\eta(D) = 0$.

**Example:** If $M$ admits an orientation-reversing isometry, then $\eta(B_M) = 0$.

The refined $\overline{\nu}$ invariant, for a torsion-free $G_2$-structure, is defined to be

$$\overline{\nu}(M,\phi) = 3\eta(B_M) - 24\eta(D_M)$$

# Towards $\nu$ of Joyce's $G_2$-manifolds

Let's return to Joyce's original construction of compact $G_2$-manifolds.

Let $\Gamma$ be a finite group acting on $T^7$ preserving the $G_2$-structure.

Write $\mathcal{O} = T^7/\Gamma$ for the resulting Riemannian orbifold.

There are orbifold $\eta$-invariants defined for $\mathcal{O}$. For example:

$$\eta(B_{\mathcal{O}})(s) = \sum_{\lambda \neq 0} \operatorname{sign}(\lambda) \dim(E_\lambda^\Gamma) |\lambda|^s$$

where $E_\lambda^\Gamma$ is the subspace of the eigenspace $E_\lambda$ which is fixed by $\Gamma$.

Simarly we can define $\eta(D_{\mathcal{O}})$. Now define $\nu(\mathcal{O})$ and $\overline{\nu}(\mathcal{O})$ by

$$\overline{\nu}(\mathcal{O}) := 3\eta(B_{\mathcal{O}}) - 24\eta(D_{\mathcal{O}})$$

Joyce's construction resolves the singularities of $\mathcal{O}$ and gives a $G_2$-manifold $M$.

**Ideally**: $\overline{\nu}(M)$ is determined by $\overline{\nu}(\mathcal{O})$. This may be too hopeful in general.

We will give conditions for which $\nu(M) \equiv \nu(\mathcal{O})$ as elements of $\mathbb{Z}/48$.

Once this is done we are in business, as $\nu(\mathcal{O})$, and in fact $\overline{\nu}(\mathcal{O})$, are computable by standard techniques. ($\Gamma$-equivariant geometry on $T^7$.)

**Assumption ($\star$)**: Each connected component of the singular set in $\mathcal{O} = T^7/\Gamma$ has a neighborhood isometric to a neighborhood of the singular set in the orbifold

$$\mathbb{C}^2/G \times T^3$$

where $T^3$ is a flat 3-torus and $G$ is a finite subgroup of $SU(2)$.

**Theorem**: With assumption ($\star$), $\nu(M) \equiv \nu(\mathcal{O}) \pmod{48}$.

The result is more general: the singular set may be a union of components $(\mathbb{C}^n/G \times T^{7-2n})/B$ where $n \in \{2, 3\}$ where $B$ is a finite group, $G \subset SU(n)$ $+$some extra conditions.

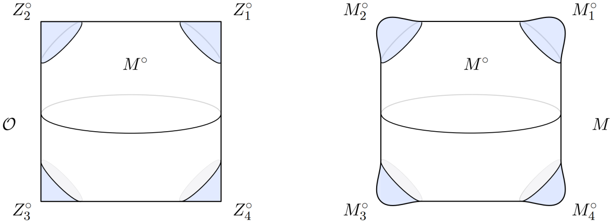
For simplicity we assume ($\star$) holds.

The resolution $M$ of $\mathcal{O}$ is obtained as follows. Write

$$\mathcal{O} = M^\circ \cup \bigcup_i Z_i^\circ$$

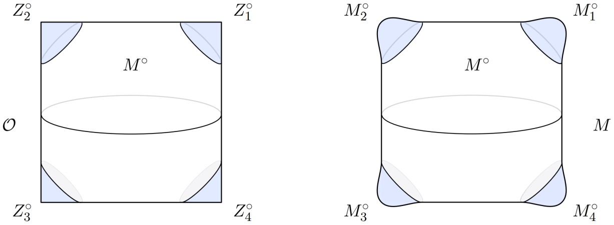
The $Z_i^\circ$ are neighborhoods of the singular components, and $M^\circ$ is smooth.

By assumption $(\star)$, $Z_i^\circ$ is identified with a neighborhood of $\mathbb{C}^2/G_i \times T^3$.




For $G_i \subset SU(2)$ choose a resolution $X_{G_i}$ of $\mathbb{C}^2/G_i$ which admits an ALE metric with holonomy $SU(2)$. Let $M_i^\circ \subset X_{G_i} \times T^3$ be such that $\partial Z_i^\circ = \partial M_i^\circ$. Then

$$M = M^\circ \cup \bigcup_i M_i^\circ$$




A metric $g_{int}$ is constructed on $M$ equal to $g_{ALE} + g_{T^3}$ on $M_i^\circ \subset X_{G_i} \times T^3$ away from the gluing region, and is the flat metric on $M^\circ$ away from the gluing region.

On the gluing regions $[r_1, r_2] \times S^3 \times T^3$ the metric is $g' + g_{T^3}$ where $g'$ is a metric on $[r_1, r_2] \times S^3$ which interpolates between $g_{ALE}$ and $dr^2 + r^2 g_{S^3}$.

**Note**: Joyce's construction is much more specific, taking care as to which ALE metric is used, and also the size and position of the gluing region.

The actual $G_2$-metric $g_{G_2}$ on $M$ that Joyce proves exists is some metric which can be chosen arbitrarily $C^0$ close to one of the above interpolation metrics $g_{int}$.

# Matching odd signature $\eta$-invariants

With this setup, the first step towards $\nu(M) \equiv \nu(\mathcal{O})$ is to show that $\eta(B_{M,g_{int}}) = \eta(B_{\mathcal{O}})$, where $M$ has the interpolation metric $g_{\text{int}}$.

Strategy: use our decompositions from above and a gluing result for $\eta$-invariants.

**Theorem** (Kirk–Lesch)
Let $X$ be a closed, oriented odd-dimensional Riemannian manifold and $Y \subset X$ a hypersurface separating $X$ into $X_+$ and $X_-$. Suppose the metric on $X$ is a product in a neighborhood of $Y$. Then:

$$\eta(B_X) = \eta_{APS}(B_{X_+}, V_+) + \eta_{APS}(B_{X_-}, V_-) + m(V_+, V_-; H^*(Y))$$

The last term is the Maslov index of the Lagrangian subspaces $V_\pm \subset H^*(Y)$ defined by $V_\pm = im(H^*(X_\pm) \to H^*(Y))$

$\eta_{APS}(B_{X_\pm}, V_\pm)$ is the $\eta$-invariant for a manifold with boundary. To define this one uses APS boundary conditions determined by a Lagrangian subspace in $H^*(\partial X_\pm)$.

**Theorem** (Kirk–Lesch)

$$\eta(B_X) = \eta_{APS}(B_{X_+}, V_+) + \eta_{APS}(B_{X_-}, V_-) + m(V_+, V_-; H^*(Y))$$

Apply this to $M = M^\circ \cup (\cup_i M_i^\circ)$ and $\mathcal{O} = M^\circ \cup (\cup_i Z_i^\circ)$ to get

$$\eta(B_{M,g_{int}}) = \eta_{APS}(M^\circ, V_+) + \eta_{APS}(\cup_i M_i^\circ, V_-) + m(V_+, V_-; H^*(\partial M^\circ))$$

$$\eta(B_{\mathcal{O}}) = \eta_{APS}(M^\circ, V_+) + \eta_{APS}(\cup_i Z_i^\circ, V'_-) + m(V_+, V'_-; H^*(\partial M^\circ))$$

Now each of $M_i^\circ \subset X_{G_i} \times T^3$ and $Z_i^\circ \subset \mathbb{C}^2/G_i \times T^3$ admit orientation reversing isometries (using the $T^3$ factor), so the middle terms vanish.

By a simple cohomology calculation, $V_- = V'_-$ so the last terms are equal.

So $\eta(B_{M,g_{int}}) = \eta(B_{\mathcal{O}})$! But is this all valid?

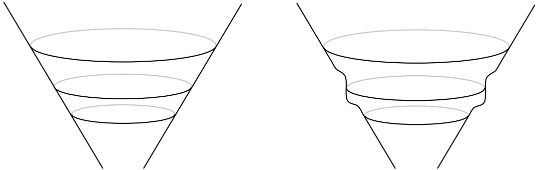
Gluing theorem applies to orbifolds if hypersurface disjoint from singularities. ✓

Remaining issue:

> **Theorem** (Kirk–Lesch)
> Let $X$ be a closed, oriented odd-dimensional Riemannian manifold and $Y \subset X$ a hypersurface separating $X$ into $X_+$ and $X_-$. Suppose the metric on $X$ is a product in a neighborhood of $Y$. Then...

Recall: on a gluing region $[r_1, r_2] \times S^3 \times T^3$ the metric is $g' + g_{T^3}$ where $g'$ is a metric on $[r_1, r_2] \times S^3$ which interpolates between $g_{ALE}$ and $dr^2 + r^2 g_{S^3}$.

This is not of the required product form, $dr^2 + g''$, where $g''$ is independent of $r$.



Would like to modify $g' + g_{T^3}$ so that it is a product near $\{r_0\} \times S^3 \times T^3$.

Useful flexibility properties of $\eta(B_X)$? It is conformally invariant (APS).
This still does not help in our situation.

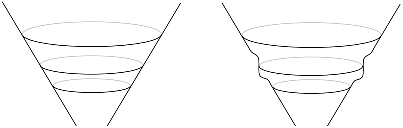
Let $X$ be a closed oriented odd-dimensional manifold, with metrics $g_1$ and $g_2$.

Suppose there is an open subset $U \subset X$ such that $U \cong V \times I_t$ and

- $g_1$, $g_2$ agree on $X \backslash U$
- $g_i = h_i + dt^2$ for metrics $h_i$ on $V$ $(i = 1, 2)$

Thus $g_1$, $g_2$ differ in a region where $X$ metrically splits off a flat factor $(I, dt^2)$.

**Lemma**: When two metrics are related as above, $\eta(B_{X,g_1}) = \eta(B_{X,g_2})$.



Back to our situation: the interpolation metric $g_{int}$ on the resolution $M$ and a modified metric which is a product in a collar neighborhood of $S^3 \times T^3$ in a gluing region are related in the manner above, thanks to the $T^3$ factor.

Thus the lemma completes the claim that $\eta(B_{M,g_{int}}) = \eta(B_{\mathcal{O}})$.

**Proof of lemma**: For $s \in [0,1]$ let $g_s$ be a smooth family of metrics with

- $g_s = g_1$ for $s \in [0,\epsilon]$ and $g_s = g_2$ for $s \in [1-\epsilon, 1]$
- $g_s = g_1 = g_2$ on $X \backslash U$
- $g_s = h_s + dt^2$ on $U \cong V \times I_t$

By the Atiyah–Patodi–Singer theorem,

$$\eta(B_{X,g_2}) - \eta(B_{X,g_1}) = \int_{[0,1]\times X} L(G)$$

where $L(G)$ is the Hirzebruch $L$-polynomial applied to the Pontryagin forms of the Riemannian manifold $([0,1] \times X, ds^2 + g_s)$.

Key property: $L(G) = 0$ if the metric $G$ splits off a flat factor.

On $[0,1] \times (X \backslash U)$, $G$ is a product metric $ds^2 + g_1$, since $g_1 = g_s = g_2$. So here it splits off the factor $([0,1], ds^2)$.

On $[0,1] \times U$, $G = ds^2 + h_s + dt^2$ so splits off the flat factor $(I, dt^2)$. ■

Similar arguments handle the Dirac $\eta$-invariants. For example, if for an elliptic operator $D$ one defines the *reduced* $\eta$-invariant

$$\xi(D) = \frac{1}{2}\left(\eta(D) + \dim \ker D\right)$$

then $\xi(D_{X,g_1}) \equiv \xi(D_{X,g_2}) \pmod{\mathbb{Z}}$, for two metrics as related before.

Further, there is a gluing formula for $\xi$ (due to Bunke) which works modulo $\mathbb{Z}$.

In applying the gluing formula for $\xi$, the tangential operator must be invertible: this holds in our case because the cross section $S^3 \times T^3$ has a metric of positive scalar curvature, and no harmonic spinors.

Removing the mod $\mathbb{Z}$ constraint from the gluing formula is a delicate matter. Achieving this in the above setup would help compute $\overline{\nu}$ for Joyce $G_2$-manifolds.

There is progress in this direction in the PhD work of Nelvis Fornasin.

Having outlined the proof that $\nu(M) \equiv \nu(\mathcal{O})$ for some of Joyce's $G_2$-manifolds, we turn to computations.

# Computations of $\nu(M)$

First, let's summarize some of the results we obtain (see also the table earlier).

- The two simply-connected torsion-free $G_2$-manifolds with $b_2 = 2$ and $b_3 = 10$ constructed in Joyce's original paper have distinct $\nu$ invariants.
- For almost all examples considered, we compute $\nu \equiv 0 \pmod{24}$.
- For all of the examples considered, we compute $\nu \equiv 0 \pmod{3}$.

**Thm** (Schelling): $\nu \equiv 0 \pmod{3}$ if and only if $M$ is $G_2$-nullbordant.

Thus all of the examples we compute are $G_2$-nullbordant.

**Q**: Are there Joyce $G_2$-manifolds that are not $G_2$-nullbordant?

Note: Goette and Nordström have found extra-twisted connected sums with $\nu \neq 0 \pmod{3}$, showing the existence of $G_2$-manifolds that are not $G_2$-nullbordant.

# Dihedral examples

Let $z_1, z_2, z_3$ be coordinates for $\mathbb{C}^3$ and $x$ for $\mathbb{R}$. Let $\Lambda \subset \mathbb{C}^3$ be a rank 6 lattice. The torus $T^7 = \mathbb{C}^3 \times \mathbb{R}/\Lambda \times \mathbb{Z}$ has a flat $G_2$-structure

$$\phi = \omega \wedge dx + \mathrm{Im}(\Omega)$$

where $\omega = \frac{i}{2}\sum_{k=1}^{3} dz_k d\bar{z}_k$ and $\Omega = dz_1 dz_2 dz_3$.

Let $u = e^{i\theta_1}$ and $v = e^{i\theta_2}$ be roots of unity, and $a$ the smallest positive integer such that $u^a = v^a = 1$. Let $\alpha$ and $\beta$ be the isometries of $\mathbb{C}^3 \times \mathbb{R}$ given by

$$\alpha(z_1, z_2, z_3, x) = (uz_1, vz_2, \overline{uv}z_3, x + \tfrac{1}{a})$$
$$\beta(z_1, z_2, z_3, x) = (-\bar{z}_1, -\bar{z}_2, -\bar{z}_3, -x)$$

If $\alpha$, $\beta$ preserve $\Lambda$, they descend to isometries on $T^7$ that preserve $\phi$. Note $\alpha^a = \beta^2 = 1$ and $\alpha\beta = \beta\alpha^{-1}$. Thus they generate a dihedral group of order $2a$:

$$\Gamma := \langle \alpha, \beta \rangle = \left\{1, \alpha, \alpha^2, \ldots, \alpha^{a-1}, \beta, \beta\alpha, \beta\alpha^2, \ldots, \beta\alpha^{a-1}\right\}$$

Then $\mathcal{O} = \mathcal{O}(\Lambda, \theta_1, \theta_2) = T^7/\Gamma$ and the Joyce $G_2$-manifold is a resolution of $\mathcal{O}$.

For $\mathcal{O} = \mathcal{O}(\Lambda, \theta_1, \theta_2)$ we can use a formula of Donnelly ('78) to compute $\eta(B_{\mathcal{O}})$.

Let $\theta_3$ satisfy $\theta_1 + \theta_2 + \theta_3 \equiv 0 \pmod{2\pi}$. Normalize $\theta_i \in (0, 2\pi)$. Then:

$$\eta(B_{\mathcal{O}}) = \frac{4}{a} \sum_{j=1}^{a-1} \cot(j\pi/a) \prod_{k=1}^{3} \sin(j\theta_k) = 2 \sum_{k=1}^{3} ((\theta_k/\pi))$$

where $((t)) = t - [t] - 1/2$ if $t \notin \mathbb{Z}$ and $((t)) = 0$ otherwise. Thus:

$$\eta(B_{\mathcal{O}}) = \begin{cases} +1, & \theta_i < \pi \text{ for } i = 1,2,3 \\ -1, & \theta_i > \pi \text{ for some } i \\ 0, & \theta_i = \pi \text{ for some } i \end{cases}$$

*Ex. 7, Joyce ('96).* $\Lambda = \mathbb{Z}^3 \oplus e^{2\pi i/3}\mathbb{Z}^3 \subset \mathbb{C}^3$, $\theta_1 = \theta_2 = \theta_3 = 2\pi/3$. $\eta(B_{\mathcal{O}}) = 1$.

*Ex. 8, Joyce ('96).* Same $\Lambda$ and $\theta_1 = \theta_2 = \pi/3$, $\theta_3 = 4\pi/3$. $\eta(B_{\mathcal{O}}) = -1$.

*Ex. 9, Joyce ('96).* $\Lambda = \mathbb{Z}^3 \oplus i\mathbb{Z}^3$ and $\theta_1 = \theta_2 = \pi/2$, $\theta_3 = \pi$. $\eta(B_{\mathcal{O}}) = 0$.

In principle $\eta(B_{\mathcal{O}})$ can also be computed by directly computing the spectrum of $B_{\mathcal{O}}$. In general this is difficult, but geometry of $\mathcal{O} = T^7/\Gamma$ makes this feasible.

We use this direct method for the Dirac operator $D_{\mathcal{O}}$. To give an idea of how this does, consider $T^7 = \mathbb{R}^7/\Lambda_0$. The $\lambda$-eigen-spinors of $D_{T^7}$ are given by

$$f_{u,w} : T^7 \to S, \qquad f_{u,w}(x) = e^{2\pi i\langle u,x\rangle} w$$

where $u \in \Lambda_0^*$ with $|u| = 2\pi/\lambda$, and $w$ is an $\mp i\lambda/2\pi$-eigenvector in $S = \mathbb{C}^8$ (standard $\mathrm{Spin}(7)$ representation) with respect to Clifford multiplication by $u$.

Then $\eta(D_{\mathcal{O}})$ where $\mathcal{O} = T^7/\Gamma$ can be computed by finding the $\Gamma$-invariant $f_{u,w}$.

This sort of direct computation (on $T^7$) goes back at least to Friedrich ('84).

For Joyce's dihedral examples the computation gives (using $\theta_i \not\equiv 0 \pmod{2\pi}$):

$$\eta(D_{\mathcal{O}}) = -\frac{1}{a}\sum_{j=1}^{a-1} \cot(\pi j/a) \sum_{k=1}^{3} \sin(j\theta_k) = -1$$

# Summary of some examples

| Ex. No. | $\lvert\Gamma\rvert$ | $\pi_1$ | $b_2$ | $b_3$ | $\eta(B_M)$ | $\overline{\eta}(D_M)$ | $\nu$ (mod 48) |
|---|---|---|---|---|---|---|---|
| 7 | 6 | 0 | 5 | 13 | 1 | $-1$ | 3 |
| 8 | 12 | 0 | 3 | 11 | $-1$ | $-1$ | 45 |
| 9 | 8 | 0 | 11 | 36 | 0 | $-1$ | 0 |
| 10 | 16 | $\mathbb{Z}/2$ | 6 | 21 | 0 | $-1$ | 0 |
| 11 | 12 | 0 | 4 | 17 | 0 | $-1$ | 0 |
| 12 | 24 | $\mathbb{Z}/2$ | 2 | 11 | 0 | $-1$ | 0 |
| 13 | 14 | 0 | 2 | 10 | $-1$ | $-1$ | 45 |
| 14 | 18 | 0 | 2 | 10 | $1/3$ | $-1/3$ | 33 |

$\nu$ invariants for Examples 7–14 from Joyce ('96). Here $\overline{\eta} := \eta \pmod{2\mathbb{Z}}$.

Examples 7, 8, 9, 11, 13 are dihedral. Examples 10, 12, 14 are obtained from dihedral ones by enlarging $\Gamma$ (adding an involution, etc.)

# Vanishing examples

For many of Joyce's original constructions, $\nu \equiv 24 \pmod{48}$. Recall

$$\phi_0 = dx_{127} + dx_{136} + dx_{145} + dx_{235} - dx_{246} + dx_{347} + dx_{567}$$

Let $\alpha$, $\beta$, $\gamma$ be the involutions

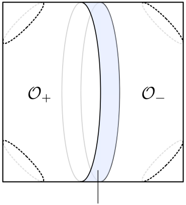
$$\begin{aligned}
\alpha(x_1, \ldots, x_7) &= (-x_1, -x_2, -x_3, -x_4, x_5, x_6, x_7) \\
\beta(x_1, \ldots, x_7) &= (b_1 - x_1, b_2 - x_2, x_3, x_4, -x_5, -x_6, x_7) \\
\gamma(x_1, \ldots, x_7) &= (c_1 - x_1, x_2, c_3 - x_3, x_4, c_5 - x_5, x_6, -x_7)
\end{aligned}$$

where $b_1$, $b_2$, $c_1$, $c_3$, $c_5 \in \{0, 1/2\}$. Let $\Gamma = \langle \alpha, \beta, \gamma \rangle$. Many of Joyce's original examples come from orbifolds $\mathcal{O} = T^7/\Gamma$ where $T^7 = \mathbb{R}^7/\mathbb{Z}^7$.

The orientation-reversing isometry $(x_1, \ldots, x_7) \mapsto (-x_1, \ldots, -x_7)$ is $\Gamma$-equivariant and thus $\eta(B_{\mathcal{O}}) = \eta(D_{\mathcal{O}}) = 0$.

# Twisted connected sum type decompositions

(suggested by Goette, Nordström)




The dihedral examples admit a decomposition $\mathcal{O} = \mathcal{O}_+ \cup \mathcal{O}_-$ where

$$\mathcal{O}_+ = \left(T^6 \times [-\tfrac{1}{4a}, \tfrac{1}{4a}]\right)/\beta, \qquad \mathcal{O}_- = \left(T^6 \times [\tfrac{1}{4a}, \tfrac{3}{4a}]\right)/\alpha\beta$$

We apply the gluing formulas for $\eta(B_{\mathcal{O}})$, $\eta(D_{\mathcal{O}})$ to this decomposition. Each $\mathcal{O}_\pm$ has spectral symmetry, so the computation reduces to Masolv indices.

We obtain the same computations of $\eta$-invariants for these examples, although the details are of a different flavor.

Thank you!