Uniqueness of Asymptotically Conical Gradient Shrinking Solitons in G_2-Laplacian Flow.

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3-FORMS AND G_2-STRUCTURES

Definition

A G_2**-structure** on a 7-manifold M is a 3-form $\varphi \in \Omega^3(M)$ such that for each $x \in M$, there is an isomorphism $\iota : T_x M \to \mathbb{R}^7$ such that $\iota^* \varphi_0 = \varphi_x$, where φ_0 is the 3-form on $\mathbb{R}^7 = \mathrm{Im}(\mathbb{O})$ given by

$$\varphi_0(u, v, w) := \langle u \times v, w \rangle = \langle uv, w \rangle.$$

- **closed** G_2-structure: $d\varphi = 0$; **coclosed** G_2-structure: $d^*\varphi = 0$.
- **torsion-free** G_2-structure: $d\varphi = 0$ and $d^*\varphi = 0$.
- The **torsion** of a G_2-structure is a two-tensor T_{ij} satisfying $\nabla\varphi = T_i{}^m (\star_\varphi \varphi)_{mjkl}$.
- If φ is closed, then T_{ij} is a two-form satisfying $T \wedge \star_\varphi \varphi = 0$. Also, $R = -|T|^2$ and $\nabla T = \mathrm{Rm} * \varphi + T^2 * \varphi$.

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A G_2-structure gives an orientation and a metric g_φ on M satisfying

$$g_\varphi(u,v)\mathrm{Vol}_{g_\varphi} = \frac{1}{6}\iota_u\varphi \wedge \iota_v\varphi \wedge \varphi.$$

Definition

If M is compact and φ_0 is closed, then the **Hitchin volume functional**, $\mathcal{H} : [\varphi_0]_+ \to \mathbb{R}$, is defined for $\varphi \in [\varphi_0]_+$ by

$$\mathcal{H}(\varphi) := \frac{1}{7}\int_M \varphi \wedge \star_\varphi \varphi = \mathrm{Vol}(M, g_\varphi).$$

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Gradient Flow of Volume

For a compact M with a closed G_2-structure φ_0,

Proposition

The upward gradient flow of $\mathcal{H}$ on $[\varphi_0]_+$ satisfies

$$\frac{\partial}{\partial t}\varphi = \Delta_\varphi \varphi.$$

This is the **Laplacian flow** *on a compact M.*

For any geometric flow of closed G_2-structures, $\pi_1\left(\frac{\partial \varphi}{\partial t}\right) = 3f_0\varphi$ and $\pi_1\left(\frac{\partial}{\partial t}(\star_\varphi \varphi)\right) = 4f_0 \star_\varphi \varphi$. For $\varphi(t) = \varphi(0) + d\eta(t)$,

$$\frac{\partial}{\partial t}\mathcal{H}(\varphi) = \int_M f_0 \varphi \wedge \star_\varphi \varphi = \frac{1}{3}\left\langle \frac{\partial \varphi}{\partial t}, \varphi \right\rangle_{L^2} = \frac{1}{3}\left\langle d\frac{\partial \eta}{\partial t}, \varphi \right\rangle_{L^2} = \frac{1}{3}\left\langle \frac{\partial \eta}{\partial t}, d^*\varphi \right\rangle_{L^2}.$$

The upward gradient flow is $\frac{\partial \eta}{\partial t} = d^*\varphi$, and thus $\frac{\partial}{\partial t}\varphi = dd^*\varphi = \Delta_\varphi \varphi$.

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Bryant's Laplacian flow

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For a closed G_2-structure φ_0 on a (possibly) noncompact M, the **Laplacian flow** of φ_0 is a solution $\varphi = \varphi(t)$ to

$$\begin{cases} \frac{\partial \varphi}{\partial t} = \Delta_\varphi \varphi, \\ d\varphi = 0, \end{cases}$$

for $t \in [0, \epsilon)$ with initial condition $\varphi(0) = \varphi_0$.

- Fixed points of the flow are exactly the torsion-free G_2-structures.
- The flow may help construct G_2-holonomy metrics. All previous constructions involve perturbing closed G_2-structures with small torsion to exactly torsion-free ones. (Joyce, Kovalev, Karigiannis, Corti-Haskins-Nordström-Pacini)

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Bryant's Laplacian flow

The metric $g = g_{\varphi(t)}$ evolves by

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij} - \frac{2}{3}|T|^2 g_{ij} - 4T_i{}^k T_{kj},$$

where T_{ij} is the torsion tensor.

- Bryant–Xu : Short-time existence and uniqueness of solutions to Laplacian flow on compact M.
- If the flow exists for $t \in [0, \infty)$, it need not converge to a torsion-free G_2-structure as $t \to \infty$.
- Lotay–Wei: Torsion-free G_2-structures are stable under Laplacian flow.
- The flow can have singularities in finite time, i.e. $|\mathrm{Rm}|_{g_t} \to \infty$ for $t \to T < \infty$. No compact examples with finite-time singularities are presently known.

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The context of other geometric flows

Ricci flow:

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij}.$$

- Hamilton: Under volume-preserving Ricci flow, any positive Ricci curvature metric on a compact 3-manifold converges to a spherical space form as $t \to \infty$. In higher dimensions, similar results hold under positive curvature assumptions which are preserved by the flow. (Huisken, Böhm-Wilking, Brendle-Schoen)
- Without any curvature restrictions, Ricci flow on 3-manifolds has finite-time singularities. Classifying finite-time singularities is central to topological results. Thurston's Geometrization Conjecture guides us, e.g. neckpinches modeled on the self-similarly shrinking $S^2 \times \mathbb{R}$ implement connected sum decompositions of the manifold. (Perelman)

An **ancient solution** to a geometric flow is a solution which exists for $t \in (-\infty, T]$. These are *singularity models* for the flow.

We study the behavior of the flow around singularities by parabolically rescaling the flow and taking limits using a compactness theory to find ancient solutions.

- In Ricci flow, $(M, g(t))$ is rescaled by choosing a sequence of spacetime points $(x_k, t_k) \in M \times \mathbb{R}$ and $C_k = |\mathrm{Rm}(x_k, t_k)| > 0$ and considering the sequence of flows $g_k(t) = C_k g(t_k + C_k^{-1} t)$. The compactness theory requires bounded curvature and injectivity radius estimates (noncollapsing: lower bounds on volume ratio).
- In Laplacian flow, there is a compactness theory due to Lotay-Wei. Just as in Ricci flow, compactness requires bounded curvature and injectivity radius estimates. Chen showed that Laplacian flow is noncollapsing under a bound on scalar curvature, $R = -|T|^2$.

The context of other geometric flows, pt. II

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Solitons in Laplacian flow

Solitons are self-similar solutions to a geometric flow, and understanding them is the first step to understanding ancient solutions.

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A **Laplacian soliton** on a 7-manifold is a closed G_2-structure φ together with a vector field X and $\lambda \in \mathbb{R}$ satisfying

$$\begin{cases} d\varphi_0 = 0 \\ \Delta_{\varphi_0}\varphi_0 = \lambda\varphi_0 + \mathcal{L}_X\varphi_0 \end{cases}$$

Let $c(t) = (1 + \frac{2}{3}\lambda t)^{3/2}$, and let $X(t) = c(t)^{-2/3}X$. Then, $\varphi(t) = c(t)\phi_t^*\varphi_0$ is a solution to Laplacian flow, and it is self-similar. Note that $c'(t) = \lambda c(t)^{1/3}$.

$$\begin{aligned} \frac{\partial\varphi}{\partial t} &= c'(t)\phi_t^*\varphi_0 + c(t)\phi_t^*(\mathcal{L}_{X(t)}\varphi_0) = c(t)^{1/3}\phi_t^*(\lambda\varphi_0 + \mathcal{L}_X\varphi_0) \\ &= c(t)^{1/3}\phi_t^*(\Delta_{\varphi_0}\varphi_0) = c(t)^{1/3}\Delta_{\phi_t^*\varphi_0}\phi_t^*\varphi_0 = \Delta_{\varphi(t)}\varphi(t) \end{aligned}$$

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Facts about Laplacian Solitons

For a Laplacian soliton φ with constant $\lambda \in \mathbb{R}$,

$\lambda < 0$: **shrinking soliton** ("shrinker"), exists for $t \in (-\infty, 0)$, models finite-time singularities.

$\lambda = 0$: **steady soliton**, exists for $t \in (-\infty, \infty)$, models infinite-time singularities and degenerate cases of finite-time singularities.

$\lambda > 0$: **expanding soliton** ("expander"), exists for $t \in (0, \infty)$, models singularity resolution.

- Solitons with X a gradient vector field, $X = \nabla f$, are special and are known as **gradient solitons**.
- Solitons on a compact manifold are either torsion-free or expanding. Shrinking solitons must be noncompact.

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Examples of Laplacian Solitons

- Fowdar's inhomogeneous gradient shrinker with metric given by

$$g(t) = 2^{-1/2}e^{u}(dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2) + 2e^{2u}(dx_5^2 + dx_6^2) + du^2.$$

- Lauret's non-gradient shrinking homogeneous soliton on a solvmanifold.
- Steady and expanding homogeneous solitons (Lauret, Fino–Raffero, Fernández–Fino–Manero, etc.)
- Ball's inhomogeneous steady gradient solitons on $\mathbb{R}xN$, where N is some admissible 6-manifolds, e.g. a particular T^2-bundle over a hyperkähler 4-manifold.
- Haskins–Nordström: cohomogeneity-one AC gradient shrinkers, steadies, and expanders. Also, a steady soliton with exponential volume growth.

Thank you for your attention!

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