Uniqueness of Asymptotically Conical Gradient Shrinking Solitons in G_2-Laplacian Flow

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Let's recall the definition of the Laplacian flow for closed G_2 structures:

Definition

Consider the smooth family of G_2 structures on a 7-manifold M, $\{\varphi_t\} \subset \Omega^3(M)_{t\in[0,T)}$. This family flows by the Laplacian flow if

$$\begin{cases} \partial_t \varphi_t = \Delta_{\varphi_t} \varphi_t \\ d\varphi_t = 0 \end{cases}$$

where $\Delta_\varphi \varphi = dd^*\varphi + d^*d\varphi$ is the Hodge Laplacian of φ with respect to the metric g determined by φ.

Metric Structures

Each G_2 structure determines a corresponding metric via the non-linear equation

$$g(u,v)\mathrm{vol}_g = \frac{1}{6}(u \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge \varphi$$

Under the flow, the metrics g_t corresponding to structures φ_t evolve by a "perturbed" Ricci flow:

$$\frac{\partial g_t}{\partial t} = -2\mathrm{Ric}(g_t) + \text{terms quadratic in torsion of } \varphi_t$$

This evolution equation gives hope for importing strategies from Ricci flow, but also gives rise to profoundly different behavior.

Gradient Shrinking Solitons

A gradient shrinking G_2 soliton φ and its corresponding metric g respectively satisfy

$$\Delta_{\varphi}\varphi = -\frac{3}{2}\varphi + \mathcal{L}_{\nabla f}\varphi, \quad -R_{ij} - \frac{1}{3}|T|^2 g_{ij} - 2T_i^k T_{kj} = -\frac{1}{2}g_{ij} + \nabla_i \nabla_j f.$$

Let Ψ_s be the flow of ∇f. Define $\varphi(t) = (-t)^{3/2}\Psi^*_{-\log(-t)}\varphi_0$ for $t < 0$.

$$\begin{aligned}
\frac{\partial \varphi}{\partial t} &= -\frac{3}{2}(-t)^{\frac{1}{2}}(\Psi_{-\log(-t)})^*\varphi_0 + (-t)^{\frac{3}{2}}(\Psi_{-\log(-t)})^*(\mathcal{L}_{\nabla f}\varphi_0)\frac{1}{t} \\
&= (-t)^{\frac{1}{2}}(\Psi_{-\log(-t)})^*\left(-\frac{3}{2}\varphi_0 + \mathcal{L}_{\nabla f}\varphi_0\right) \\
&= (-t)^{\frac{1}{2}}(\Psi_{-\log(-t)})^*(\Delta_{\varphi_0}\varphi_0) \\
&= \left((-t)^{\frac{3}{2}}\right)^{\frac{1}{3}}(\Delta_{(\Psi_{-\log(-t)})^*\varphi_0}(\Psi_{-\log(-t)})^*\varphi_0) \\
&= \Delta_{\varphi(t)}\varphi(t),
\end{aligned}$$

Asymptotically conical shrinkers

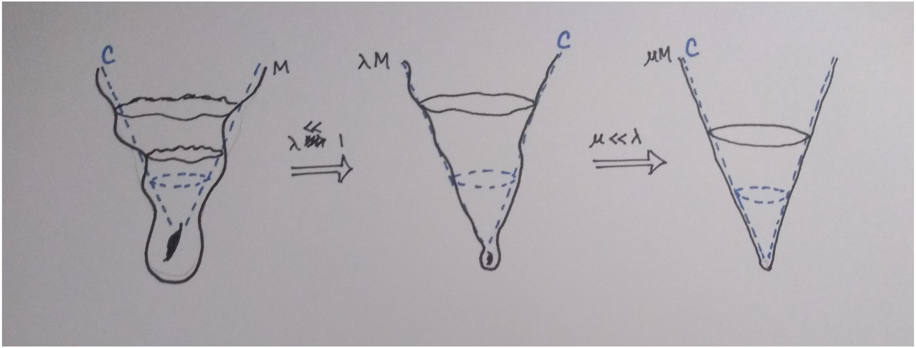
Let (M, φ) be a 7-dimensional manifold with a closed G_2-structure.

- An *end* of M: unbounded connected component V of $M \setminus K$, K compact.
- A *closed G_2-cone*: a 7-manifold (C, φ_C) with closed G_2 structure such that $C = (0, \infty) \times \Sigma^6$ (where (Σ^6, g_Σ) is a closed Riemannian 6-manifold) and has induced metric $g = dr^2 + r^2 g_\Sigma$.
- The dilation map ρ_λ is the map $(r, \sigma) \in (\mathcal{R}, \infty) \times \Sigma \mapsto (\lambda r, \sigma) \in (\mathcal{R}, \infty) \times \Sigma$

Definition

Let V be an end of M. We say that (M, φ) is *asymptotic to the G_2-cone (C, φ_C) along V* if, for some $\mathcal{R} > 0$, there is a diffeomorphism $\Phi : (\mathcal{R}, \infty) \times \Sigma \to V$ such that $\lambda^{-3} \rho_\lambda^* \Phi^* \varphi \to \varphi_C$ as $\lambda \to \infty$ in $C^3_{\text{loc}}(C, g_C)$.

Visualization of AC condition



Haskins-Nordström Examples

Theorem (Haskins-Nordström)

There exists a complete AC shrinker with rate -1 *on* $\Lambda^2_- S^4$ *and on* $\Lambda^2_- \mathbb{C}P^2$

Theorem (Haskins-Nordström)

Let G *be* $SU(3)$ *or* $Sp(2)$*. For every closed* G*-invariant* G_2*-cone* C *there exists a unique* G*-invariant shrinking AC end asymptotic to* C*.*

The latter theorem yields continuous families of AC shrinker ends (1- and 2-dimensional for $Sp(2)$ and $SU(3)$ respectively).

Main Result

Theorem

Let (M_1, φ_1) and (M_2, φ_2) be two 7-manifolds with closed G_2 structure asymptotic to the G_2 cone (C, φ_C) along the ends $V_1 \subset M_1$ and $V_2 \subset M_2$, respectively. Then there exist ends $W_1 \subset V_1$ and $W_2 \subset V_2$ and a diffeomorphism $\Psi : W_1 \to W_2$ such that $\Psi^* \varphi_2 = \varphi_1$.

Analogous to the result for gradient Ricci shrinkers proved by Kotschwar-Wang.

How to prove? Static problem $\implies$ backwards uniqueness for associated parabolic problem

Strategy of proof

Follow the strategy of Kotschwar-Wang. To φ_1 and φ_2 are associated flows $\varphi_1(t)$, $\varphi_2(t)$ for $t < 0$ which limit to the conical G_2 structure φ_C as $t \to 0$. Prove backwards uniqueness for $\varphi_1(t) - \varphi_2(t)$, which (heuristically) satisfies a heat-type equation.
Model problem:

Theorem (Escauriaza-Seregin-Šverák)

Let u be a smooth function on $Q_{R,T} = (\mathbb{R}^n \setminus B_R(0)) \times [0, T]$ which satisfies

$$|\partial_t u + \Delta u| \leq N(|u| + |\nabla u|), \quad u(x,0) = 0, \quad \textit{and} \quad |u(x,t)| \leq Ne^{N|x|^2}.$$

Then $u \equiv 0$.

Carleman Estimates

Carleman estimate to prove unique continuation for sufficiently decaying function:

Lemma

Let $Q_{R,T}$ be as before. There is a constant $\alpha^*(R,n)$ such that for all $u \in C_c^\infty(Q_{R,T})$ with $u(\cdot, 0) \equiv 0$ and $\alpha \geq \alpha*$,

$$||e^{\alpha(T-t)(|x|-R)+|x|^2} u||_{L^2(Q_{R,T})} + ||e^{\alpha(T-t)(|x|-R)+|x|^2} \nabla u||_{L^2(Q_{R,T})}$$
$$\leq ||e^{\alpha(T-t)(|x|-R)+|x|^2} (\partial_t + \Delta) u||_{L^2(Q_{R,T})} + ||e^{|x|^2} \nabla u(\cdot, T)||_{L^2(\mathbb{R}^n \setminus B_R)}.$$

Carleman estimate to prove exponential decay:

Lemma

Let $\sigma_a = (t+a)e^{-(t+a)/3}$. *There is a constant* $C(n)$ *such that for all* $u \in C_c^\infty(\mathbb{R}^n \times [0,1))$ *with* $u(\cdot, 0) \equiv 0$, $y \in \mathbb{R}^n$, $a \in (0,1)$ *and* $\alpha \geq 0$,

$$\sqrt{\alpha}||\sigma_a^{-\alpha-1/2} e^{-\frac{|x-y|^2}{8(t+a)}} u||_{L^2(\mathbb{R}^n\times[0,1))} + ||\sigma_a^{-\alpha} e^{-\frac{|x-y|^2}{8(t+a)}} \nabla u||_{L^2(\mathbb{R}^n\times[0,1))} \leq C||\sigma_a^{-\alpha} e^{-\frac{|x-y|^2}{8(t+a)}} (\partial_t + \Delta)u||_{L^2(\mathbb{R}^n\times[0,1))}$$

Note that both inequalities are valid only for compactly supported functions.

Sketch of Proof

Assume that $u \in C^\infty(Q_{R,T}(\mathbb{R}^n \setminus B_R(0)))$ satisfies $|\partial_t u + \Delta u| \leq \epsilon(|u| + |\nabla u|)$. Smoothly cut off u to obtain u_r supported in $(B_r(0) \setminus B_R(0)) \times [0, T]$. Then

$$\sqrt{\alpha}||\sigma_a^{-\alpha-1/2} e^{-\frac{|x-y|^2}{8(t+a)}} u_r||_{L^2(\mathbb{R}^n\times[0,1))} + ||\sigma_a^{-\alpha} e^{-\frac{|x-y|^2}{8(t+a)}} \nabla u_r||_{L^2(\mathbb{R}^n\times[0,1))}$$
$$\leq C||\sigma_a^{-\alpha} e^{-\frac{|x-y|^2}{8(t+a)}} (\partial_t + \Delta)u_r||_{L^2(\mathbb{R}^n\times[0,1))}$$
$$\leq C\epsilon||\sigma_a^{-\alpha} e^{-\frac{|x-y|^2}{8(t+a)}} (u_r + \nabla u_r)||_{L^2(\mathbb{R}^n\times[0,1))}$$
$$+ \text{ remainder terms involving derivatives of the cutoff functions}$$

Prove that remainder terms become small as $r \to \infty$.

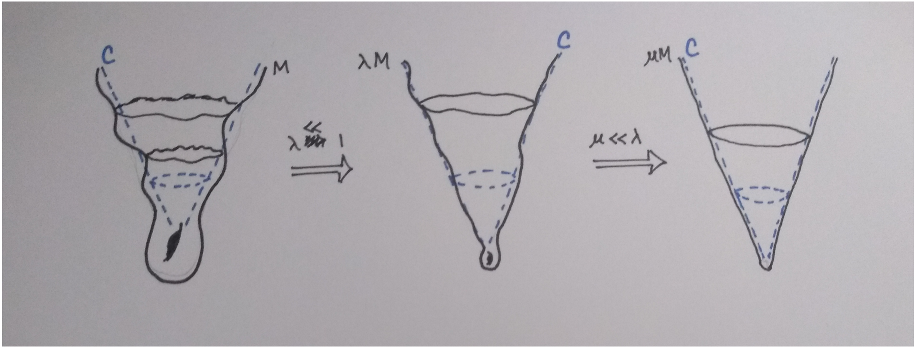
Obstacles to generalization

Gauge selection. Given φ_1, φ_2, associated Laplacian flows must be modified by diffeomorphisms so that

- limit structures at $t = 0$ coincide with φ_C defined on the asymptotic cone.
- time-dependent potential function $f(x, t)$ is comparable to radial distance on the limit cone (C, φ_C).
- time-independent curvature estimates

Nonetheless, $\varphi_1(t) - \varphi_2(t)$ does *not* satisfy strictly parabolic equation: general issue for backwards uniqueness.

Intuition about gauge selection



$$\varphi(t) = (-t)^{3/2}\Psi^*_{-\log(-t)}\varphi_0$$

ODE-PDE System

PDE part: a tensor field $\mathbf{X}$ consisting of $\nabla^{(k)}(T_1 - T_2)$, $\nabla^{(k)}(\mathrm{Rm}_1 - \mathrm{Rm}_2)$ satisfies strictly parabolic equation, up to lower order terms in $\nabla^{(k)}(\varphi_1 - \varphi_2), \nabla^{(k)}(g_1 - g_2)$, and their derivatives.

ODE part: a tensor field $\mathbf{Y}$ consisting of $\nabla^{(k)}(\varphi_1 - \varphi_2), \nabla^{(k)}(g_1 - g_2)$, and their derivatives satisfy ODE type inequality involving $|\mathbf{X}|$ and $|\mathbf{Y}|$.

$$\left|\frac{\partial \mathbf{Y}}{\partial t}\right| \leq C(|\mathbf{X}| + |\nabla \mathbf{X}|) + \frac{C}{r_c}|\mathbf{Y}|,$$

$$\left|\frac{\partial \mathbf{X}}{\partial t} + \Delta \mathbf{X}\right| \leq \frac{C}{r_c}(|\mathbf{X}| + |\nabla \mathbf{X}| + |\mathbf{Y}|),$$

where r_c is radial distance on (C, φ_C). Estimates come from evolution equations combined with curvature decay estimates.

Carleman Estimates, Revisited

Integration by parts holds $\implies$ can obtain essentially the same Carleman estimates on the PDE part. Compatible ODE estimates hold as well (see Kotschwar-Wang). In Ricci soliton case, can follow the same basic strategy as sketched earlier.

In the case of G_2 solitons, there are unique difficulties ultimately arising from the failure of the following Ricci soliton identities to hold.

$$\mathrm{Ric}(g) + \nabla\nabla f = \frac{1}{2}g, \quad R + |\nabla f|^2 = f$$

Instead, for a G_2 Laplacian shrinker:

$$-R_{ij} - \frac{1}{3}|T|^2 g_{ij} - 2T_i{}^k T_{kj} = -\frac{1}{2}g_{ij} + \nabla_i\nabla_j f.$$

and

$$\begin{aligned}&\nabla_i(R + 3|\nabla f|^2 - 3f)\\ &\qquad = -2|T|^2\nabla_i f - 12T_i{}^l T_{lk}\nabla_k f + \nabla_j(2|T|^2 g_{ij} + 12T_i{}^k T_{kj}).\end{aligned}$$

$$\implies R + 3|\nabla f|^2 - 3f = O(\log r_c)$$

Carleman estimates are weighted-L^2 estimates, with weights

$$e^{\alpha(T-t)(|x|-R)+|x|^2}, \text{ and } e^{-\frac{|x-y|^2}{8(t+a)}}.$$

On $Q_{R,T}$, consistent notion of length on time-slices $(\mathbb{R}^n \setminus B_R) \times \{t\}$: $|x|$.
For each time t, we must choose a notion of radial length on $(M, \varphi(t))$:

$$h(x,t) := 2\sqrt{t\Psi_t^* f(x)} \to r_c(x), r \to 0.$$

(Soliton identity $\implies$ f is "like" r^2.) New weights, $W_1(h(x,t),t)$ and $W_2(h(x,t),t)$ give analogous Carleman estimates(*).

Consequences of the Soliton Identities

Since the Carleman estimates arise from divergence identities in vector calculus (Rellich-Nečas), we will have to differentiate the weights $W(x, t)$, which will give us terms involving the derivatives $\partial_t h$, ∇h, and Δh. These all involve log-type growth.

Example/Heuristics: $f - r_c^2/4 = O(r_c^{-2})$ in the Ricci case. In ours, $f - r_c^2/4 = O(\log r_c)$.

Upshot: Carleman estimate 1 is more or less unaffected. Carleman estimate 2 changes drastically.

Carleman Estimate 2

We obtain an estimate more or less analogous to

$$\sqrt{\alpha}||\sigma_a^{-\alpha-1/2}e^{-\frac{|x-y|^2}{8(t+a)}}u_\rho||_{L^2(Q_{R,T})} + ||\sigma_a^{-\alpha}e^{-\frac{|x-y|^2}{8(t+a)}}\nabla u_\rho||_{L^2(Q_{R,T})}$$
$$\leq C||\sigma_a^{-\alpha}e^{-\frac{|x-y|^2}{8(t+a)}}(\partial_t+\Delta)u_\rho||_{L^2(Q_{R,T})}+||\sigma_a^{-\alpha-1/2}e^{-\frac{|x-y|^2}{8(t+a)}}(\log r)u_\rho||_{L^2(Q_{R,T})}$$

Choose α judiciously to absorb the log term on an annulus $A(c\rho, C\rho)$. Obtain decay estimate of the form

$$|||\mathbf{X}| + |\nabla \mathbf{X}| + |\mathbf{Y}|||_{L^2(A(c\rho,C\rho))} \leq N(\rho)e^{-\frac{\rho^2}{Cs}}.$$

Then, by our choice of α, we can prove that $N(\rho)$ grows slowly enough to increase C slightly and obtain

$$|||\mathbf{X}| + |\nabla \mathbf{X}| + |\mathbf{Y}|||_{L^2(A(c\rho,C\rho))} \leq N(n)e^{-\frac{\rho^2}{Cs}}.$$

A General Framework

The central problem is geometric: however, solution is analytic. In principle, techniques are adaptable to flows satisfying

$$\frac{\partial g_t}{\partial t} = -2\mathrm{Ric}(g_t) + \text{quadratic error}.$$

Possible application to other unique continuation problems.

Thank you for your attention!

References

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