Uniqueness of Asymptotically Conical Gradient Shrinking Solitons in G_2-Laplacian Flow

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Let's recall the definition of the Laplacian flow for closed G_2 structures:

Definition

Consider the smooth family of G_2 structures on a 7-manifold M, $\{\varphi_t\} \subset \Omega^3(M)_{t\in[0,T)}$. This family flows by the Laplacian flow if

$$\begin{aligned} \partial_t \varphi_t &= \Delta_{\varphi_t} \varphi_t \\ d\varphi_t &= 0 \end{aligned}$$

where $\Delta_\varphi \varphi = dd^*\varphi + d^*d\varphi$ is the Hodge Laplacian of φ with respect to the metric g determined by φ.

Each G_2 structure determines a corresponding metric via the non-linear equation

$$g(u, v)\mathrm{vol}_g = \frac{1}{6}(u \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge \varphi$$

Under the flow, the metrics g_t corresponding to structures φ_t evolve by a "perturbed" Ricci flow:

$$\frac{\partial g_t}{\partial t} = \quad 2\mathrm{Ric}(g_t) + \text{terms quadratic in torsion of } \varphi_t$$

This evolution equation gives hope for importing strategies from Ricci flow, but also gives rise to profoundly different behavior.

Gradient Shrinking Solitons

A gradient shrinking G_2 soliton φ and its corresponding metric g respectively satisfy

$$\Delta_\varphi \varphi = -\frac{3}{2}\varphi + L_{\nabla f}\varphi; \qquad R_{ij} - \frac{1}{3}|T|^2 g_{ij} - 2T_i^{\ k}T_{kj} = -\frac{1}{2}g_{ij} + \nabla_i\nabla_j f.$$

Let Ψ_s be the flow of ∇f. Define $\varphi(t) = (-t)^{3/2}\Psi^*_{-\log(-t)}\varphi_0$ for $t < 0$.

$$\begin{aligned}
\frac{\partial\varphi}{\partial t} &= -\frac{3}{2}(-t)^{\frac{1}{2}}(\Psi_{-\log(-t)})^*\varphi_0 + (-t)^{\frac{3}{2}}(\Psi_{-\log(-t)})^*(L_{\nabla f}\varphi_0)\frac{1}{t} \\
&= (-t)^{\frac{1}{2}}(\Psi_{-\log(-t)})^*\left(-\frac{3}{2}\varphi_0 + L_{\nabla f}\varphi_0\right) \\
&= (-t)^{\frac{1}{2}}(\Psi_{-\log(-t)})^*(\Delta_{\varphi_0}\varphi_0) \\
&= (-t)^{\frac{3}{2}\cdot\frac{1}{3}}(\Delta_{(\Psi_{-\log(-t)})^*\varphi_0}(\Psi_{-\log(-t)})^*\varphi_0) \\
&= \Delta_{\varphi(t)}\varphi(t);
\end{aligned}$$

Let $(M; \varphi)$ be a 7-dimensional manifold with a closed G_2-structure.

- An *end* of M: unbounded connected component V of $M \setminus K$, K compact.
- A *closed G_2-cone*: a 7-manifold $(C; \varphi_C)$ with closed G_2 structure such that $C = (0; \infty) \quad \Sigma^6$ (where $(\Sigma^6; g_\Sigma)$ is a closed Riemannian 6-manifold) and has induced metric $g = dr^2 + r^2 g_\Sigma$.
- The dilation map $\quad$ is the map $(r; \ ) \in (R; \infty) \quad \Sigma \mapsto (\ r; \ ) \in (R; \infty) \quad \Sigma$

DEFINITION

Let V be an end of M. We say that $(M; \varphi)$ is *asymptotic to the G_2-cone $(C; \varphi_C)$ along V* if, for some $R > 0$, there is a diffeomorphism $\Phi : (R; \infty) \quad \Sigma \to V$ such that $\ ^3 \ \Phi \ \varphi \to \varphi_C$ as $\ \to \infty$ in $C^3_{\mathrm{loc}}(C; g_C)$.

Visualization of AC condition

Haskins-Nordström Examples

Theorem (Haskins-Nordström)

There exists a complete AC shrinker with rate 1 on $\Lambda^2 S^4$ and on $\Lambda^2 CP^2$

Theorem (Haskins-Nordström)

Let G be $SU(3)$ or $Sp(2)$. For every closed G-invariant G_2-cone C there exists a unique G-invariant shrinking AC end asymptotic to C.

The latter theorem yields continuous families of AC shrinker ends (1- and 2-dimensional for $Sp(2)$ and $SU(3)$ respectively).

Main Result

Theorem

Let (M_1, φ_1) and (M_2, φ_2) be two 7-manifolds with closed G_2 structure asymptotic to the G_2 cone (C, φ_C) along the ends $V_1 \subset M_1$ and $V_2 \subset M_2$, respectively. Then there exist ends $W_1 \subset V_1$ and $W_2 \subset V_2$ and a diffeomorphism $\Psi : W_1 \to W_2$ such that $\Psi^* \varphi_2 = \varphi_1$.

Analogous to the result for gradient Ricci shrinkers proved by Kotschwar-Wang.

How to prove? Static problem $\Longrightarrow$ backwards uniqueness for associated parabolic problem

Follow the strategy of Kotschwar-Wang. To φ_1 and φ_2 are associated flows $\varphi_1(t)$, $\varphi_2(t)$ for $t < 0$ which limit to the conical G_2 structure φ_C as $t \to 0$. Prove backwards uniqueness for $\varphi_1(t) - \varphi_2(t)$, which (heuristically) satisfies a heat-type equation.
Model problem:

Theorem (Escauriaza-Seregin-Šverák)

Let u be a smooth function on $Q_{R,T} = (\mathbb{R}^n \setminus B_R(0)) \times [0, T]$ which satisfies

$$|\partial_t u + \Delta u| \leq N(|u| + |\nabla u|); \quad u(x, 0) = 0; \quad \text{and} \quad |u(x, t)| \leq N e^{N|x|^2}.$$

Then $u \equiv 0$.

Carleman Estimates

Carleman estimate to prove unique continuation for sufficiently decaying function:

Lemma

Let $Q_{R;T}$ be as before. There is a constant $(R;n)$ such that for all $u \in C_c^{\infty}(Q_{R;T})$ with $u(\cdot;0) \quad 0$ and ,

$$\|e^{\ (T-t)(|x|-R)+|x|^2} u\|_{L^2(Q_{R;T})} + \|e^{\ (T-t)(|x|-R)+|x|^2}\, \nabla u\|_{L^2(Q_{R;T})}$$
$$\quad \|e^{\ (T-t)(|x|-R)+|x|^2}(\partial_t + \Delta)u\|_{L^2(Q_{R;T})} + \|e^{|x|^2}\, \nabla u(\cdot;T)\|_{L^2(\mathbb{R}^n \setminus B_R)}.$$

Carleman estimate to prove exponential decay:

Lemma

Let $\sigma_a = (t+a)e^{-(t+a)/3}$. *There is a constant* $C(n)$ *such that for all* $u \in C_c^\infty(\mathbb{R}^n \times [0,\infty))$ *with* $u(\cdot, 0) \equiv 0$, $y \in \mathbb{R}^n$, $a \in (0,1)$ *and* $\alpha \geq 0$,

$$\sqrt{\alpha}\,\|\sigma_a^{-\alpha-1/2} e^{-\frac{|x-y|^2}{8(t+a)}} u\|_{L^2(\mathbb{R}^n \times [0,\infty))} + \|\sigma_a^{-\alpha} e^{-\frac{|x-y|^2}{8(t+a)}} \nabla u\|_{L^2(\mathbb{R}^n \times [0,\infty))} \leq C\|\sigma_a^{-\alpha} e^{-\frac{|x-y|^2}{8(t+a)}} (\partial_t + \Delta)u\|_{L^2(\mathbb{R}^n \times [0,\infty))}$$

Note that both inequalities are valid only for compactly supported functions.

Assume that $u \in C^\infty(Q_{R;T}(\mathbb{R}^n \setminus B_R(0)))$ satisfies $|\partial_t u + \Delta u| \leq (|u| + |\nabla u|)$. Smoothly cut off u to obtain u_r supported in $(B_r(0) \setminus B_R(0)) \times [0;T]$. Then

$$\sqrt{\ }\,\|\,{}_a^{-1/2} e^{\frac{|x-y|^2}{8(t+a)}} u_r\|_{L^2(\mathbb{R}^n \times [0;1))} + \|\,{}_a\, e^{\frac{|x-y|^2}{8(t+a)}} \nabla u_r\|_{L^2(\mathbb{R}^n \times [0;1))}$$

$$\leq C\|\,{}_a\, e^{\frac{|x-y|^2}{8(t+a)}} (\partial_t + \Delta) u_r\|_{L^2(\mathbb{R}^n \times [0;1))}$$

$$\leq C\,\|\,{}_a\, e^{\frac{|x-y|^2}{8(t+a)}} (u_r + \nabla u_r)\|_{L^2(\mathbb{R}^n \times [0;1))}$$

$+$ remainder terms involving derivatives of the cutoff functions

Prove that remainder terms become small as $r \to \infty$.

Obstacles to generalization

Gauge selection. Given $'_1$, $'_2$, associated Laplacian flows must be modified by diffeomorphisms so that

- limit structures at $t = 0$ coincide with $'_C$ defined on the asymptotic cone.
- time-dependent potential function $f(x; t)$ is comparable to radial distance on the limit cone $(C; '_C)$.
- time-independent curvature estimates

Nonetheless, $'_1(t) \quad '_2(t)$ does *not* satisfy strictly parabolic equation: general issue for backwards uniqueness.

Intuition about gauge selection

$$' (t) = (\quad t)^{3=2} \quad \log(\quad t)' \; 0$$

ODE-PDE System

PDE part: a tensor field X consisting of $\nabla^{(k)}(T_1 - T_2)$, $\nabla^{(k)}(Rm_1 - Rm_2)$ satisfies strictly parabolic equation, up to lower order terms in $\nabla^{(k)}(\varphi_1 - \varphi_2); \nabla^{(k)}(g_1 - g_2)$, and their derivatives.

ODE part: a tensor field Y consisting of $\nabla^{(k)}(\varphi_1 - \varphi_2); \nabla^{(k)}(g_1 - g_2)$, and their derivatives satisfy ODE type inequality involving $|X|$ and $|Y|$.

$$\left|\frac{\partial Y}{\partial t}\right| \leq C(|X| + |\nabla X|) + \frac{C}{r_c}|Y|;$$

$$\left|\frac{\partial X}{\partial t} + \Delta X\right| \leq \frac{C}{r_c}(|X| + |\nabla X| + |Y|);$$

where r_c is radial distance on (C, φ_C). Estimates come from evolution equations combined with curvature decay estimates.

Carleman Estimates, Revisited

Integration by parts holds $\Rightarrow$ can obtain essentially the same Carleman estimates on the PDE part. Compatible ODE estimates hold as well (see Kotschwar-Wang). In Ricci soliton case, can follow the same basic strategy as sketched earlier.

In the case of G_2 solitons, there are unique difficulties ultimately arising from the failure of the following Ricci soliton identities to hold.

$$\mathrm{Ric}(g) + \nabla\nabla f = \frac{1}{2}g; \quad R + |\nabla f|^2 = f$$

Metric identities for Laplacian Shrinkers

Instead, for aG_2 Laplacian shrinker:

$$R_{ij} \quad \frac{1}{3} jTj^2 g_{ij} \quad 2T_i{}^k T_{kj} = \quad \frac{1}{2} g_{ij} + r_i r_j f:$$

and

$$r_i (R + 3 jr f j^2 \quad 3f)$$

$$= \quad {\color{red} 2jTj^2 r_i f \quad 12T_i{}^l T_{lk} r_k f} + r_j (2jTj^2 g_{ij} + 12 T_i{}^k T_{kj}):$$

$$=) \quad R + 3 jr f j^2 \quad 3f = O(\log r_c)$$

Notions of Length

Carleman estimates are weighted L^2-estimates, with weights

$$e^{-(T-t)(|x|-R)+|x|^2}; \text{ and } e^{-\frac{|x-y|^2}{8(t+a)}}.$$

On $Q_{R,T}$, consistent notion of length on time-slices $(\mathbb{R}^n \setminus B_R) \times \{t\}$: $|x|$.
For each time t, we must choose a notion of radial length on $M(\Phi(t))$:

$$h(x,t) := 2\sqrt{-t\,{}_t f(x)} \to r_c(x),\ r \to 0.$$

(Soliton identity $\Rightarrow$ f is "like" r^2.) New weights, $W_1(h(x,t),t)$ and $W_2(h(x,t),t)$ give analogous Carleman estimates(*).

Consequences of the Soliton Identities

Since the Carleman estimates arise from divergence identities in vector calculus (Rellich-Nečas), we will have to differentiate the weights $W(x,t)$, which will give us terms involving the derivatives $\partial_t h$, ∇h, and Δh. These all involve log-type growth.

Example/Heuristics: $f - r_c^2/4 = O(r_c^{-2})$ in the Ricci case. In ours, $f - r_c^2/4 = O(\log r_c)$.

Upshot: Carleman estimate 1 is more or less unaffected. Carleman estimate 2 changes drastically.

Carleman Estimate 2

We obtain an estimate more or less analogous to

$$\sqrt{\ }\,\| \ \ a \ ^{1/2} e^{\frac{|x \ \ y|^2}{8(t+a)}} u \|_{L^2(Q_{R;T})} + \| \ \ a \ \ e^{\frac{|x \ \ y|^2}{8(t+a)}} \nabla u \|_{L^2(Q_{R;T})}$$

$$C \| \ \ a \ \ e^{\frac{|x \ \ y|^2}{8(t+a)}} (\partial_t + \ ) \ u \|_{L^2(Q_{R;T})} + \| \ \ a \ ^{1/2} e^{\frac{|x \ \ y|^2}{8(t+a)}} (\log r) u \|_{L^2(Q_{R;T})}$$

Choose judiciously to absorb the log term on an annulus $A(c \ ; C \ )$.
Obtain decay estimate of the form

$$\| \, |X| + |\nabla X| + |Y| \, \|_{L^2(A(c \ ; C \ ))} \quad N(\ ) e^{\frac{\ ^2}{Cs}} :$$

Then, by our choice of , we can prove that $N(\ )$ grows slowly enough to increase C slightly and obtain

$$\| \, |X| + |\nabla X| + |Y| \, \|_{L^2(A(c \ ; C \ ))} \quad N(n) e^{\frac{\ ^2}{Cs}} :$$

A General Framework

The central problem is geometric: however, solution is analytic. In principle, techniques are adaptable to flows satisfying

$$\frac{\partial g_t}{\partial t} = -2\mathrm{Ric}(g_t) + \text{quadratic error}.$$

Possible application to other unique continuation problems.

Thank you for your attention!

References

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