A prolegomena to finding higher ~~homotopy~~ topology in G_2-moduli spaces

Diarmuid Crowley (University of Melbourne)

Geometry, topology & singular spaces

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Pages 17-23 edited after talk

Errors re ν_2 & ν_3 on p 23 corrected

jt w S. Goette & J. Nordström

- We last met in Sept 2018
  Stony Brook & New York

- Early report on work in progress:

"Conjecture" = theorem we think we have proved

"Conjecture" – usual use of the term

7-mflds & G_2-structures

- M^7 closed, smooth, oriented 7-mfld

- φ a G_2-structure on M
  - $\varphi \in \Omega^3_+(M)$ geometric
  - $\varphi = Q \subseteq FM$ topological

    FM (oriented) frame bundle

    Q a principal G_2 sub-bundle

    $G_2 \subseteq SO_7 \subseteq SL_7(\mathbb{R})$

Spaces of G_2-structures

- $\mathcal{G}_2^*(M)$ = space of $*$-G_2 strs on M

$*$ = $+$ positive

cc co-closed

c closed

tf torsion free

may drop M

$$\begin{tikzcd}
 & \mathcal{G}_2^{cc} \arrow[dr] \arrow[r, phantom, "\simeq"] & \\
\mathcal{G}_2^{tf} \arrow[ur] \arrow[dr] & & \mathcal{G}_2^{+} \\
 & \mathcal{G}_2^{c} \arrow[ur] &
\end{tikzcd}$$

C-N (2015)

NB differs from Scaduto talk

Spaces of diffeomorphisms:

$f : M \to M$

- $\mathrm{Diff}(M)$ – space of o.p. diffeos of M
- $\mathrm{Diff}(M)_f$ – component f ; $\mathrm{Diff}(M)_0 = \mathrm{Diff}(M)_{id}$
- $\mathrm{Diff}_*(M)$ – $f(*) = *$

$\cup\!\!\mid$

- $\mathrm{Diff}_{\underline{v}}(M)$ – $Tf(\underline{v}) = \underline{v}$ $\left\{ \begin{array}{l} \underline{v} \text{ a basis of} \\ T_*M \end{array} \right.$

- $\mathrm{Diff}_*(M) \to \mathrm{Diff}(M) \to M$

- $MCG(M) := \pi_0 \mathrm{Diff}(M) = \mathrm{Diff}(M)/\mathrm{Diff}(M)_0$

Moduli spaces:

- $\bar{\mathcal{G}}_2^*(M)_0 := \mathcal{G}_2^*(M)/\mathrm{Diff}(M)_0$
- $\bar{\mathcal{G}}_2^*(M) := \mathcal{G}_2^*(M)/\mathrm{Diff}(M)$

$MCG(M)$ acts

Thm (Joyce) $\bar{\mathcal{G}}_2^{-tf}(M)_0$ is a non-singular manifold of dim $= b_3(M)$

Goal: say something about $\pi_* \bar{\mathcal{G}}_2^*(M)$

For π_0: see ν & $\bar{\nu}$

What about π_i $i>0$ (or H_i etc)

The ν-approach:

build here

$$\begin{tikzcd}
\pi_i \mathcal{G}_2^{tf}(M) \arrow[r] \arrow[d] & \pi_i \mathcal{G}_2^{+}(M) \arrow[r] \arrow[d] & \Omega_i \arrow[d] \\
\pi_i \bar{\mathcal{G}}_2^{tf}(M) \arrow[r] & \pi_i \bar{\mathcal{G}}_2^{+}(M) \arrow[r, "\nu"] & \Omega_i = \Omega_{8+i}^{\mathrm{spin}_{7,1}, \mathcal{G}_2}
\end{tikzcd}$$

$\pi_i \mathcal{G}_2^{tf}(M) \circlearrowleft \mathrm{Diff}(M)$, $\pi_i \mathcal{G}_2^{+}(M) \circlearrowleft \mathrm{Diff}(M)$, $i = 0$, $\Omega_i \cong \mathbb{Z}$

detect here

$\nu = \nu_0$: $\pi_0 \mathcal{G}_2^{+}(M) / \pi_0 \mathrm{Diff}(M) \xrightarrow{\cong} \pi_0 \bar{\mathcal{G}}_2(M) \cong \mathbb{Z}/48$ quotient

hope injective

ν_i: $\pi_i \mathcal{G}_2^{+}(M) / \pi_i \mathrm{Diff}(M) \longrightarrow \pi_i \bar{\mathcal{G}}_2(M)$

Aspects of the approach:

1) Topology of $\mathcal{G}_2^*(M)$

2) Topology of $\mathrm{Diff}(M)$ or $\mathrm{BDiff}(M)$

3) Singularities in $\mathcal{G}_2^*(M) \to \mathcal{G}_2^*(M)/\mathrm{Diff}(M)$

This talk:

- Preliminaries on (1) for $\mathcal{G}_2^+(M)$ & (2)
- Re (3) help wanted

Aside: homotopy quotients & universal families

Defn: let G act on X

$$X /\!/ G := X \times_G EG \qquad \text{(Borel construct}^{\underline{n}}\text{)}$$

htpy equivalence if the action is free

$\exists$. map $X /\!/ G \longrightarrow X/G$

$$[x, e] \longmapsto [x]$$

- fibration sequence

$$X \longrightarrow X /\!/ G \longrightarrow BG$$

classifying space for principal G-bundles

Aside cbel:

observer moduli space $\rightarrow$

marked point $\rightarrow$

$$\begin{tikzcd}
\overline{\mathcal{G}}^*_{2,\underline{v}}(M) \arrow[d] & \mathcal{G}_2^*(M)/\!/\mathrm{Diff}_{\underline{v}}(M) \arrow[l, "\sim"'] \arrow[d] \\
\overline{\mathcal{G}}^*_{2,*}(M) \arrow[d] & \mathcal{G}_2^*(M)/\!/\mathrm{Diff}(M)_* \arrow[l] \arrow[r, equal, "?"] & B\mathcal{G}_2^*(M) \\
\overline{\mathcal{G}}_2(M)_0 \arrow[r] & \overline{\mathcal{G}}_2(M) & & & M \arrow[d] \\
& & \mathcal{G}_2^*(M) \arrow[r] & B\mathcal{G}_2^*(M) \arrow[r] & B\mathrm{Diff}_*(M) \arrow[d] \\
& & & & B\mathrm{Diff}(M)
\end{tikzcd}$$

universal family of $\mathcal{G}_2^*$-sols on M?

universal M-family

Topology of $\mathcal{G}_2^+(M)$

g a Riem. metric

Consider $\mathcal{G}_2^+(M,g) \overset{i}{\hookrightarrow} \mathcal{G}_2(M) \to R(M) \simeq *$

Then ① i is a homotopy equivalence

② $\mathcal{G}_2^+(M,g) \cong \Gamma(PSM) = \{[S]\}$

$PSM = SM/\mathbb{Z}_2$, SM unit spinor bundle for any spin structure on (M,g)

$$\begin{tikzcd} \mathbb{R}P^7 \arrow[r] & PSM \arrow[r] \arrow[l, bend right] & M \arrow[l, bend right, "{[S]}"'] \end{tikzcd}$$

Topology of $\Gamma(PSM)$: π_0

spin sts

$\mathcal{G}_2$-sts on a given spin str

$$\pi_0 \Gamma(PSM) \cong H^1(M;\mathbb{Z}/2) \times H^7(M;\mathbb{Z})$$

$\mathbb{Z}/2\,\mathbb{Z}$

$\pi_0 \tilde{\mathcal{G}}_2^+(M)$

input for ν

Proof sketch:

$$\begin{array}{c} \mathbb{RP}^7 \\ \downarrow \\ PSM \\ \downarrow \\ M \end{array} \qquad [s_0],[s_1]$$

$$\mathrm{Ob}([s_0],[s_1]) \in H^1(M;\pi_1(\mathbb{RP}^7)) = \mathbb{Z}/2$$

$$\in H^7(M;\pi_7(\mathbb{RP}^7)) = \mathbb{Z}$$

Topology of $\Gamma(PSM)$ π_1:

$\pi_1 \mathrm{Spin}(M) \cong \mathbb{Z}_2$ $\quad \mathbb{Z}_2$

$\cong \nearrow j_1$

Conjecture: $\pi_1 \Gamma(PSM) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \underline{\underline{H_1(M;\mathbb{Z})}}$

where $\exists$ space $\mathrm{Spin}(M)$ of spin s/s on M

$$\begin{array}{c} S^1 \times SM \\ \downarrow p \\ S^1 \times M \end{array}$$

Proof idea: $S^1 \longrightarrow \Gamma(SM)$

regard it as a p. spinor on $S^1 \times M$

Ob $H^1(M \times S^1; \mathbb{Z}_2)$

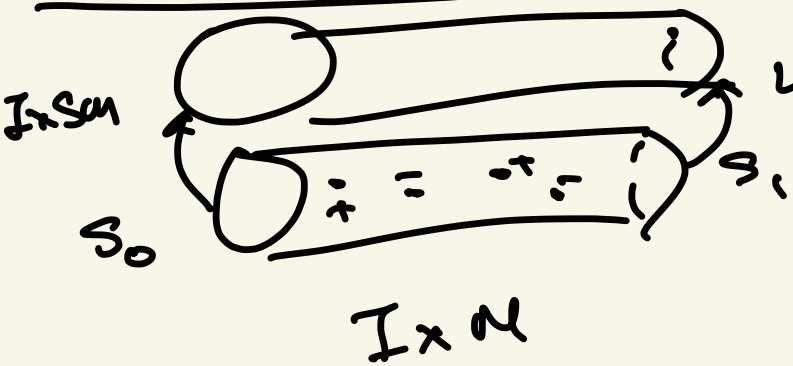
$H^7(M \times S^1; \mathbb{Z})$

$H^8(M \times S^1; \pi_8(\mathbb{R}P^7)) = H^8(M \times S^1; \mathbb{Z}_2)$

$\mathbb{Z}_2$
$\shortparallel$

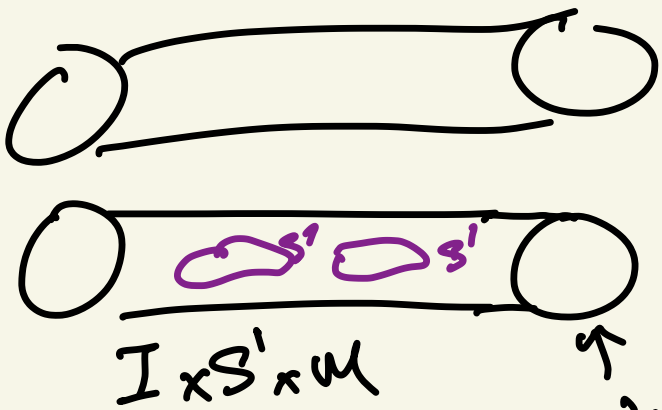
π

Relative J-inv revisited:



$\nu(\varphi_0, \varphi_1) \in \mathbb{Z}$

$\underline{\underline{f^* \varphi_1}}$



$I \times S^1 \times SM$

$H^*(M \times S^1; \mathbb{Z}_2)$

Zero set

$\Omega_1^{fr} = \mathbb{Z}_2$

Defn of the ν_1-inv:

$\varphi_t : S^1 \to \mathcal{G}_2^+(M)$, $t \in S^1$

$(S^1 \times M, \varphi_t)$ has a G_2-str.

$T(S^1 \times M) = \underline{\mathbb{R}} \times \pi^*(TM)$ $\quad \pi : S^1 \times M \to M$

If φ_t is compatible with a spin str

compare φ_t & φ_0 = constant family

$W = I \times S^1 \times M$

$\nu_1(\varphi_t) \in \Omega_9^{\text{spin}, G_2} \cong \mathbb{Z}/2$

$:= (W, \chi^{st}, \varphi_t \cup -\varphi_*)$

<u>Claim</u>: there's a way to do this even if φ_t induces the non-trivial elt of $\pi_1(\mathrm{Spin}(M))$: use stable Spin_7 cobordism $S^1 \times W \to S^1 \times M$

<u>Egs of loops in $\mathfrak{g}_2^*(M)$</u>:

Let x be an SU_3-str on (M, φ)

x is equivalent to a section of TM.

$$\varphi: \quad SM \cong \underline{\mathbb{R}}(1) \oplus TM$$

$$\begin{array}{ccc} \underline{\mathbb{R}}(1) & \oplus & TM \\ \downarrow & & \uparrow \\ S_0 & & S_1 \\ M & & \end{array}$$

$\varphi_x : t \longmapsto \cos(\pi t) s_0 + \sin(\pi t) s_1$

<u>Facts</u> ① $[\varphi_x] \mapsto 1 \in \pi_1(\mathrm{Spin}(M))$

② $\Omega_7^{SU_3} \longrightarrow \Omega_7^{G_2}$ ν maps (stable bordism cps

$\mathbb{Z}/2 \longleftarrow$ <u>shrinking</u>

$\mathbb{Z}/6 \longrightarrow \mathbb{Z}/3$

Pick X non-bounding: $\longrightarrow$ <u>odd vinst</u>

<u>Conjecture</u>: $\nu_1(\varphi_x, \varphi_*) = 1 \in \mathbb{Z}/2$ $+\nu_0(\varphi)$

$\Longleftrightarrow X$ is non-bounding

Example: $S^1 \times N = M^7$ — N is a Calabi–Yau 3-fold.

$g = g_{flat} \times g_N$

$hol(g) = SU_3$

Fact: $[S^1 \times N, x] \in \Omega_7^{SU_3} \cong \mathbb{Z}/2$

" $c_3(N) \bmod 4$.

Eg $N = X^3_{5,1} \subseteq \mathbb{CP}^5$ (check).

Get a loop $S^1 \xrightarrow{\varphi_t} \mathcal{G}_2^{tf}(S^1 \times N)$

with $\nu_1(\varphi_t) = 1$.

The map $\pi_1 \mathrm{Diff}(M) \longrightarrow \Omega_q^{\mathrm{spin}_7, g_2}$

Recall $\nu_1(\varphi_\epsilon) \in \Omega_q^{\mathrm{spin}_7, g_2} \cong \mathbb{Z}/2$

Conjecture:

$$\pi_1 \mathrm{Diff}(M) \to \pi_1 \mathcal{G}_2^*(M) \xrightarrow{\nu_1} \Omega_q^{\mathrm{spin}_7, g_2}$$

is zero.

If M is 2-clf
or $M = S^1 \times N$
with $\pi_1(N) = \{e\}$

Proof idea: There is an exact sequence

calculation

$$(*) \quad \Omega_q^{g_2} \to \Omega_q^{\mathrm{spin}_7} \to \Omega_q^{\mathrm{spin}_7, g_2} \to 0$$

(*) fits into the commutative diagram:

$$
\begin{tikzcd}
\Omega_q^{G_2} \arrow[r] \arrow[d, equal] & \Omega_q^{spin_7} \arrow[r] \arrow[d] & \Omega_q^{spin_7, G_2} \arrow[r] \arrow[d, equal] & 0 \\
\Omega_q^{G_2} \arrow[r] & \Omega_q^{spin} \arrow[r] \arrow[d, equal] & \Omega_q^{spin, G_2} \arrow[r] \arrow[d, equal] & 0 \\
\Theta_q^{\mathbb{Z}_2^3} \arrow[u] \arrow[r] \arrow[d] & \Omega_q^{spin} \arrow[r] \arrow[d, equal] & \mathbb{Z}_2 \arrow[r] \arrow[d, equal] & 0 \\
\Omega_q^{SU^{\mathbb{Z}_2}} \arrow[r] & \Omega_q^{spin^{\mathbb{Z}_2^2}} \arrow[r, "\pi'"] & \mathbb{Z}_2 \arrow[r] & 0
\end{tikzcd}
$$

$\Omega_q^{spin} \cong \mathbb{Z}_2^2$

group of homotopy q-spheres

Theorem:

C-Yang arXiv: 2011.0980

π' index inv

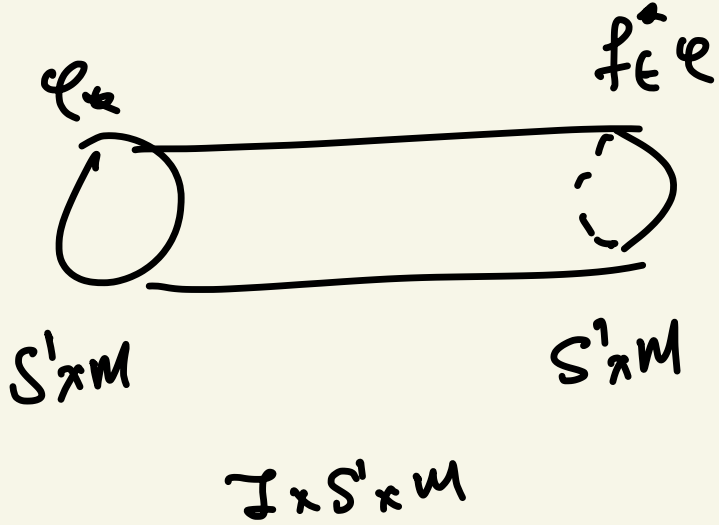
$\pi' = \mathrm{ind}(\mathcal{D}_{TM}) - \mathrm{ind}(\mathcal{D})$

α-inv

In C-Yang there is topological interpretation of π^1

Idea given f_t:

$$\begin{aligned} S^1 &\longrightarrow \mathrm{Diff}(M) \\ t &\longmapsto f_t \end{aligned} \implies \begin{aligned} S^1 &\longrightarrow \mathcal{G}_2^*(M) \\ \varphi &\longmapsto f_t^* \varphi \end{aligned}$$



$$f_t : S^1 \times M \to S^1 \times M$$
$$(t, m) \mapsto (t, f_t(m))$$

Take the mapping torus $S^1 \times M \to T_{f_t} \to S^1$

$$[T_{f_t}] \in \Omega_q^{spin}$$

<u>Show:</u> $\pi'([T_{f_t}]) = 0$

<u>Conclusion:</u> if the above conjectures hold

$\exists$ φ_X, a loop of G_2-strs associated to an SL_3-str X s.t. $[\varphi_X] \neq 0 \in \dfrac{\pi_1 \mathcal{G}_2^+(M)}{\pi_1 \mathrm{Diff}(M)}$

Can even replace '+' with 'tf' for $M = S^1 \times X^3_{1,5}$

/20

Diagram for the conclusion:

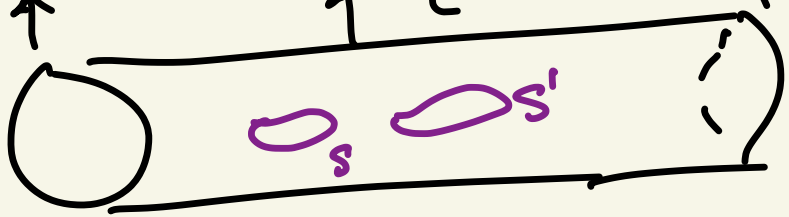
$$\begin{tikzcd}
{[\varphi_x]} \in \pi_1(\mathcal{G}_2^+(M)) / \pi_1 \mathrm{Diff}(M) \arrow[d, mapsto] \arrow[d, "\bar{\nu}_1"] \\
\bar{\nu}_1([\varphi_x]) \neq 0 \in \left(\Omega_q^{\mathrm{spin}_{\tau_1,\varphi_2}} / \nu_1(\mathrm{Diff}(M))\right) = \mathbb{Z}_2
\end{tikzcd}$$

End of talk

Questions: relationship between $\Omega_1^{fr} \cong \pi_8(S^7)$ & $\Omega_q^{Spin_7, G_2}$

$$BG_2 \longrightarrow BSpin_7 \longleftarrow BG_2$$

$$\uparrow \qquad \uparrow \tau^{st} \qquad \uparrow$$



$TM \cong SM$ stably

$S^1 \times M \times \{0\}$

$S^1 \times M \times \{1\}$

$$I \times S^1 \times M = W^q \in \Omega_q^{Spin_7, G_2}$$

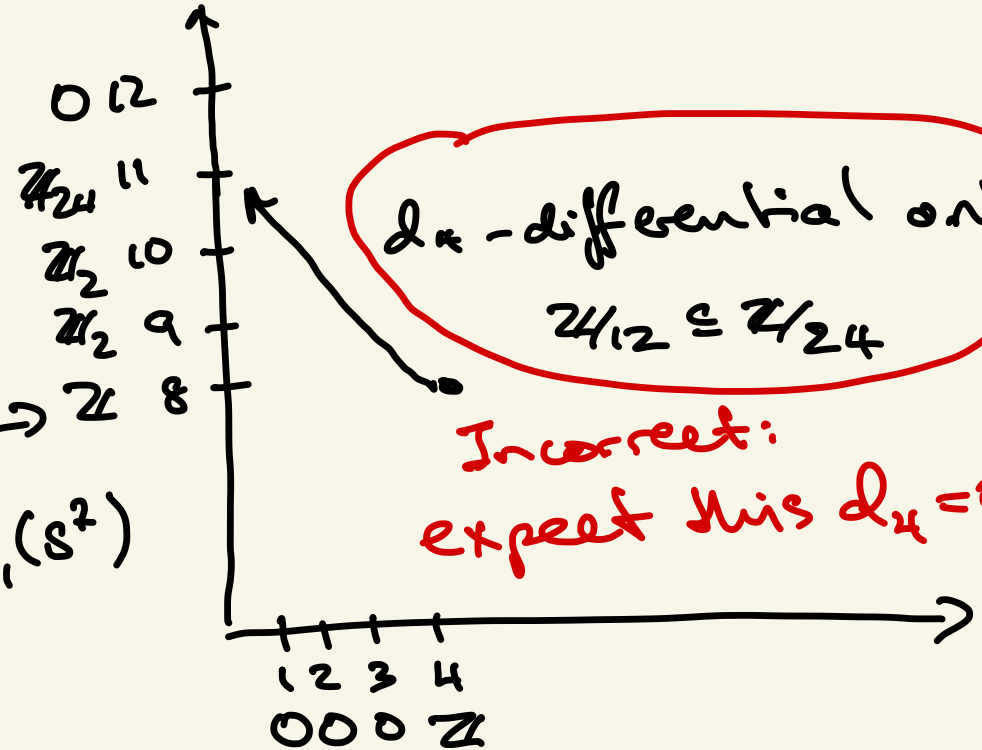
$$[W, \psi^{st}, \varphi_0^{st} \cup \varphi_1^{st}]$$

More on $\Omega_*^{\mathrm{Spin}_7, G_2}$

$$\begin{tikzcd}
S^7 \arrow[d] & \pi^* E\mathrm{Spin}_7 \arrow[d] \arrow[r, equal] & \\
BG_2 \arrow[d, "\pi"'] & EG_2 \arrow[l] \arrow[d] & \\
B\mathrm{Spin}_7 & E_{\mathrm{Spin}_7} \arrow[l] &
\end{tikzcd}$$

S^7 - fibre bundle

James SS for $\Omega_*^{\mathrm{Spin}_7, G_2}$



d_* - differential onto $\mathbb{Z}/12 \subseteq \mathbb{Z}/24$

Incorrect: expect this $d_4 = 0$

$\Omega_*^{\mathrm{Spin}_7, G_2} \cong \mathbb{Z}/2$ for $* = 8, 9, 10$

$\nu_2 = 0$ on $\pi_2 \mathcal{G}_2^+(M)$ – stated in question by DC as wrong

<u>Correct statement:</u>

$$\begin{tikzcd} \nu_i : \pi_i(\mathcal{G}_2^+(M)) \arrow[rr] \arrow[dr] & & \Omega_{i+8}^{\mathrm{Spin}_7, C_2} \\ & \pi_i^s(\mathcal{G}_2^+(M)) \arrow[ur] & \end{tikzcd}$$

factors over

So ν_1 & ν_2 factor over $H_i(\mathcal{G}_2^+(M); \mathbb{Z})$ $i = 1, 2$.