Given a non-compact Calabi-Yau 3-fold singularity $X$, M-theory engineers a 5d superconformal field theory (SCFT) $T_X$.

We claim there is a general procedure to geometrically predict the 0-form symmetry, $G_F$ (with center $Z_G$), and the 2-group structure of $T_X$ (up to some subtleties).

This refines earlier results in the literature which calculate the 1-form symmetry as (assuming electric polarization)

$$\hat{A}^\vee = H_1(\partial X, \mathbb{Z})$$

(all bulk compact cycles blown-down)
1. Derive our general procedure geometrically by examining equivalence classes of line operators and explain why it captures the 0-form, 1-form, and 2-group symmetry
2. Examine some examples where $X = \mathbb{C}^3/\Gamma$
   Number of flavor branes = 0, 1, 2, or 3
   \[ \Gamma \cong \mathbb{Z}_n \text{ or } \mathbb{Z}_n \times \mathbb{Z}_m \quad \text{where } m \text{ divides } n \]
1-Form Symmetry

- Recall (Jonathan’s talk): 1-form symmetry acts on line operators
- Ex. Let $A = \mathbb{Z}_n$
- Then (equivalence classes of) line operators labeled by $A^\vee = \mathbb{Z}_n$
- Action:
  \[ \mathcal{O}_k(L) \mapsto e^{2\pi i k/n} \mathcal{O}_k(L) \]
- Equivalence relation:
  \[ \mathcal{O}(L) \circlearrowleft \mathcal{O}^{loc.}_p \mathcal{O}'(L) \quad \leftrightarrow \quad \mathcal{O}(L) \sim \mathcal{O}'(L) \]
• One can define a more refined group from a coarser equivalence relation [Lee, Ohmori, Tachikawa ‘21]

\[ \mathcal{O}(L) \quad \mathcal{O}'(L) \quad \overset{\sim'}{\rightleftharpoons} \quad \mathcal{O}(L) \sim' \mathcal{O}'(L) \]

\[ \mathcal{O}_p^{\text{genuine loc.}} \]

\[ \widetilde{\mathcal{A}}^\vee := \{\text{line operators}\}/ \sim' \]
NAÏVE 1-FORM SYMMETRY

\( \mathcal{O}_{p}^{\text{genuine loc.}} \) is a local operator which is faithfully acted upon by the 0-form symmetry

The term *genuine* is motivated by considering the converse

A non-genuine local operator transforms projectively under \( Z_{G} \), which implies it transforms faithfully under a finite cover \( Z_{\tilde{G}} \rightarrow \pi_{1} Z_{G} \).

Since \( Z_{\tilde{G}} \) is not the true 0-form symmetry, by definition, a non-genuine operator is not truly local.

\( \mathcal{O}_{p}^{\text{non-gen.}} \)

Topological line operator (or, conceptually, a cut)
Line operator equivalence classes give an extension:

\[ 0 \to \mathcal{C}^\vee \to \tilde{\mathcal{A}}^\vee \to \mathcal{A}^\vee \to 0 \]

Finite cover of the group acting faithfully written as an extension:

\[ 0 \to \mathcal{C} \to Z_{\tilde{G}} \to Z_G \to 0 \]

\[ \iff 0 \to \mathcal{A} \to \tilde{\mathcal{A}} \to Z_{\tilde{G}} \to Z_G \to 0 \]

\[ P \in H^3(BZ_G, \mathcal{A}) \]

(Postnikov Class)
WHAT IS A 2-GROUP?

• Abstract nonsense definition: a 2-groupoid with one element [Baez, Lauda ‘03]

• Less abstract definition: A tuple \((\pi_1, \pi_2, \rho, P)\) such that

\[ \rho : \pi_1 \rightarrow \text{Aut}(\pi_2) \quad P \in H^3(B\pi_1, (\pi_2)\rho) \]

• For us, \(\pi_1 = G_F\)

• Our methods are sensitive to \(Z_G \subset G_F\) so we take \(\pi_1 = Z_G\)

• \(\rho\) is trivial in the 5d SCFT examples that we consider

(see also [Kapustin, Thorngren ‘13] and [Benini, Cordova, Psin ‘18])
NOW LET’S SEE THIS PICTURE FROM THE GEOMETRY!
In the case $X = \mathbb{C}^3/\Gamma$, the flavor loci are $S^1$s in $\partial X = S^5/\Gamma$.
Focusing on one singularity, we want to geometrically obtain an element in

\[ C^\gamma = \{ \text{line operators which are trivial under } \sim, \text{ but not under } \sim' \} \]

\[ = \{ \text{line operators that can end on non-genuine local operators} \} \]

\[ O^{(\gamma)}(L) \]
For an A-type flavor brane along a locus $K$, the local neighborhood is of the form $\mathbb{C}^2/\mathbb{Z}_n \times K$, with boundary $S^3/\mathbb{Z}_n \times K$

$$\text{Tor } H_1(S^3/\mathbb{Z}_n \times K) = \mathbb{Z}_n \simeq \mathbb{Z}_n = Z(SU(n))$$

$\mathcal{O}_p^{(D)}$ is in a representation of $\mathbb{Z}_n \subset SU(n)_{\text{flavor}}$
Example which becomes a trivial element in $\mathcal{C}^\vee \subset \tilde{A}$ and $\gamma$ is contractible in this case.

$$\tilde{\mathcal{A}}^\vee = \text{Tor} \; H_1(\partial X^\circ, \mathbb{Z})$$

where

$$\partial X^\circ := \partial X \setminus K$$

Thus, given $j_1 : \partial X^\circ \hookrightarrow \partial X$ we have

$$\mathcal{C}^\vee = \text{Tor Ker}(j_1)$$

Furthermore, we are naturally led to define

$$Z_{\tilde{G}} = \text{Tor} \; H_1(\partial X^\circ \cap T(K)) = \bigoplus_i \text{Ab}(\Lambda_{g_i})$$

Tubular neighborhood
• We can then derive the four-term exact sequence defining the 2-group from the long exact sequence of Mayer-Vietoris

\[ \cdots \rightarrow H_2(\partial X) \xrightarrow{\partial_2} H_1(\partial X^\circ \cap T(K)) \xrightarrow{\iota_1} H_1(\partial X^\circ) \oplus H_1(T(K)) \xrightarrow{j_1 - \ell_1} H_1(\partial X) \xrightarrow{\partial_1} 0 \]

\[ \implies \]

\[ 0 \rightarrow \text{Ker}(\iota_1) \rightarrow H_1(\partial X^\circ \cap T(K)) \rightarrow H_1(\partial X^\circ) \oplus H_1(T(K)) \rightarrow H_1(\partial X) \rightarrow 0 \]

• Taking the Pontryagin dual of this sequence then reproduces

\[ 0 \rightarrow A \rightarrow \tilde{A} \rightarrow Z_{\tilde{G}} \rightarrow Z_G \rightarrow 0 \]
We can equivalently phrase this result in terms of orbifold homology. Orbifold homology is equivalent to equivariant homology when there is a globally defined group action

\[ H_*^{\text{orb.}}(X/G) = H_*^{\text{equiv.}}(X) \]

It can be evaluated on orbifolds that do not necessarily have a presentation as a global quotient.

For first homology, there is a useful relation (when the orbifold singularities have real codimension>2) [Thurston]

\[ H_1^{\text{orb.}}(X) = H_1(X^\circ) \]

This assists us in evaluating the naïve 1-form symmetry in the case when

\[ \partial X = S^5/\Gamma_{SU(3)} \]
CASE STUDY: $X = \mathbb{C}^3/\Gamma$
\[ X = \mathbb{C}^3 / \Gamma \text{ AND M-THEORY} \]

- Constructs 5d N=1 SCFT localized at the origin (8 supercharges)
- Physics is usually elucidated by resolving or deforming singularity at origin (Coulomb branch/Higgs branch respectively), non-Lagrangian at fixed point
- 3d McKay Correspondence [Ito Reid ‘94] is a central tool, (for some modern physics references see [Tian, Wang ‘21] and [Del Zotto, Heckman, Meynet, Moscrop, Zhang ‘22])
- Physics is usually the strongly coupled completion of 5d gauge theories
  \[ g_I^2 \sim \frac{1}{\text{Vol}(\mathbb{P}^1_I)} \]
- Ranks of gauge group and flavor group Lie algebras given in terms of group theoretic data of \( \Gamma \)
$X = \mathbb{C}^3/\Gamma$ GEOMETRY

- We will focus on abelian $\Gamma$
- Two possibilities: $\Gamma \cong \mathbb{Z}_n$ or $\mathbb{Z}_n \times \mathbb{Z}_m$ (m divides n)
- When $\Gamma \cong \mathbb{Z}_n$ the action is given by

$$\begin{align*}
(z_1, z_2, z_3) &\mapsto (\omega^{k_1} z_1, \omega^{k_2} z_2, \omega^{k_3} z_3) \\
&\quad (\sum_i k_i = 1, \omega^n = 1)
\end{align*}$$

- Can have 0, 1, 2, or 3 flavor branes on loci parametrized by $\text{Arg}(z_i)$
- For $\Gamma = \mathbb{Z}_n \times \mathbb{Z}_m$, we have additional generator

$$\begin{align*}
(z_1, z_2, z_3) &\mapsto (z_1, \eta z_2, \eta^{-1} z_3) \\
&\quad (\eta^m = 1)
\end{align*}$$

- Always have 3 flavor branes
\( X = \mathbb{C}^3/\Gamma \) GEOMETRY

• Can be presented as toric model \( T^3/\Gamma \hookrightarrow S^5/\Gamma \hookrightarrow B \)

\[ |z_1| = 1 \quad |z_2| = 1 \quad |z_3| = 1 \]

\( = \) A-type singularities
EXAMPLE 1: $T_N$ THEORY

- Take $X = \mathbb{C}^3/(\mathbb{Z}_n \times \mathbb{Z}_n)$, can define action as
  $$(z_1, z_2, z_3) \sim (\omega z_1, z_2, \omega^{-1} z_3) \sim (z_1, \eta z_2, \eta^{-1} z_3) \quad (\eta^N = \omega^N = 1)$$

- Physics: After compactifying on $S^1$ becomes 4d $T_N$ Theory
  which is the building block of class S 4d N=2 SCFTs [Gaiotto’10]

$\bullet = A_{N-1}$ singularity along $S^1$ fiber
Using the relation to orbifold homology

\[ H_1(\partial X^\circ) = \mathbb{Z}_N^2 = \tilde{A} \]

While from Armstrong’s theorem, which states that the fundamental group is the quotient of \( \Gamma \) by the subgroup which acts non-freely [Armstrong ‘68], it follows that

\[ \pi_1(\partial X) = H_1(\partial X) = 0 = \mathcal{A} \]

Thus

\[ 0 \rightarrow \mathcal{C}^\vee \rightarrow \tilde{\mathcal{A}}^\vee \rightarrow \mathcal{A}^\vee \rightarrow 0 \quad \implies \quad \mathcal{C} = \mathbb{Z}_N^2 \]

\[ 0 \rightarrow \mathcal{C} \rightarrow \mathbb{Z}_{\tilde{G}} \rightarrow \mathbb{Z}_G \rightarrow 0 \quad \implies \quad \mathbb{Z}_G = \mathbb{Z}_N \]

\[ G_F = SU(N)^3/\mathbb{Z}_N^2 \]

This agrees with earlier QFT results [Bhardwaj ‘20], up to some subtleties with symmetry enhancement.
EXAMPLE 2: SU(N)_N

- Consider \( \Gamma \cong \mathbb{Z}_{2N} \) with \( (k_1, k_2, k_3) = (1, 1, 2N - 2) \)
- Have one fixed point of type \( A_1 \) parametrized by \( \text{Arg}(z_3) \)
- Physics (for even \( N \)) is UV completion of pure \( \mathcal{N}=1 \) SU(N) gauge theory with Chern-Simons level \( N \)

\[
N \text{Tr} A \wedge F \wedge F \subset \mathcal{L}_{5d \text{ theory}}
\]

- From the geometry:

\[
\tilde{\mathcal{A}} = \mathbb{Z}_{2N} \quad \mathcal{A} = \mathbb{Z}_{N} \quad \mathbb{Z}_{\tilde{G}} = \mathbb{Z}_2
\]
\[ \tilde{A} = \mathbb{Z}_{2N} \quad A = \mathbb{Z}_N \quad \tilde{Z}_G = \mathbb{Z}_2 \]

\[ 0 \to A \to \tilde{A} \to C \to 0 \quad \Longleftrightarrow \quad 0 \to \mathbb{Z}_N \to \mathbb{Z}_{2N} \to \mathbb{Z}_2 \to 0 \]

\[ 0 \to C \to \tilde{Z}_G \to Z_G \to 0 \quad \Longleftrightarrow \quad 0 \to \mathbb{Z}_2 \to \mathbb{Z}_2 \to 0 \to 0 \]

Results:
\[ C = \mathbb{Z}_2 \quad Z_G = 0 \quad G_F = SO(3) \]

- Flavor symmetry consistent with [Apruzzi, Bhardwaj, Oh, Schäfer-Nameki ‘21]
- Can still in principle have a non-trivial 2-group for N even since \( H^3(BSO(3), \mathbb{Z}_N) \neq 0 \)
EX.3: NON-TRIVIAL 2-GROUP

- Consider $\Gamma \cong \mathbb{Z}_9 \times \mathbb{Z}_3$ with generator weights
  \[(k_1, k_2, k_3) = (1, 1, 7) \quad \text{and} \quad (0, 1, 2)\]

- Neat IR physics description is currently unknown to us, can still understand the 2-group!

- We have three $A_2$ singularities which means $Z_{\tilde{G}} = \mathbb{Z}_3^3$

\[\tilde{A} = \mathbb{Z}_9 \times \mathbb{Z}_3 \quad A = \mathbb{Z}_3\]
\[ \tilde{\mathcal{A}} = \mathbb{Z}_9 \times \mathbb{Z}_3 \quad \mathcal{A} = \mathbb{Z}_3 \quad Z_{\tilde{G}} = \mathbb{Z}_3^3 \]

\[ 0 \to \mathcal{A} \to \tilde{\mathcal{A}} \to \mathcal{C} \to 0 \quad \Rightarrow \quad 0 \to \mathbb{Z}_3 \to \mathbb{Z}_9 \times \mathbb{Z}_3 \to \mathbb{Z}_3 \times \mathbb{Z}_3 \to 0 \]

\[ 0 \to \mathcal{C} \to Z_{\tilde{G}} \to Z_G \to 0 \quad \Rightarrow \quad 0 \to \mathbb{Z}_3^2 \to \mathbb{Z}_3^3 \to \mathbb{Z}_3 \to 0 \]

Results: \[ \mathcal{C} = \mathbb{Z}_3 \times \mathbb{Z}_3 \quad \mathcal{Z}_G = \mathbb{Z}_3 \quad G_F = SU(3)^3/\mathbb{Z}_3^2 \]

\[ P \neq 0 \in H^3(BZ_G, \mathcal{A}) \quad \Rightarrow \quad \text{This sequence does not split:} \quad 0 \to \mathcal{A} \to \tilde{\mathcal{A}} \to Z_{\tilde{G}} \to Z_G \to 0 \]
SUMMARY/OUTLOOK

• We have introduced a general procedure to calculate the 0-form, 1-form, and 2-group symmetries

• Can be applied to M/F-theory geometric engineering setups of various dimensions

• Is not yet sensitive to symmetry enhancements: still dealing with classical geometry. Is there a generalization that sees this?

• Compact models (see Max’s talk)

• Further study how this constrains dynamics of theories engineered from $G_2$ (see Mirjam+Max talks) and Spin(7)

• N-group? More general categorical symmetries?
DANKE!