GETTING HIGH ON GLUING ORBIFOLDS

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SOME OF WHAT YOU JUST HEARD

- Given a non-compact Calabi-Yau 3-fold singularity X, M-theory engineers a 5d superconformal field theory (SCFT) T_X
- We claim there is a general procedure to geometrically predict the 0-form symmetry, G_F (with center Z_G), and the 2-group structure of T_X (up to some subtleties)
- This refines earlier results in the literature which calculate the 1-form symmetry as (assuming electric polarization)

$$\mathcal{A}^\vee = H_1(\partial X, \mathbb{Z})$$

(all bulk compact cycles blown-down)

PLAN OF THIS TALK

1. Derive our general procedure geometrically by examining equivalence classes of line operators and explain why it captures the 0-form, 1-form, and 2-group symmetry
2. Examine some examples where $X = \mathbb{C}^3/\Gamma$

   Number of flavor branes = 0, 1, 2, or 3

   $\Gamma \cong \mathbb{Z}_n$ or $\mathbb{Z}_n \times \mathbb{Z}_m$ m divides n

1-FORM SYMMETRY

- Recall (Jonathan's talk): 1-form symmetry acts on line operators
- Ex. Let $\mathcal{A} = \mathbb{Z}_n$
- Then (equivalence classes of) line operators labeled by $\mathcal{A}^\vee = \mathbb{Z}_n$
- Action:

$$\mathcal{O}_k(L) \mapsto e^{2\pi i k/n}\mathcal{O}_k(L)$$

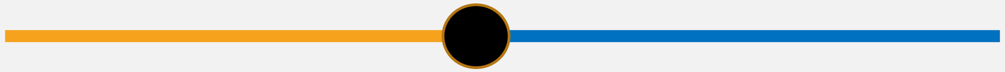
- Equivalence relation:

$$\Longleftrightarrow \quad \mathcal{O}(L) \sim \mathcal{O}'(L)$$

NAÏVE 1-FORM SYMMETRY

- One can define a more refined group from a coarser equivalence relation [Lee, Ohmori, Tachikawa '21]

$$\mathcal{O}(L) \qquad \mathcal{O}'(L)$$



$$\mathcal{O}_p^{\text{genuine loc.}}$$

$$\Longleftrightarrow \quad \mathcal{O}(L) \sim' \mathcal{O}'(L)$$

$$\widetilde{\mathcal{A}}^\vee := \{\text{line operators}\}/\sim'$$

NAÏVE 1-FORM SYMMETRY

$\mathcal{O}_p^{\text{genuine loc.}}$ is a local operator which is faithfully acted upon by the 0-form symmetry

The term *genuine* is motivated by considering the converse

A non-genuine local operator transforms projectively under Z_G, which implies it transforms faithfully under a finite cover $Z_{\tilde{G}} \xrightarrow{\pi} Z_G$

Since $Z_{\tilde{G}}$ is not the true 0-form symmetry, by definition, a non-genuine operator is not truly local

$\mathcal{O}_p^{\text{non-gen.}}$

Topological line operator (or, conceptually, a cut)

Line operator equivalence classes give an extension:

$$0 \to \mathcal{C}^\vee \to \widetilde{\mathcal{A}}^\vee \to \mathcal{A}^\vee \to 0$$

Finite cover of the group acting faithfully written as an extension:

$$0 \to \mathcal{C} \to Z_{\widetilde{G}} \to Z_G \to 0$$

$$\Longrightarrow \qquad 0 \to \mathcal{A} \to \widetilde{\mathcal{A}} \to Z_{\widetilde{G}} \to Z_G \to 0$$

$$\Longleftrightarrow \qquad P \in H^3(BZ_G, \mathcal{A})$$

(Postnikov Class)

WHAT IS A 2-GROUP?

- Abstract nonsense definition: a 2-groupoid with one element [Baez, Lauda '03]
- Less abstract definition: A tuple $(\pi_1, \pi_2, \rho, \mathrm{P})$ such that

$$\rho : \pi_1 \to \mathrm{Aut}(\pi_2) \qquad P \in H^3(B\pi_1, (\pi_2)_\rho)$$

- For us, $\pi_1 = G_F$ $\qquad \pi_2 = \mathcal{A}$
- Our methods are sensitive to $Z_G \subset G_F$ so we take $\pi_1 = Z_G$
- ρ is trivial in the 5d SCFT examples that we consider

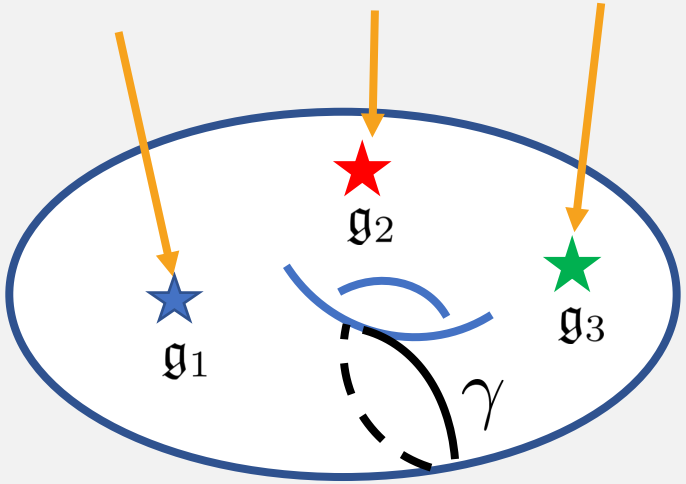
(see also [Kapustin, Thorngren '13] and [Benini, Cordova, Psin '18])

NOW LET’S SEE THIS PICTURE FROM THE GEOMETRY!

BOUNDARY GEOMETRY

$\mathbb{C}^2/\Lambda_{\mathfrak{g}_i}$ singularities (flavor branes)

∂X



$\Lambda_{\mathfrak{g}_i} \subset SU(2)$

ADE classification

M2 on Cone(γ) = element in $\mathcal{A}^\vee$

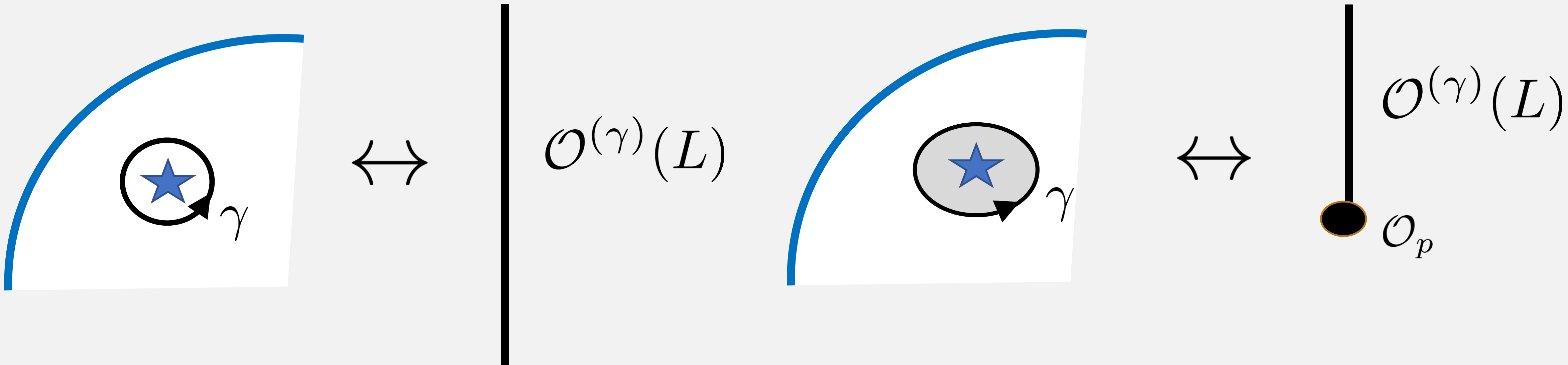
In the case X=$\mathbb{C}^3/\Gamma$, the flavor loci are S^1s in $\partial X = S^5/\Gamma$

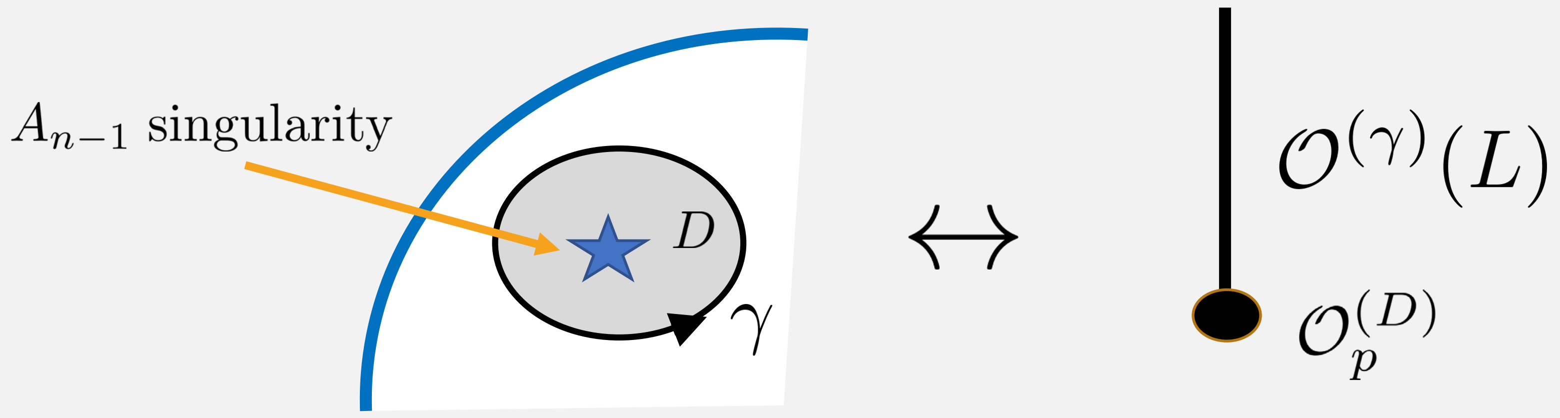
BOUNDARY GEOMETRY SINGULARITY

Focusing on one singularity, we want to geometrically obtain an element in

$$\mathcal{C}^{\vee} = \{\text{line operators which are trivial under } \sim\text{, but not under } \sim'\}$$

$$= \{\text{line operators that can end on non-genuine local operators}\}$$

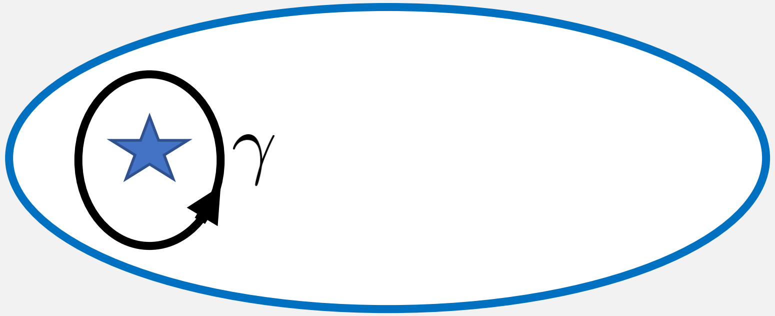




- For an A-type flavor brane along a locus K, the local neighborhood is of the form $\mathbb{C}^2/\mathbb{Z}_n \times K$, with boundary $S^3/\mathbb{Z}_n \times K$

$$\mathrm{Tor}\, H_1(S^3/\mathbb{Z}_n \times K) = \mathbb{Z}_n^\vee \simeq \mathbb{Z}_n = Z(SU(n)) \implies \mathcal{O}_p^{(D)} \text{ is in a representation of } \mathbb{Z}_n \subset SU(n)_{flavor}$$

Example which becomes a trivial element in $\mathcal{C}^\vee \subset \tilde{\mathcal{A}}$



γ is contractible in this case

$\Downarrow$

$$\widetilde{\mathcal{A}}^\vee = \mathrm{Tor}\, H_1(\partial X^\circ, \mathbb{Z})$$ where $$\partial X^\circ := \partial X \backslash K$$

Thus, given $j_1 : \partial X^\circ \hookrightarrow \partial X$ we have $\mathcal{C}^\vee = \mathrm{TorKer}(j_1)$

Furthermore, we are naturally led to define $Z_{\widetilde{G}} = \mathrm{Tor}\ H_1(\partial X^\circ \cap T(K)) = \oplus_i \mathrm{Ab}(\Lambda_{\mathfrak{g}_i})$

$\downarrow$ Tubular neighborhood

MAYER-VIETORIS SEQUENCE

- We can then derive the four-term exact sequence defining the 2-group from the long exact sequence of Mayer-Vietoris

$$\cdots \longrightarrow H_2(\partial X) \xrightarrow{\partial_2} H_1(\partial X^\circ \cap T(K)) \xrightarrow{\iota_1} H_1(\partial X^\circ) \oplus H_1(T(K)) \xrightarrow{j_1 - \ell_1} H_1(\partial X) \xrightarrow{\partial_1} 0$$

$$\Longrightarrow$$

$$0 \to \mathrm{Ker}(\iota_1) \to H_1(\partial X^\circ \cap T(K)) \to H_1(\partial X^\circ) \oplus H_1(T(K)) \to H_1(\partial X) \to 0$$

- Taking the Pontryagin dual of this sequence then reproduces

$$0 \to \mathcal{A} \to \widetilde{\mathcal{A}} \to Z_{\widetilde{G}} \to Z_G \to 0$$

ORBIFOLD HOMOLOGY

- We can equivalently phrase this result in terms of orbifold homology
- Orbifold homology is equivalent to equivariant homology when there is a globally defined group action

$$H_*^{orb.}(X/G) = H_G^{equiv.}(X)$$

- It can be evaluated on orbifolds that do not necessarily have a presentation as a global quotient
- For first homology, there is a useful relation (when the orbifold singularities have real codimension>2) [Thurston]

$$H_1^{orb.}(X) = H_1(X^\circ)$$

- This assists us in evaluating the naïve 1-form symmetry in the case when

$$\partial X = S^5/\Gamma_{SU(3)}$$

CASE STUDY: $X = \mathbb{C}^3/\Gamma$

$X = \mathbb{C}^3/\Gamma$ AND M-THEORY

- Constructs 5d N=1 SCFT localized at the origin (8 supercharges)
- Physics is usually elucidated by resolving or deforming singularity at origin (Coulomb branch/Higgs branch respectively), non-Lagrangian at fixed point
- 3d McKay Correspondence [Ito Reid '94] is a central tool, (for some modern physics references see [Tian, Wang '21] and [Del Zotto, Heckman, Meynet, Moscrop, Zhang '22])
- Physics is usually the strongly coupled completion of 5d gauge theories

$$g_I^2 \sim \frac{1}{\mathrm{Vol}(\mathbb{P}^1_I)}$$

- Ranks of gauge group and flavor group Lie algebras given in terms of group theoretic data of Γ

$X = \mathbb{C}^3/\Gamma$ GEOMETRY

- We will focus on abelian Γ
- Two possibilities: $\Gamma \cong \mathbb{Z}_n$ or $\mathbb{Z}_n \times \mathbb{Z}_m$ (m divides n)
- When $\Gamma \cong \mathbb{Z}_n$ the action is given by

$$(z_1, z_2, z_3) \mapsto (\omega^{k_1} z_1, \omega^{k_2} z_2, \omega^{k_3} z_3) \qquad (\Sigma_i k_i = 1,\ \omega^n = 1)$$

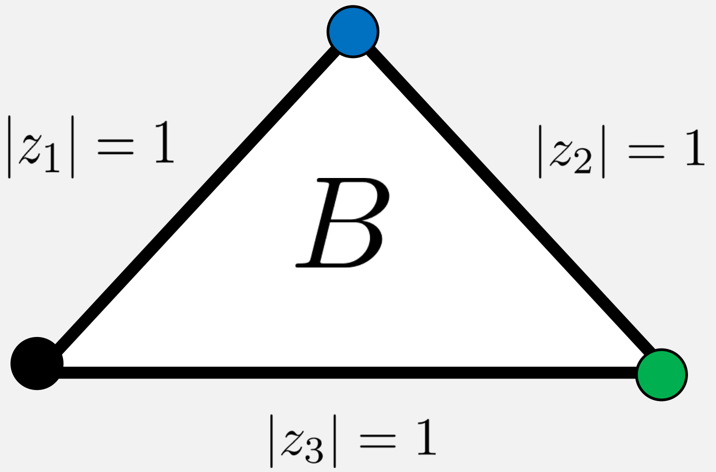
- Can have 0, 1, 2, or 3 flavor branes on loci parametrized by $Arg(z_i)$
- For $\Gamma = \mathbb{Z}_n \times \mathbb{Z}_m$, we have additional generator

$$(z_1, z_2, z_3) \mapsto (z_1, \eta z_2, \eta^{-1} z_3) \qquad (\eta^m = 1)$$

- Always have 3 flavor branes

$X = \mathbb{C}^3/\Gamma$ GEOMETRY

- Can be presented as toric model $T^3/\Gamma \hookrightarrow S^5/\Gamma \to B$



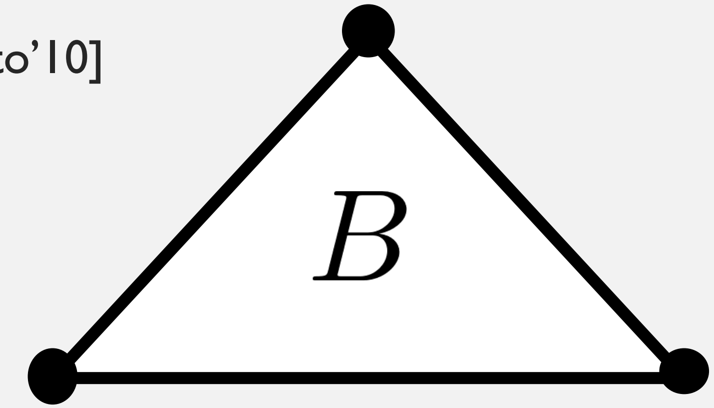
EXAMPLE 1: T_N THEORY

- Take $X = \mathbb{C}^3/(\mathbb{Z}_n \times \mathbb{Z}_n)$, can define action as

$$(z_1, z_2, z_3) \sim (\omega z_1, z_2, \omega^{-1} z_3)$$
$$\sim (z_1, \eta z_2, \eta^{-1} z_3)$$

$$(\eta^N = \omega^N = 1)$$

- Physics: After compactifying on S^1 becomes 4d T_N Theory which is the building block of class S 4d N=2 SCFTs [Gaiotto'10]



● $= A_{N-1}$ singularity along S^1 fiber

Using the relation to orbifold homology

$$H_1(\partial X^\circ) = \mathbb{Z}_N^2 = \widetilde{\mathcal{A}}$$

While from Armstrong's theorem, which states that the fundamental group is the quotient of Γ by the subgroup which acts non-freely [Armstrong '68], it follows that

$$\pi_1(\partial X) = H_1(\partial X) = 0 = \mathcal{A}$$

Thus

$$0 \to \mathcal{C}^\vee \to \widetilde{\mathcal{A}}^\vee \to \mathcal{A}^\vee \to 0 \implies \mathcal{C} = \mathbb{Z}_N^2$$

$$0 \to \mathcal{C} \to Z_{\widetilde{G}} \to Z_G \to 0 \implies Z_G = \mathbb{Z}_N$$

$$G_F = SU(N)^3/\mathbb{Z}_N^2$$

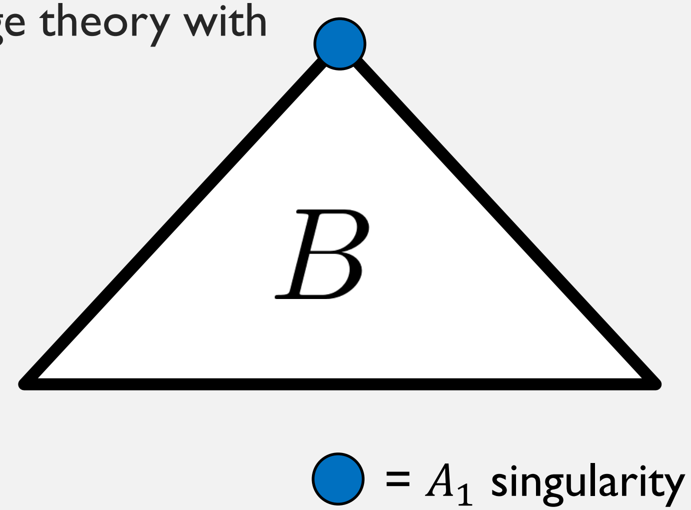
This agrees with earlier QFT results [Bhardwaj '20], up to some subtleties with symmetry enhancement

EXAMPLE 2:SU(N)N

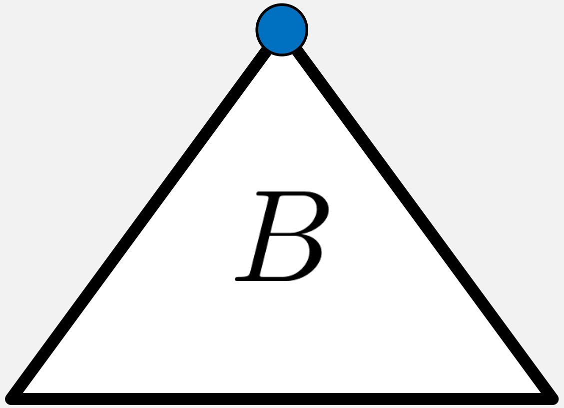
- Consider $\Gamma \cong \mathbb{Z}_{2N}$ with $(k_1, k_2, k_3) = (1, 1, 2N-2)$
- Have one fixed point of type A_1 parametrized by Arg(z_3)
- Physics (for even N) is UV completion of pure $\mathcal{N}$=1 SU(N) gauge theory with Chern-Simons level N

$$N \mathrm{Tr} A \wedge F \wedge F \subset \mathcal{L}_{\text{5d theory}}$$

- From the geometry:



$$\widetilde{\mathcal{A}} = \mathbb{Z}_{2N} \qquad \mathcal{A} = \mathbb{Z}_N \qquad Z_{\widetilde{G}} = \mathbb{Z}_2$$



$$\widetilde{\mathcal{A}} = \mathbb{Z}_{2N} \qquad \mathcal{A} = \mathbb{Z}_N \qquad Z_{\widetilde{G}} = \mathbb{Z}_2$$

$$0 \to \mathcal{A} \to \widetilde{\mathcal{A}} \to \mathcal{C} \to 0 \implies 0 \to \mathbb{Z}_N \to \mathbb{Z}_{2N} \to \mathbb{Z}_2 \to 0$$

$$0 \to \mathcal{C} \to Z_{\widetilde{G}} \to Z_G \to 0 \implies 0 \to \mathbb{Z}_2 \to \mathbb{Z}_2 \to 0 \to 0$$

Results: $\mathcal{C} = \mathbb{Z}_2 \qquad Z_G = 0 \qquad G_F = SO(3)$

- Flavor symmetry consistent with [Apruzzi, Bhardwaj, Oh, Schäfer-Nameki ‘21]
- Can still in principle have a non-trivial 2-group for N even since $H^3(BSO(3), \mathbb{Z}_N) \neq 0$

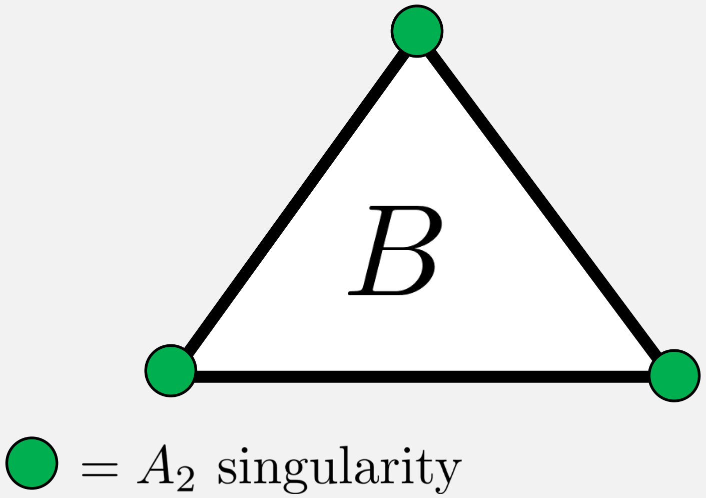
EX.3: NON-TRIVIAL 2-GROUP

- Consider $\Gamma \cong \mathbb{Z}_9 \times \mathbb{Z}_3$ with generator weights

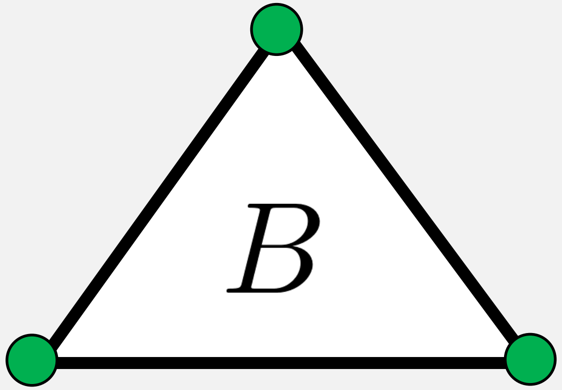
$$(k_1, k_2, k_3) = (1, 1, 7) \quad \text{and} \quad (0, 1, 2)$$

- Neat IR physics description is currently unknown to us, can still understand the 2-group!
- We have three A_2 singularities which means $Z_{\widetilde{G}} = \mathbb{Z}_3^3$

$$\widetilde{\mathcal{A}} = \mathbb{Z}_9 \times \mathbb{Z}_3 \qquad \mathcal{A} = \mathbb{Z}_3$$



$$\widetilde{\mathcal{A}} = \mathbb{Z}_9 \times \mathbb{Z}_3 \qquad \mathcal{A} = \mathbb{Z}_3 \qquad Z_{\widetilde{G}} = \mathbb{Z}_3^3$$



$$0 \to \mathcal{A} \to \widetilde{\mathcal{A}} \to \mathcal{C} \to 0 \quad \Longrightarrow \quad 0 \to \mathbb{Z}_3 \to \mathbb{Z}_9 \times \mathbb{Z}_3 \to \mathbb{Z}_3 \times \mathbb{Z}_3 \to 0$$

$$0 \to \mathcal{C} \to Z_{\widetilde{G}} \to Z_G \to 0 \quad \Longrightarrow \quad 0 \to \mathbb{Z}_3^2 \to \mathbb{Z}_3^3 \to \mathbb{Z}_3 \to 0$$

Results: $\mathcal{C} = \mathbb{Z}_3 \times \mathbb{Z}_3 \qquad Z_G = \mathbb{Z}_3 \qquad G_F = SU(3)^3/\mathbb{Z}_3^2$

$P \neq 0 \in H^3(BZ_G, \mathcal{A}) \Longrightarrow$ This sequence does not split:

$$0 \to \mathcal{A} \to \widetilde{\mathcal{A}} \to Z_{\widetilde{G}} \to Z_G \to 0$$

SUMMARY/OUTLOOK

- We have introduced a general procedure to calculate the 0-form, 1-form, and 2-group symmetries
- Can be applied to M/F-theory geometric engineering setups of various dimensions
- Is not yet sensitive to symmetry enhancements: still dealing with classical geometry. Is there a generalization that sees this?
- Compact models (see Max's talk)
- Further study how this constrains dynamics of theories engineered from G_2 (see Mirjam+Max talks) and Spin(7)
- N-group? More general categorical symmetries?


DANKE!