The generalized Kähler Calabi-Yau problem

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Part I: Background and Statements

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$$\Omega = \sigma^{-1} \in \Lambda_I^{2,0+0,2} T^* M \cap \Lambda_J^{2,0+0,2} T^* M,$$

is the real part of a holomorphic symplectic form.

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$$\mathbb{J}_\omega := \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}, \qquad L^{1,0} = \{X - \sqrt{-1}\omega(X, \cdot) \mid X \in TM\},$$
$$\mathbb{J}_J := \begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix}, \qquad L^{1,0} = T^{1,0}M \oplus \Lambda^{0,1}(M).$$

Theorem

(Gualtieri 2004) Given (M, H_0)*, a generalized Kähler structure* (g, b, I, J) *is equivalent to a pair* $(\mathbb{J}_1, \mathbb{J}_2)$ *of generalized complex structures further satisfying*

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Explicitly,

$$\mathbb{J}_{1/2} = \frac{1}{2}e^b \begin{pmatrix} I \pm J & -(\omega_I^{-1} \mp \omega_J^{-1}) \\ \omega_I \mp \omega_J & -(I^* \pm J^*) \end{pmatrix} e^{-b},$$

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The spinor is closed if $d_{H_0}\psi = 0$.

Examples

Kähler metric (g, J) as GK:

$$(g, 0, J, J), \qquad \mathbb{J}_1 = \begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix}, \qquad \mathbb{J}_2 = \begin{pmatrix} 0 & -\omega_J^{-1} \\ \omega_J & 0 \end{pmatrix}$$

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$$(g, b, I, J), \qquad \mathbb{J}_1 = e^b \left(\mathbb{J}_{I_{T_+}} \oplus \mathbb{J}_{\omega_-} \right) e^{-b}, \qquad \mathbb{J}_2 = e^b \left(\mathbb{J}_{\omega_+} \oplus \mathbb{J}_{I_{T_-}} \right) e^{-b}$$

$$\psi_1 = \overline{\Theta}_+ \wedge e^{b+\sqrt{-1}\omega_-} \qquad \psi_2 = e^{b+\sqrt{-1}\omega_+} \wedge \overline{\Theta}_-$$

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This is GK via:

$$(g, b, I, J), \qquad \mathbb{J}_1 = e^{-2\Omega}\begin{pmatrix} 0 & -F_-^{-1} \\ F_- & 0 \end{pmatrix}, \qquad \mathbb{J}_2 = \begin{pmatrix} 0 & -F_+^{-1} \\ F_+ & 0 \end{pmatrix}$$

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Calabi-type problem: Find a ‘canonical’ representative of a generalized Kähler class. We will impose the further restriction that the underlying generalized complex structures have holomorphically trivial canonical bundles, that is that they are globally defined by closed pure spinors.

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Conjecture

Let (M^{2n}, g, b, I, J) be a compact generalized Kähler manifold with holomorphically trivial canonical bundles. Then there exists a unique generalized Calabi-Yau geometry $(g_{CY}, b_{CY}, I, J_{CY}) \in [(g, b, I, J)]$, and furthermore (g_{CY}, I) and (g_{CY}, J_{CY}) are both Kähler Ricci-flat.

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In terms of the local potential theory, the equation $\Phi \equiv \lambda$ corresponds to a (generally) nonconvex fully nonlinear PDE.

Uniqueness and Kähler rigidity

For a generalized Kähler structure (g, b, I, J) we have the associated Bismut connections

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Given a GK structure $(\mathbb{J}_{\psi_1}, \mathbb{J}_{\psi_2})$, one has

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Thus gCY structures have vanishing Bismut Ricci forms.

Theorem

(Apostolov, Fu, S, Ustinovskiy, 2022) Compact generalized Calabi-Yau geometries (g, b, I, J) *satisfy that* (g, I) *and* (g, J) *are Kähler, Ricci-flat,*

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Aside: non-Kähler examples

Consider the standard Hopf surface

$$M^4 = \left(\mathbb{C}^2 - \{0\}\right) / \langle (z_1, z_2) \to (2z_1, 2z_2) \rangle \cong S^3 \times S^1.$$

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Theorem

(Cao, 1986) Let (M^{2n}, g, J) be a compact Kähler manifold with $c_1(M, J) = 0$. The solution to Kähler-Ricci flow with initial condition g exists for all time and converges to the unique Calabi-Yau metric in $[\omega]$.

Conjecture

Let (M^{2n}, g, b, I, J) *be a compact generalized Kähler manifold with holomorphically trivial canonical bundles. Then the GKRF with this initial data preserves the generalized Kähler class, exists for all time, and converges to the unique generalized Calabi-Yau geometry in this class.*

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Corollary

Let (M^{2n}, g, b, I, J) be a compact generalized Kähler manifold satisfying

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Let (M^{2n}, g, I) *be a Kähler Calabi-Yau manifold which is part of a generalized Calabi-Yau geometry* (g, b, I, J)*.*

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The structure of generalized Kähler classes

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Let (M^{2n}, g, I) be a Kähler Calabi-Yau manifold which is part of a generalized Calabi-Yau geometry (g, b, I, J). Then

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Let (M^{4n}, g, I, J, K) be a hyperKähler manifold. Let

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Then $\mathrm{Ham}_0^+(\omega_K) \subset \mathrm{Ham}^+(\omega_K)$ *is contractible:*

$$\mathrm{Ham}_0^+(\omega_K) \cong \star.$$

Part II: Proofs of Main Theorems

Formal structure of GKRF

Recall again the generalized Kähler-Ricci flow:

$$\frac{\partial}{\partial t} g = \ -2\,\mathrm{Rc} + \frac{1}{2} H^2, \qquad \frac{\partial}{\partial t} b = -d_g^* H, \qquad H = H_0 + db,$$
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Pulling back by the diffeomorphisms generated by $\theta_I^\sharp$ yields

$$\frac{\partial}{\partial t}\omega_I = -2\rho_I^{1,1}, \qquad \frac{\partial}{\partial t} J = L_{\theta_J^\sharp - \theta_I^\sharp} J, \qquad \frac{\partial}{\partial t}\beta = -2\rho_I^{2,0},$$

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$$\frac{\partial}{\partial t} \mathbb{J}_1 = -2[\mathbb{J}_1, \rho_I \mathbb{J}_1], \qquad \frac{\partial}{\partial t} \mathbb{J}_2 = -2[\mathbb{J}_2, \rho_I \mathbb{J}_2].$$

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Ricci potential estimates

Using the perturbation theorem we prove for instance the transgression formulas:

$$\rho_I = -\frac{1}{2} dJd\Phi, \qquad \rho_J = -\frac{1}{2} dId\Phi, \qquad \Phi = -\log \frac{(\psi_1, \overline{\psi}_1)}{(\psi_2, \overline{\psi}_2)}.$$

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Given $k \geq 0$ there exists a constant C such that

$$\sup_{M \times \{t\}} t \sum_{j=0}^{k} \left| \nabla_h^j \Upsilon(g,h) \right|^{\frac{2}{1+j}} \leq C,$$

where h is a background metric and $\Upsilon = \nabla^g - \nabla^h$ is the difference of Chern connections.

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$$G = \begin{pmatrix} g_{i\bar{j}} + \beta_{ik} \overline{\beta}_{\bar{j}\bar{l}} g^{\bar{l}k} & \sqrt{-1} \beta_{ip} g^{\bar{l}p} \\ -\sqrt{-1} \overline{\beta}_{\bar{j}\bar{p}} g^{\bar{p}k} & g^{\bar{l}k} \end{pmatrix}.$$

This is a Hermitian metric on $\mathcal{Q}$.

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This is a Hermitian metric on $\mathcal{Q}$. Surprisingly, the Bismut curvature of g is naturally identified with the Chern curvature of G. Moreover, if S^G denotes the Hermitian-Yang-Mills curvature of G, one has

$$S^G \equiv 0 \qquad \longleftrightarrow \qquad \rho_B \equiv 0$$

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It follows that, using uniform ellipticity bounds one can show

$$\square \left|\Upsilon(G, G_0)\right|^2_{g,G} \leq C\left(\left|\Upsilon(G, G_0)\right|^2_{g,G} + 1\right),$$

a generalization of Yau's C^3 estimate for the Calabi-Yau Theorem.

C^0 metric estimates

Next we establish the uniform parabolicity estimates when the flow is on a Kähler background.

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Note that this PDE has a gauge ambiguity replacing α by $\alpha + \partial\phi$. To resolve this gauge ambiguity we explicitly define a new system:

$$\Box\eta = -T\circ\overline{\partial}\eta,$$

$$\Box f = \operatorname{tr}_{\omega_t}\omega_{CY} + \log\frac{\det\omega_t}{\det\omega_{CY}}.$$

and define $\alpha = \eta - \sqrt{-1}\partial f$.

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Lemma

One has

$$\begin{aligned}
\Box \tfrac{\partial f}{\partial t} &= \left\langle \tfrac{\partial g}{\partial t}, \overline{\partial}\eta + \partial\overline{\eta} \right\rangle, \\
\Box |\eta|^2 &= -|\nabla\eta|^2 - \left|\overline{\nabla}\eta\right|^2 - \langle Q, \eta \otimes \overline{\eta}\rangle + 2\Re\left\langle \eta, T \circ \overline{\partial}\eta \right\rangle \leq 0, \\
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$$Q_{i\bar{j}} = g^{\bar{l}k} g^{\bar{q}p} T_{ik\bar{q}} T_{\bar{j}\bar{l}p}.$$

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$$r^k(M,g) := \inf_x \min \left\{ r_h(x,g)\,,\ \left(\sum_{i=0}^{k} |\nabla_g^i \operatorname{Rm}|^{\frac{1}{i+2}}(x) \right)^{-1},\ \left(\sum_{i=0}^{k} |\nabla_g^i H|^{\frac{1}{i+1}}(x) \right)^{-1} \right\},$$

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Proposition

There exists $\epsilon(n)$ so that if $(M^{2n}, g_t, b_t, I, J_t)$ is a GKRF on $[-4,0]$, satisfying

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Proof

Argue by contradiction, assuming there exist counterexamples as $\epsilon_i \to 0$. Point picking/blowup argument: produce a new sequence of solutions on $[-1,0]$ where:

$$r(g_0) = 1, \qquad r(g_\tau) = \tfrac{1}{2}, \qquad r(g_t) \geq \tfrac{1}{4},\ \ t \in [\tau - \tfrac{1}{4}, 0], \qquad |\nabla\Phi| \leq \epsilon_i$$

Convergence

$$r^k(M,g) := \inf_x \min\left\{ r_h(x,g)\,,\ \left(\sum_{i=0}^{k} |\nabla_g^i \operatorname{Rm}|^{\frac{1}{i+2}}(x)\right)^{-1},\ \left(\sum_{i=0}^{k} |\nabla_g^i H|^{\frac{1}{i+1}}(x)\right)^{-1} \right\},$$

Proposition

There exists $\epsilon(n)$ so that if $(M^{2n}, g_t, b_t, I, J_t)$ is a GKRF on $[-4,0]$, satisfying

1. $\sup_{M\times[-2,0]} |\nabla\Phi| \leq \epsilon(n)$,
2. $r^k(M, g_0) \geq 1$.

Then for every $t \in [-1,0]$, we have $r^k(M, g_t) > \frac{1}{2}$.

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These solutions have a limit which on the one hand is static since $\nabla\Phi \equiv 0$, but on the other hand must move due to the change in regularity scale.