The generalized Kähler Calabi-Yau problem

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January 2023
Part I: Background and Statements
Generalized Kähler Geometry

Given a smooth manifold $M$ with a closed three-form $H_0$, 

1. $I$ and $J$ are integrable complex structures
2. $g$ is compatible with both $I$ and $J$, yielding Kähler forms $\omega^I, \omega^J$.
3. One has $-d c^I \omega^I = H_0 + db = d c^J \omega^J$.

By Pontecorvo/Hitchin, there is an associated Poisson structure $\sigma = \frac{1}{2} [I, J] g^{-1} \in \Lambda^2 T^* M \cap \Lambda^2, 0+0, 2 I T^* M \cap \Lambda^2, 0+0, 2 J T^* M$.

A special case occurs when the pairing induced by $\sigma$ is nondegenerate. In this case one has that $\Omega = \sigma^{-1} \in \Lambda^2 T^* M \cap \Lambda^2, 0+0, 2 I T^* M \cap \Lambda^2, 0+0, 2 J T^* M$ is the real part of a holomorphic symplectic form.
Given a smooth manifold $M$ with a closed three-form $H_0$, a generalized Kähler structure is a quadruple $(g, b, I, J)$ such that

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$$\langle X + \xi, Y + \eta \rangle = \frac{1}{2}(\xi(Y) + \eta(X))$$

$$[X + \xi, Y + \eta] = [X, Y] + L_X \eta - i_Y d\xi + i_Y i_X H_0$$
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On this background a generalized complex structure is an almost complex structure $J$ on $E$ which is $\langle \cdot, \cdot \rangle$-orthogonal and whose $\sqrt{-1}$-eigenbundle $L^{1,0}$ is closed under the Dorfman bracket.
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On this background a generalized complex structure is an almost complex structure $J$ on $E$ which is $\langle \cdot, \cdot \rangle$-orthogonal and whose $\sqrt{-1}$-eigenbundle $L^{1,0}$ is closed under the Dorfman bracket. Two fundamental examples are:

\[
J_\omega := \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}, \quad L^{1,0} = \{ X - \sqrt{-1} \omega(X, \cdot) \mid X \in TM \},
\]
\[
J_J := \begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix}, \quad L^{1,0} = T^{1,0} M \oplus \wedge^{0,1}(M).
\]
Theorem
(Gualtieri 2004) Given \((M, H_0)\), a generalized Kähler structure \((g, b, I, J)\) is equivalent to a pair \((\mathcal{J}_1, \mathcal{J}_2)\) of generalized complex structures further satisfying

1. \([\mathcal{J}_1, \mathcal{J}_2] = 0\),
2. \(<-\mathcal{J}_1 \mathcal{J}_2 \cdot, \cdot>\) is positive definite.
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Explicitly,

\[
\mathbb{J}_{1/2} = \frac{1}{2} e^b \left( \begin{array}{cc} I \pm J & -\left(\omega^{-1}_I \mp \omega^{-1}_J\right) \\ \omega_I \mp \omega_J & -(I^* \pm J^*) \end{array} \right) e^{-b},
\]
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A spinor $\psi$ defines a generalized complex structure $J$ on $T \oplus T^*$ if
\[\text{Ker}(J - \sqrt{-1} \text{Id}) = \{X + \xi \in (T \oplus T^*) \otimes \mathbb{C} \mid (X + \xi) \cdot \psi = 0\}.\]
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Locally every generalized complex structure is described in this way, where $\psi$ is a nonvanishing section of the canonical bundle of $J$, then denoted $J_\psi$. 
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Integrability of $\mathbb{J}$ is equivalent to the existence of a section $X + \xi$ such that

$$d_{H_0} \psi = (X + \xi) \cdot \psi, \quad d_{H_0} := d + H_0 \wedge.$$
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The spinor is closed if $d_{H_0} \psi = 0$. 

Examples

Kähler metric \((g, J)\) as GK:

\[(g, 0, J, J), \quad \mathbb{J}_1 = \begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix}, \quad \mathbb{J}_2 = \begin{pmatrix} 0 & -\omega_j^{-1} \\ \omega_j & 0 \end{pmatrix}\]

\[\psi_1 = \bar{\Theta}, \quad \psi_2 = e^{\sqrt{-1}\omega_j}\]
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\[
(g, b, I, J), \quad \mathbb{J}_1 = e^b \left( \mathbb{J}_{I+} \oplus \mathbb{J}_{I-} \right) e^{-b}, \quad \mathbb{J}_2 = e^b \left( \mathbb{J}_{I+} \oplus \mathbb{J}_{I-} \right) e^{-b}
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\[
\psi_1 = \bar{\Theta}_+ \wedge e^{b+\sqrt{-1}\omega_-} \quad \psi_2 = e^{b+\sqrt{-1}\omega_+} \wedge \bar{\Theta}_-
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Examples

HyperKähler metric \((g, I, J, K)\) as GK.

Define 
\[ F^\pm = -2g(I \pm J) - \frac{1}{2}, \quad b = \frac{1}{2}F^+ (J - I), \quad \Omega = \sigma - 1 = 2g[I, J]. \]

This is GK via:
\[ (g, b, I, J_1, J_2) = e^{-2\Omega} (0 - F^- 1 - F^- 0), \quad \psi_1 = e^{2\Omega + \sqrt{-1}F^-}, \quad \psi_2 = e^{\sqrt{-1}F^-}. \]

Nondegenerate GK structure \((g, b, I, J)\). Define \(F^\pm, b\) as above and let \[ \Omega = \sigma - 1. \]

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HyperKähler metric \((g, I, J, K)\) as GK. Define

\[ F_\pm = -2g(I \pm J)^{-1}, \quad b = \frac{1}{2} F_+(J - I), \quad \Omega = \sigma^{-1} = 2g[I, J]^{-1} = \omega_K. \]
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(g, b, I, J), \quad J_1 = e^{-2\Omega} \begin{pmatrix} 0 & -F_-^{-1} \\ F_- & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & -F_+^{-1} \\ F_+ & 0 \end{pmatrix}
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Deformations

Kähler case: $(g, 0, I, I):

$$\omega_f I := \omega I + \frac{1}{2} \dd c I f$$

Commuting case: $(g, b, I, J)$:

$$\left(g + b\right) f I := \omega I + b I + \frac{1}{2} \dd c J f$$

In terms of the splitting $T = T_1 + T_2 - T_3$ this yields

$$\omega_f I = \omega I + \sqrt{-1} \left(\partial_1 + \partial_2 + \partial_3 - \partial_4 - \partial_5\right) f$$

$$b_f I = b I + \sqrt{-1} \left(\partial_1 - \partial_2 + \partial_3 + \partial_4 - \partial_5\right) f$$

Nondegenerate case: (Ω-Hamiltonian deformations after Joyce):

$$(g, b, I, J, \Omega) = \sigma - 1.$$ Given $f_t$ define $X_f t$ via $df_t = -X_f t \Omega$.

Let $\phi_t$ be the 1-parameter family of diffeomorphisms of $M$ generated by $X_f t$.

Then $(I, \phi_t^* J, \Omega)$ defines a unique nondegenerate generalized Kähler structure.

Note that the metric $g$ is recovered algebraically from the triple $(I, \phi_t^* J, \Omega)$, and is not given by diffeomorphism modification.
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In terms of the splitting \(T = T + T -\) this yields

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Kähler case: \((g, 0, l, l)\): \(\omega^f_I := \omega_I + \frac{1}{2} dd^c_I f\)

Commuting case: \((g, b, l, J)\):
\[(g + b)^f l := \omega_l + bl + \frac{1}{2} dd^c_l f\]

In terms of the splitting \(T = T_+ + T_-\) this yields
\[
\omega^f_I = \omega_I + \sqrt{-1} \left( \partial_+ \bar{\partial}_+ - \partial_- \bar{\partial}_- \right) f
\]
\[
b^f l = bl + \sqrt{-1} \left( \partial_+ \bar{\partial}_- - \bar{\partial}_+ \bar{\partial}_- \right) f
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Let \(\phi_t\) be the 1-parameter family of diffeomorphisms of \(M\) generated by \(X_{f_t}\).
Deformations

Kähler case: \((g, 0, I, I)\): \(\omega^f_I := \omega_I + \frac{1}{2} dd^c_I f\)

Commuting case: \((g, b, I, J)\):

\[(g + b)^f I := \omega_I + bI + \frac{1}{2} dd^c_J f\]

In terms of the splitting \(T = T_+ + T_-\) this yields

\[\omega^f_I = \omega_I + \sqrt{-1} \left( \partial_+ \bar{\partial}_+ - \partial_- \bar{\partial}_- \right) f\]

\[b^f I = bI + \sqrt{-1} \left( \partial_+ \partial_- - \bar{\partial}_+ \bar{\partial}_- \right) f\]

Nondegenerate case: (\(\Omega\)-Hamiltonian deformations after Joyce):

\((g, b, I, J), \Omega = \sigma^{-1}\). Given \(f_t\) define \(X_{f_t}\) via

\[df_t = - X_{f_t} \lrcorner \Omega.\]

Let \(\phi_t\) be the 1-parameter family of diffeomorphisms of \(M\) generated by \(X_{f_t}\). Then \((I, \phi_t^* J, \Omega)\) defines a unique nondegenerate generalized Kähler structure.
Deformations

Kähler case: \((g, 0, l, l)\): \(\omega^f_I := \omega_I + \frac{1}{2} dd^c_I f\)

Commuting case: \((g, b, l, J)\):

\[(g + b)^f l := \omega_I + bl + \frac{1}{2} dd^c_J f\]

In terms of the splitting \(T = T_+ + T_-\) this yields

\[
\omega^f_I = \omega_I + \sqrt{-1} \left( \partial_+ \partial_+ - \partial_- \overline{\partial}_- \right) f
\]

\[
b^f l = bl + \sqrt{-1} \left( \partial_+ \partial_- - \overline{\partial}_+ \overline{\partial}_- \right) f
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Nondegenerate case: \((\Omega\text{-Hamiltonian deformations after Joyce})\):

\((g, b, l, J), \Omega = \sigma^{-1}\). Given \(f_t\) define \(X_{f_t}\) via

\[df_t = - X_{f_t} \Omega.\]

Let \(\phi_t\) be the 1-parameter family of diffeomorphisms of \(M\) generated by \(X_{f_t}\). Then \((l, \phi^*_t J, \Omega)\) defines a unique nondegenerate generalized Kähler structure. Note that the metric \(g\) is recovered algebraically from the triple \((l, \phi^*_t J, \Omega)\), and is not given by diffeomorphism modification.
Deformations

General case:

\[ \partial \frac{\partial}{\partial t} \left( g_t + b_t \right) I_t = \frac{1}{2} \delta d c J_t f, \quad \partial \frac{\partial}{\partial t} J_t = \frac{1}{2} \sigma d c J_t f. \]

More generally, we replace \( \frac{1}{2} \delta d c J_t f \) by any family of exact 2-forms \( K_t \in \Lambda^1, J_t \).

Equivalently,

\[ \partial \frac{\partial}{\partial t} J_{1} = [J_{1}, K J_{1}], \quad \partial \frac{\partial}{\partial t} J_{2} = [J_{2}, K J_{2}]. \]

Definition

Given a GK structure \( (g, b, I, J) \), its generalized Kähler class is its equivalence class under canonical deformations, and is denoted \([ (g, b, I, J) ]\).
Deformations

General case: given $f_t$ a one-parameter family of functions, this defines a canonical deformation $(g_t, b_t, I, J_t)$ by

$$\frac{\partial}{\partial t} (g + b) \omega = \frac{1}{2} dd_{J_t}^c f, \quad \frac{\partial}{\partial t} J = \frac{1}{2} \sigma dd_{J_t}^c f$$
Deformations

General case: given $f_t$ a one-parameter family of functions, this defines a canonical deformation $(g_t, b_t, I, J_t)$ by

$$\frac{\partial}{\partial t} (g + b)I = \frac{1}{2} dd^c_{J_t} f, \quad \frac{\partial}{\partial t} J = \frac{1}{2} \sigma dd^c_{J_t} f$$

More generally, we replace $\frac{1}{2} dd^c_{J_t} f$ by any family of exact 2-forms $K_t \in \Lambda^{1,1}_{J_t}$. 
Deformations

General case: given $f_t$ a one-parameter family of functions, this defines a canonical deformation $(g_t, b_t, l, J_t)$ by

$$\frac{\partial}{\partial t} (g + b)l = \frac{1}{2} dd^c_{J_t} f, \quad \frac{\partial}{\partial t} J = \frac{1}{2} \sigma dd^c_{J_t} f$$

More generally, we replace $\frac{1}{2} dd^c_{J_t} f$ by any family of exact 2-forms $K_t \in \Lambda^1_{J_t}$. Equivalently,

$$\frac{\partial}{\partial t} J_1 = [J_1, KJ_1], \quad \frac{\partial}{\partial t} J_2 = [J_2, KJ_2].$$
Deformations

General case: given $f_t$ a one-parameter family of functions, this defines a canonical deformation $(g_t, b_t, I, J_t)$ by

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Definition
Given a GK structure $(g, b, I, J)$, its generalized Kähler class
Deformations

General case: given $f_t$ a one-parameter family of functions, this defines a canonical deformation $(g_t, b_t, I, J_t)$ by

$$\frac{\partial}{\partial t} (g + b) I = \frac{1}{2} dd_J^c f, \quad \frac{\partial}{\partial t} J = \frac{1}{2} \sigma dd_J^c f$$

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Definition

Given a GK structure $(g, b, I, J)$, its generalized Kähler class is its equivalence class under canonical deformations,
Deformations

General case: given $f_t$ a one-parameter family of functions, this defines a canonical deformation $(g_t, b_t, l, J_t)$ by

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More generally, we replace $\frac{1}{2} dd^c J_t f$ by any family of exact 2-forms $K_t \in \Lambda_{j_t}^{1,1}$. Equivalently,

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Definition

Given a GK structure $(g, b, l, J)$, its generalized Kähler class is its equivalence class under canonical deformations, and is denoted $[(g, b, l, J)]$. 
A Generalized Calabi-Yau problem

Calabi-type problem: Find a ‘canonical’ representative of a generalized Kähler class.
Calabi-type problem: Find a ‘canonical’ representative of a generalized Kähler class. We will impose the further restriction that the underlying generalized complex structures have **holomorphically trivial canonical bundles**, 

\[ \Phi := -\log (\psi_1, \psi_1) (\psi_2, \psi_2) = \lambda. \] 

**Conjecture** 

Let \((M, g, b, I, J)\) be a compact generalized Kähler manifold with holomorphically trivial canonical bundles. Then there exists a unique generalized Calabi-Yau geometry \((g_{CY}, b_{CY}, I, J_{CY}) \in [(g, b, I, J)]\), and furthermore \((g_{CY}, I)\) and \((g_{CY}, J_{CY})\) are both Kähler Ricci-flat.
A Generalized Calabi-Yau problem

Calabi-type problem: Find a ‘canonical’ representative of a generalized Kähler class. We will impose the further restriction that the underlying generalized complex structures have holomorphically trivial canonical bundles, that is that they are globally defined by closed pure spinors.
A Generalized Calabi-Yau problem

Calabi-type problem: Find a ‘canonical’ representative of a generalized Kähler class. We will impose the further restriction that the underlying generalized complex structures have holomorphically trivial canonical bundles, that is that they are globally defined by closed pure spinors. For spinors $\varphi, \psi$, the Mukai pairing is

$$ (\varphi, \psi) = (2\sqrt{-1})^{-n}[\varphi \wedge s(\psi)]_{\text{top}}. $$

For $(J_\varphi \psi_1, J_\psi \psi_2)$ a GK structure with holomorphically trivial canonical bundles, Gualtieri defines the generalized Calabi-Yau equation:

$$ \Phi := -\log (\varphi_1, \varphi_1)(\psi_2, \psi_2) = \lambda. $$
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For \((J_1 \psi_1, J_2 \psi_2)\) a GK structure with holomorphically trivial canonical bundles, Gualtieri defines the generalized Calabi-Yau equation:

\[
\Phi := -\log \frac{(\psi_1, \psi_1)}{(\psi_2, \psi_2)} = \lambda.
\]
A Generalized Calabi-Yau problem

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$$\Phi := -\log \frac{(\psi_1, \overline{\psi}_1)}{(\psi_2, \overline{\psi}_2)} = \lambda.$$

Conjecture

Let $(M^{2n}, g, b, l, J)$ be a compact generalized Kähler manifold with holomorphically trivial canonical bundles. Then there exists a unique generalized Calabi-Yau geometry $(g_{CY}, b_{CY}, l, J_{CY}) \in [(g, b, l, J)]$, and furthermore $(g_{CY}, l)$ and $(g_{CY}, J_{CY})$ are both Kähler Ricci-flat.
A Generalized Calabi-Yau problem

Kähler setting: $\psi_1 = \Theta, \psi_2 = e^{\sqrt{-1} \omega_J}$,
A Generalized Calabi-Yau problem

Kähler setting: $\psi_1 = \overline{\Theta}$, $\psi_2 = e^{\sqrt{-1} \omega J}$, and

$$\Phi = \log \frac{\omega_j^n}{\Theta \wedge \Theta}.$$
A Generalized Calabi-Yau problem

Kähler setting: $\psi_1 = \Theta$, $\psi_2 = e^{\sqrt{-1}\omega J}$, and

$$\Phi = \log \frac{\omega^m}{\Theta \wedge \Theta}.$$ 

Commuting type: $\psi_1 = \Theta_+ \wedge e^{b+\sqrt{-1}\omega_-}$, $\psi_2 = e^{b+\sqrt{-1}\omega_+} \wedge \Theta_-$,
A Generalized Calabi-Yau problem

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\]

Commuting type: \( \psi_1 = \overline{\Theta} \wedge e^{b+\sqrt{-1}\omega_-}, \psi_2 = e^{b+\sqrt{-1}\omega_+} \wedge \overline{\Theta}, \) and
\[
\Phi = \log \frac{\omega^k_+ \wedge \Theta_- \wedge \overline{\Theta}_-}{\Theta_+ \wedge \overline{\Theta}_+ \wedge \omega_-^l}.
\]
A Generalized Calabi-Yau problem

Kähler setting: \( \psi_1 = \overline{\Theta}, \psi_2 = e^{\sqrt{-1} \omega_J}, \) and

\[
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\]

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\[
\Phi = \log \frac{\omega^k_+ \wedge \Theta_- \wedge \overline{\Theta}_-}{\Theta_+ \wedge \overline{\Theta}_+ \wedge \omega^-_+}
\]

Nondegenerate case: \( \psi_1 = e^{2\Omega+\sqrt{-1} F_-}, \psi_2 = e^{\sqrt{-1} F_+}, \)
A Generalized Calabi-Yau problem

Kähler setting: \( \psi_1 = \Theta, \psi_2 = e^{\sqrt{-1} \omega J}, \) and

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Nondegenerate case: \( \psi_1 = e^{2\Omega+\sqrt{-1}F_-}, \psi_2 = e^{\sqrt{-1}F_+}, \) and

\[
\Phi = \log \frac{F^{2n}_+}{F^{2n}_-} = \log \frac{\det(I + J)}{\det(I - J)}
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A Generalized Calabi-Yau problem

Kähler setting: \( \psi_1 = \Theta, \psi_2 = e^{\sqrt{-1}\omega_J}, \) and

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\]

Nondegenerate case: \( \psi_1 = e^{2\Omega+\sqrt{-1}F_-}, \psi_2 = e^{\sqrt{-1}F_+}, \) and

\[
\Phi = \log \frac{F_+^{2n}}{F_-^{2n}} = \log \frac{\det(I+J)}{\det(I-J)}.
\]

In terms of the local potential theory, the equation \( \Phi \equiv \lambda \) corresponds to a (generally) nonconvex fully nonlinear PDE.
Uniqueness and Kähler rigidity

For a generalized Kähler structure \((g, b, l, J)\) we have the associated Bismut connections

\[
\nabla^l = \nabla + \frac{1}{2} g^{-1} H, \quad \nabla^J = \nabla - \frac{1}{2} g^{-1} H.
\]
Uniqueness and Kähler rigidity

For a generalized Kähler structure \((g, b, I, J)\) we have the associated Bismut connections

\[
\nabla^I = \nabla + \frac{1}{2} g^{-1} H, \quad \nabla^J = \nabla - \frac{1}{2} g^{-1} H.
\]

Denote the curvatures by \(\Omega^I, \Omega^J\), and define Bismut Ricci forms

\[
\rho_I = \frac{1}{2} \text{tr} \Omega^I \circ I, \quad \rho_J = \frac{1}{2} \text{tr} \Omega^J \circ J.
\]
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These are closed forms such that \(\rho_I \in 2\pi c_1(M, I), \rho_J \in 2\pi c_1(M, J)\).
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Proposition

Given a GK structure \((J_{\psi_1}, J_{\psi_2})\), one has

\[
\rho_I = -\frac{1}{2} dJ d\Phi, \quad \rho_J = -\frac{1}{2} dI d\Phi.
\]
Uniqueness and Kähler rigidity

For a generalized Kähler structure \((g, b, l, J)\) we have the associated Bismut connections

\[
\nabla^l = \nabla + \frac{1}{2} g^{-1} H, \quad \nabla^J = \nabla - \frac{1}{2} g^{-1} H.
\]

Denote the curvatures by \(\Omega^l, \Omega^J\), and define Bismut Ricci forms

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\rho_l = \frac{1}{2} \text{tr} \Omega^l \circ l, \quad \rho_J = \frac{1}{2} \text{tr} \Omega^J \circ J.
\]

These are closed forms such that \(\rho_l \in 2\pi c_1(M, l), \rho_J \in 2\pi c_1(M, J)\). A key starting point for our work is the following transgression formula:

**Proposition**

*Given a GK structure \((\mathcal{J}_\psi_1, \mathcal{J}_\psi_2)\), one has*

\[
\rho_l = -\frac{1}{2} dJd\Phi, \quad \rho_J = -\frac{1}{2} dld\Phi.
\]

Thus gCY structures have vanishing Bismut Ricci forms.
Theorem

(Apostolov, Fu, S, Ustinovskiy, 2022) Compact generalized Calabi-Yau geometries \((g, b, I, J)\) satisfy that \((g, I)\) and \((g, J)\) are Kähler, Ricci-flat, and are furthermore unique in their GK class.

Proof

The proof exploits the partial Ricci potentials \(\Psi_i = -\log (\psi_i, \psi_i) dV_g\), \(\Phi = \Psi_1 - \Psi_2\).

For a gCY geometry one has

\[-\Delta \Psi_1 + |d \Psi_1|^2 = \frac{1}{6} |H|^2.\]

By the strong maximum principle, \(\Psi_1\) is constant and \(H\) vanishes.

Uniqueness follows by showing that a gCY structure has \([\omega_I]\) and \([\omega_J]\) uniquely determined by the spinor classes \([\psi_1]\), \([\psi_2]\), using the Beauville-Bogomolov decomposition.
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Aside: non-Kähler examples

Consider the standard Hopf surface

\[ M^4 = \left( \mathbb{C}^2 - \{0\} \right) / \langle (z_1, z_2) \rightarrow (2z_1, 2z_2) \rangle \cong S^3 \times S^1. \]
Aside: non-Kähler examples

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Let \( I \) denote induced complex structure and let \( J := j^* I \), where

\[ j(z_1, z_2) = \left( \frac{\overline{z_2}}{|z_1|^2 + |z_2|^2}, \frac{z_1}{|z_1|^2 + |z_2|^2} \right). \]
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Furthermore let $g = g_{S^3} \oplus g_{S^1}.$
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Let \( I \) denote induced complex structure and let \( J := j^* I \), where

\[ j(z_1, z_2) = \begin{pmatrix} \overline{z_2} & z_1 \\ \frac{|z_1|^2 + |z_2|^2}{|z_1|^2 + |z_2|^2} & \frac{|z_1|^2 + |z_2|^2}{|z_1|^2 + |z_2|^2} \end{pmatrix}. \]

Furthermore let \( g = g_{S^3} \oplus g_{S^1} \). Then \((M, g, I, J)\) is a GK structure, which is Bismut-Ricci flat.
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Furthermore let \( g = g_{S^3} \oplus g_{S^1} \). Then \( (M, g, I, J) \) is a GK structure, which is Bismut-Ricci flat. The Poisson tensor \( \sigma \) is

\[ \sigma = -\Re \left( z_1 \frac{\partial}{\partial z_1} \wedge z_2 \frac{\partial}{\partial z_2} \right). \]
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The degeneracy loci are the elliptic curves \( \{z_1 = 0\} \), \( \{z_2 = 0\} \) where \( I = -J \) and \( I = J \), respectively.
Aside: non-Kähler examples

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Furthermore let \( g = g_{S^3} \oplus g_{S^1} \). Then \( (M, g, I, J) \) is a GK structure, which is Bismut-Ricci flat. The Poisson tensor \( \sigma \) is

\[ \sigma = -\Re \left( z_1 \frac{\partial}{\partial z_1} \wedge z_2 \frac{\partial}{\partial z_2} \right). \]

The degeneracy loci are the elliptic curves \( \{z_1 = 0\}, \{z_2 = 0\} \) where \( I = -J \) and \( I = J \), respectively. Near these loci the structure cannot be described by closed spinors.
Generalized Kähler-Ricci flow

A one-parameter family \((g_t, b_t, I_t, J_t)\) satisfies generalized Kähler-Ricci flow

\[
\frac{\partial}{\partial t} g_t = -2 \text{Rc} + \frac{1}{2} H^2,
\]

\[
\frac{\partial}{\partial t} b_t = - d^* g H,
\]

\[
H^2(X, Y) = \langle i_X H, i_Y H \rangle,
\]

\[
\frac{\partial}{\partial t} I_t = L_{\theta I} I_t,
\]

\[
\frac{\partial}{\partial t} J_t = L_{\theta J} J_t.
\]

where \(H^2(X, Y)\) and \(\theta_I, \theta_J\) are the Lee forms.

• One can modify by diffeomorphisms to fix \(I\).
• This flow is a special case of pluriclosed flow, which in turn is a special case of generalized Ricci flow (RG flow).

**Theorem** (Cao, 1986) Let \((M^{2n}, g, J)\) be a compact Kähler manifold with \(c_1(M, J) = 0\).

The solution to Kähler-Ricci flow with initial condition \(g\) exists for all time and converges to the unique Calabi-Yau metric in \([\omega]\).
A one-parameter family \((g_t, b_t, I_t, J_t)\) satisfies **generalized Kähler-Ricci flow** if

\[
\frac{\partial}{\partial t} g = -2 \text{Rc} + \frac{1}{2} H^2, \quad \frac{\partial}{\partial t} b = -d^* H, \quad H = H_0 + db,
\]

\[
\frac{\partial}{\partial t} I = L_{\theta_I^*} I, \quad \frac{\partial}{\partial t} J = L_{\theta_J^*} J,
\]

where \(H^2(X, Y) = \langle i_X H, i_Y H \rangle\), and \(\theta_I = Id_g^* \omega_I, \theta_J = Id_g^* \omega_J\) are the Lee forms.
Generalized Kähler-Ricci flow

A one-parameter family \((g_t, b_t, l_t, J_t)\) satisfies generalized Kähler-Ricci flow if

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\]

\[
\frac{\partial}{\partial t} l = L_{\theta^*_l} l, \quad \frac{\partial}{\partial t} J = L_{\theta^*_J} J,
\]

where \(H^2(X, Y) = \langle i_X H, i_Y H \rangle\), and \(\theta_l = \text{Id}^*_g \omega_l, \theta_J = Jd^*_g \omega_J\) are the Lee forms.

- One can modify by diffeomorphisms to fix \(l\).
Generalized Kähler-Ricci flow

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\frac{\partial}{\partial t} g = -2 \text{Rc} + \frac{1}{2} H^2, \quad \frac{\partial}{\partial t} b = -d^* H, \quad H = H_0 + db, \\
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- One can modify by diffeomorphisms to fix \(I\).
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Theorem

(Cao, 1986) Let \((M^{2n}, g, J)\) be a compact Kähler manifold with \(c_1(M, J) = 0\).
Generalized Kähler-Ricci flow

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\frac{\partial}{\partial t} l = L_{\theta_l}^\# l, \quad \frac{\partial}{\partial t} J = L_{\theta_J}^\# J,
\]

where \(H^2(X, Y) = \langle i_X H, i_Y H \rangle\), and \(\theta_l = l d_g^* \omega_l, \theta_J = J d_g^* \omega_J\) are the Lee forms.

- One can modify by diffeomorphisms to fix \(l\).
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Theorem

(Cao, 1986) Let \((M^{2n}, g, J)\) be a compact Kähler manifold with \(c_1(M, J) = 0\).

The solution to Kähler-Ricci flow with initial condition \(g\) exists for all time and converges to the unique Calabi-Yau metric in \([\omega]\).
Conjecture

Let \((M^{2n}, g, b, I, J)\) be a compact generalized Kähler manifold with holomorphically trivial canonical bundles. Then the GKRF with this initial data preserves the generalized Kähler class, exists for all time, and converges to the unique generalized Calabi-Yau geometry in this class.
Generalized Kähler-Ricci flow

Conjecture

Let \((M^{2n}, g, b, I, J)\) be a compact generalized Kähler manifold with holomorphically trivial canonical bundles. Then the GKRF with this initial data preserves the generalized Kähler class, exists for all time, and converges to the unique generalized Calabi-Yau geometry in this class.

Theorem

(AFSU 2022) Given a solution \((g_t, b_t, I_t, J_t)\) to generalized Kähler-Ricci flow with initial data defined by holomorphically trivial canonical bundles one has:
Generalized Kähler-Ricci flow

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Theorem
(AFSU 2022) Given a solution \((g_t, b_t, I, J_t)\) to generalized Kähler-Ricci flow with initial data defined by holomorphically trivial canonical bundles one has:

1. The canonical bundles of \(J^\xi_i\) are holomorphically trivial for all time, defined by closed pure spinors \(\psi^\xi_i \in [\psi^0_i]\). Furthermore the GK class is preserved.
Generalized Kähler-Ricci flow

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Let \((M^{2n}, g, b, l, J)\) be a compact generalized Kähler manifold with holomorphically trivial canonical bundles. Then the GKRF with this initial data preserves the generalized Kähler class, exists for all time, and converges to the unique generalized Calabi-Yau geometry in this class.

Theorem
(AFSU 2022) Given a solution \((g_t, b_t, l, J_t)\) to generalized Kähler-Ricci flow with initial data defined by holomorphically trivial canonical bundles one has:

1. The canonical bundles of \(J_i^c\) are holomorphically trivial for all time, defined by closed pure spinors \(\psi_i^c \in [\psi_i^0]\). Furthermore the GK class is preserved.
2. One has Ricci potential bounds

\[
\sup_{M \times \{t\}} \left( \Phi^2 + t |\nabla \Phi|^2 \right) \leq \sup_{M \times \{0\}} \Phi^2.
\]
Generalized Kähler-Ricci flow

Conjecture
Let \((M^{2n}, g, b, I, J)\) be a compact generalized Kähler manifold with holomorphically trivial canonical bundles. Then the GKRF with this initial data preserves the generalized Kähler class, exists for all time, and converges to the unique generalized Calabi-Yau geometry in this class.

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(AF SU 2022) Given a solution \((g_t, b_t, I, J_t)\) to generalized Kähler–Ricci flow with initial data defined by holomorphically trivial canonical bundles one has:

1. The canonical bundles of \(\mathbb{J}_i^t\) are holomorphically trivial for all time, defined by closed pure spinors \(\psi_i^t \in [\psi_i^0]\). Furthermore the GK class is preserved.
2. One has Ricci potential bounds

\[
\sup_{M \times \{t\}} \left( \Phi^2 + t |\nabla \Phi|^2 \right) \leq \sup_{M \times \{0\}} \Phi^2.
\]

3. There exist Mabuchi-type functionals \(M_i := \int_M \Phi(\psi_i, \bar{\psi}_i)\) whose only critical points are generalized Calabi-Yau geometries, and which are bounded and monotone along GKRF.
Generalized Kähler-Ricci flow

Theorem
(AFSU 2022) Let \((M^{2n}, g, b, I, J)\) be a compact generalized Kähler manifold with holomorphically trivial canonical bundles.
Generalized Kähler-Ricci flow

Theorem
(AFU 2022) Let \((M^{2n}, g, b, I, J)\) be a compact generalized Kähler manifold with holomorphically trivial canonical bundles.

1. Suppose \((M, I)\) is a Kähler manifold.
Theorem
(AF SU 2022) Let \((M^{2n}, g, b, I, J)\) be a compact generalized Kähler manifold with holomorphically trivial canonical bundles.

1. \textit{Suppose} \((M, I)\) is a Kähler manifold. \textit{Then the solution to generalized Kähler-Ricci flow with initial data} \((g, b, I, J)\) \textit{exists for all time, and the Mabuchi energies converge to their topologically determined extreme values.}
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(AFSU 2022) Let \((M^{2n}, g, b, I, J)\) be a compact generalized Kähler manifold with holomorphically trivial canonical bundles.

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2. Suppose there exists a generalized Calabi-Yau geometry in \([g, b, I, J]\).

Corollary
Let \((M^{2n}, g, b, I, J)\) be a compact generalized Kähler manifold satisfying

1. \(\sigma = 0\),
2. \((M, I)\) is Kähler and \(c_1(M, I) = 0\).

Then the solution to generalized Kähler-Ricci flow with initial data \((g, b, I, J)\) exists for all time and converges to a Kähler Calabi-Yau metric.
Generalized Kähler-Ricci flow

Theorem
(AF SU 2022) Let \((M^{2n}, g, b, I, J)\) be a compact generalized Kähler manifold with holomorphically trivial canonical bundles.

1. Suppose \((M, I)\) is a Kähler manifold. Then the solution to generalized Kähler-Ricci flow with initial data \((g, b, I, J)\) exists for all time, and the Mabuchi energies converge to their topologically determined extreme values.

2. Suppose there exists a generalized Calabi-Yau geometry in \([(g, b, I, J)]\). Then the solution to generalized Kähler-Ricci flow with initial data \((g, b, I, J)\) converges exponentially to this necessarily unique generalized Calabi-Yau geometry.
Generalized Kähler-Ricci flow

Theorem
(AF SU 2022) Let \((M^{2n}, g, b, I, J)\) be a compact generalized Kähler manifold with holomorphically trivial canonical bundles.

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Corollary
Let \((M^{2n}, g, b, I, J)\) be a compact generalized Kähler manifold satisfying

1. \(\sigma = 0\),
Generalized Kähler-Ricci flow

**Theorem**
(AF SU 2022) Let $(M^{2n}, g, b, I, J)$ be a compact generalized Kähler manifold with holomorphically trivial canonical bundles.

1. **Suppose** $(M, I)$ is a Kähler manifold. Then the solution to generalized Kähler-Ricci flow with initial data $(g, b, I, J)$ exists for all time, and the Mabuchi energies converge to their topologically determined extreme values.

2. **Suppose there exists a generalized Calabi-Yau geometry in** $[(g, b, I, J)]$. Then the solution to generalized Kähler-Ricci flow with initial data $(g, b, I, J)$ converges exponentially to this necessarily unique generalized Calabi-Yau geometry.

**Corollary**
Let $(M^{2n}, g, b, I, J)$ be a compact generalized Kähler manifold satisfying

1. $\sigma = 0$,
2. $(M, I)$ is Kähler and $c_1(M, I) = 0$. 

Generalized Kähler-Ricci flow

Theorem

(AFSU 2022) Let \((M^{2n}, g, b, I, J)\) be a compact generalized Kähler manifold with holomorphically trivial canonical bundles.

1. Suppose \((M, I)\) is a Kähler manifold. Then the solution to generalized Kähler-Ricci flow with initial data \((g, b, I, J)\) exists for all time, and the Mabuchi energies converge to their topologically determined extreme values.

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Let \((M^{2n}, g, b, I, J)\) be a compact generalized Kähler manifold satisfying

1. \(\sigma = 0\),

2. \((M, I)\) is Kähler and \(c_1(M, I) = 0\).

Then the solution to generalized Kähler-Ricci flow with initial data \((g, b, I, J)\) exists for all time and converges to a Kähler Calabi-Yau metric.
The structure of generalized Kähler classes

Corollary

Let \((M^{2n}, g, I)\) be a Kähler Calabi-Yau manifold which is part of a generalized Calabi-Yau geometry \((g, b, I, J)\).
Corollary

Let $(M^{2n}, g, I)$ be a Kähler Calabi-Yau manifold which is part of a generalized Calabi-Yau geometry $(g, b, I, J)$. Then

$$[(g, b, I, J)] \cong \ast.$$
Corollary
Let \((M^{2n}, g, l)\) be a Kähler Calabi-Yau manifold which is part of a generalized Calabi-Yau geometry \((g, b, l, J)\). Then
\[
[(g, b, l, J)] \cong *. 
\]

Corollary
Let \((M^{4n}, g, l, J, K)\) be a hyperKähler manifold. Let
\[
\text{Ham}^+(\omega_K) := \{ \phi \in \text{Ham}(\omega_K) \mid \phi^* \omega_I(X, lX) > 0 \text{ for nonzero } X \in TM \}.
\]
The structure of generalized Kähler classes

Corollary
Let \((M^{2n}, g, I)\) be a Kähler Calabi-Yau manifold which is part of a generalized Calabi-Yau geometry \((g, b, I, J)\). Then
\[
[(g, b, I, J)] \cong \ast.
\]

Corollary
Let \((M^{4n}, g, I, J, K)\) be a hyperKähler manifold. Let
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\Ham^+(\omega_K) := \{ \phi \in \Ham(\omega_K) | \phi^* \omega_I(X, IX) > 0 \text{ for nonzero } X \in TM \}.
\]
Then \(\Ham_0^+(\omega_K) \subset \Ham^+(\omega_K)\) is contractible:
\[
\Ham_0^+(\omega_K) \cong \ast.
\]
Part II: Proofs of Main Theorems
Formal structure of GKRF

Recall again the generalized Kähler-Ricci flow:

\[
\frac{\partial}{\partial t} g = -2 \text{Rc} + \frac{1}{2} H^2, \quad \frac{\partial}{\partial t} b = -d^* H, \quad H = H_0 + db,
\]

\[
\frac{\partial}{\partial t} I = L_{\theta^\#} I, \quad \frac{\partial}{\partial t} J = L_{\theta^\#} J,
\]
Formal structure of GKRF

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$$\frac{\partial}{\partial t} g = -2 \text{Rc} + \frac{1}{2} H^2, \quad \frac{\partial}{\partial t} b = -d^* H, \quad H = H_0 + db,$$

$$\frac{\partial}{\partial t} I = L_{\theta^\#_I} I, \quad \frac{\partial}{\partial t} J = L_{\theta^\#_J} J,$$

Pulling back by the diffeomorphisms generated by $\theta^\#_I$ yields

$$\frac{\partial}{\partial t} \omega_I = -2 \rho^{1,1}_I, \quad \frac{\partial}{\partial t} J = L_{\theta^\#_J - \theta^\#_I} J, \quad \frac{\partial}{\partial t} \beta = -2 \rho^{2,0}_I,$$

where $\beta = \sqrt{-1} b^{2,0}_I$. 
Formal structure of GKRF

Recall again the generalized Kähler-Ricci flow:

\[
\frac{\partial}{\partial t} g = -2 \text{Rc} + \frac{1}{2} H^2, \quad \frac{\partial}{\partial t} b = -d_g^* H, \quad H = H_0 + db,
\]

\[
\frac{\partial}{\partial t} I = L_{\theta_i} I, \quad \frac{\partial}{\partial t} J = L_{\theta_j} J,
\]

Pulling back by the diffeomorphisms generated by \(\theta_i\) yields

\[
\frac{\partial}{\partial t} \omega_i = -2\rho_i^{1,1}, \quad \frac{\partial}{\partial t} J = L_{\theta_j} - \theta_i J, \quad \frac{\partial}{\partial t} \beta = -2\rho_i^{2,0},
\]

where \(\beta = \sqrt{-1} b_2^{2,0}\). Equivalently,

\[
\frac{\partial}{\partial t} \mathbb{J}_1 = -2[\mathbb{J}_1, \rho_i \mathbb{J}_1], \quad \frac{\partial}{\partial t} \mathbb{J}_2 = -2[\mathbb{J}_2, \rho_i \mathbb{J}_2].
\]
Formal structure of GKRF

Recall again the generalized Kähler-Ricci flow:

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\]

\[
\frac{\partial}{\partial t} l = L_{\theta^\#} l, \quad \frac{\partial}{\partial t} J = L_{\theta^\#} J,
\]

Pulling back by the diffeomorphisms generated by $\theta^\#$ yields

\[
\frac{\partial}{\partial t} \omega_1 = -2 \rho_1^{1,1}, \quad \frac{\partial}{\partial t} J = L_{\theta^\# - \theta_i^\#} J, \quad \frac{\partial}{\partial t} \beta = -2 \rho_1^{2,0},
\]

where $\beta = \sqrt{-1} b_i^{2,0}$. Equivalently,

\[
\frac{\partial}{\partial t} \mathcal{J}_1 = -2[\mathcal{J}_1, \rho_1 \mathcal{J}_1], \quad \frac{\partial}{\partial t} \mathcal{J}_2 = -2[\mathcal{J}_2, \rho_1 \mathcal{J}_2].
\]

Equivalently,

\[
\frac{\partial}{\partial t} \psi_1 = -2 \sqrt{-1} \rho_1 \wedge \psi_1, \quad \frac{\partial}{\partial t} \psi_2 = -2 \sqrt{-1} \rho_1 \wedge \psi_2.
\]
The local structure of GK manifolds

Theorem

Let $(J_1, J_2)$ be a generalized Kähler structure with both $J_1$ and $J_2$ of even type. Near any point where both $J_1$ and $J_2$ have locally constant type, there exist spinors $\psi_1, \psi_2$ defining $J_1$ and $J_2$ and a sequence of nondegenerate generalized Kähler structures $(J_{\psi_j 1}, J_{\psi_j 2})$ such that

$$\lim_{j \to \infty} J_{\psi_j 1} = J_1,$$

$$\lim_{j \to \infty} J_{\psi_j 2} = J_2.$$

1. Gualtieri's Darboux theorem: locally $J \sim J_{\text{complex}} \oplus J_{\text{symplectic}}$

2. Hitchin: deform GC structure of complex type to one of symplectic type:

$$\Omega = \eta \rightarrow \psi_t = t \eta \exp(\eta/t),$$

$$\lim_{t \to 0} \psi_t = \Omega.$$

3. Goto's Kodaira-Spencer-type stability theorem: augment variation of $J_1$ with a variation of $J_2$ which preserves GK. We prove a local version of this, relying on a $H^0$-twisted Hodge decomposition on the closed ball.
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Let \((\mathcal{J}_1, \mathcal{J}_2)\) be a generalized Kähler structure with both \(\mathcal{J}_1\) and \(\mathcal{J}_2\) of even type. Near any point where both \(\mathcal{J}_1\) and \(\mathcal{J}_2\) have locally constant type,

1. Gualtieri’s Darboux theorem: locally \(\mathcal{J} \sim \mathcal{J}_{\text{complex}} \oplus \mathcal{J}_{\text{symplectic}}\)
2. Hitchin: deform GC structure of complex type to one of symplectic type: \(\Omega = \eta \rightarrow \psi_t = t^k \exp(\eta/t), \lim_{t \to 0} \psi_t = \Omega\)
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\[
\lim_{j \to \infty} J_{\psi_1}^j = J_1, \quad \lim_{j \to \infty} J_{\psi_2}^j = J_2.
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Theorem

Let \((\mathbb{J}_1, \mathbb{J}_2)\) be a generalized Kähler structure with both \(\mathbb{J}_1\) and \(\mathbb{J}_2\) of even type. Near any point where both \(\mathbb{J}_1\) and \(\mathbb{J}_2\) have locally constant type, there exist spinors \(\psi_1, \psi_2\) defining \(\mathbb{J}_1\) and \(\mathbb{J}_2\) and a sequence of nondegenerate generalized Kähler structures \((\mathbb{J}_{\psi_1}^j, \mathbb{J}_{\psi_2}^j)\) such that

\[
\lim_{j \to \infty} \mathbb{J}_{\psi_1}^j = \mathbb{J}_1, \quad \lim_{j \to \infty} \mathbb{J}_{\psi_2}^j = \mathbb{J}_2.
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The local structure of GK manifolds

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Let $(\mathbb{J}_1, \mathbb{J}_2)$ be a generalized Kähler structure with both $\mathbb{J}_1$ and $\mathbb{J}_2$ of even type. Near any point where both $\mathbb{J}_1$ and $\mathbb{J}_2$ have locally constant type, there exist spinors $\psi_1$, $\psi_2$ defining $\mathbb{J}_1$ and $\mathbb{J}_2$ and a sequence of nondegenerate generalized Kähler structures $(\mathbb{J}_{\psi_1^j}, \mathbb{J}_{\psi_2^j})$ such that

$$\lim_{j \to \infty} \mathbb{J}_{\psi_1^j} = \mathbb{J}_1, \quad \lim_{j \to \infty} \mathbb{J}_{\psi_2^j} = \mathbb{J}_2.$$

1. Gualtieri’s Darboux theorem: locally

$$\mathbb{J} \sim \mathbb{J}_{\text{complex}} \oplus \mathbb{J}_{\text{symplectic}}$$

2. Hitchin: deform GC structure of complex type to one of symplectic type:

$$\Omega = \eta^k \longrightarrow \psi_t = t^k \exp(\eta/t), \quad \lim_{t \to 0} \psi_t = \Omega.$$

3. Goto’s Kodaira-Spencer-type stability theorem: augment variation of $\mathbb{J}_1$ with a variation of $\mathbb{J}_2$ which preserves GK.
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3. Goto’s Kodaira-Spencer-type stability theorem: augment variation of \(\mathbb{J}_1\) with a variation of \(\mathbb{J}_2\) which preserves GK. We prove a local version of this, relying on a \(H_0\)-twisted Hodge decomposition on the closed ball.
Ricci potential estimates

Using the perturbation theorem we prove for instance the transgression formulas:

\[ \rho_I = -\frac{1}{2} dJd\Phi, \quad \rho_J = -\frac{1}{2} dld\Phi, \quad \Phi = -\log \frac{(\psi_1, \overline{\psi}_1)}{(\psi_2, \overline{\psi}_2)}. \]
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\[ \Box \left( t |\nabla \Phi|^2 + \Phi^2 \right) \leq 0 \]
Higher order regularity

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Theorem
(S, 2014, Jordan, Garcia-Fernandez, S, 2021) Let \((M^{2n}, J)\) be a compact complex manifold. Suppose \((g_t, \beta_t)\) is a solution to the pluriclosed flow on \([0, 1]\) such that

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\lambda g_0 \leq g_t \leq \Lambda g_0, \quad |\beta| \leq \Lambda.
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Given \(k \geq 0\) there exists a constant \(C\) such that

\[
\sup_{M \times \{t\}} t \sum_{j=0}^{k} \left| \nabla^j h \Upsilon (g, h) \right|^{2 (1+j)} \leq C,
\]

where \(h\) is a background metric and \(\Upsilon = \nabla^g - \nabla^h\) is the difference of Chern connections.
Pluriclosed flow and holomorphic Courant algebroids

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Given now another pluriclosed metric, suppose $\partial \omega - \partial \omega_0 = \partial \beta$, and define $G = (g_{ij} + \beta_{ik} \beta_{jl} g_{lk} - \beta_{jp} g_{pk} g_{lk})$.

This is a Hermitian metric on $Q$.

Surprisingly, the Bismut curvature of $g$ is naturally identified with the Chern curvature of $G$.

Moreover, if $S_G$ denotes the Hermitian-Yang-Mills curvature of $G$, one has $S_G \equiv 0 \iff \rho_B \equiv 0$. 
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$$\overline{\partial}^{\omega_0}(X + \xi) = \overline{\partial}X + \overline{\partial}\xi + \sqrt{-1}iX \partial \omega_0.$$
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$$ S^G \equiv 0 \quad \longleftrightarrow \quad \rho_B \equiv 0 $$
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Furthermore, if \((\omega_t, \beta_t)\) is a solution of pluriclosed flow, then

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Furthermore, if \((\omega_t, \beta_t)\) is a solution of pluriclosed flow, then
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It follows that, using uniform ellipticity bounds one can show
\[ \square |\uptau(G, G_0)|^2_{g,G} \leq C \left( |\uptau(G, G_0)|^2_{g,G} + 1 \right), \]

a generalization of Yau’s \(C^3\) estimate for the Calabi-Yau Theorem.
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Note that this PDE has a gauge ambiguity replacing $\alpha$ by $\alpha + \partial \phi$. To resolve this gauge ambiguity we explicitly define a new system:

$$\Box \eta = - T \circ \bar{\partial} \eta,$$

$$\Box f = \text{tr}_{\omega_t} \omega_{CY} + \log \frac{\det \omega_t}{\det \omega_{CY}}.$$

and define $\alpha = \eta - \sqrt{-1} \partial f$. 

\[C^0\text{ metric estimates}\]
Lemma
One has

\[\Box \frac{\partial f}{\partial t} = \langle \frac{\partial g}{\partial t}, \overline{\partial \eta} + \partial \overline{\eta} \rangle,\]

\[\Box |\eta|^2 = -|\nabla \eta|^2 - |\nabla \overline{\eta}|^2 - \langle Q, \eta \otimes \overline{\eta} \rangle + 2\Re \langle \eta, T \circ \overline{\partial \eta} \rangle \leq 0,\]

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Maximum principle: a priori upper bound for $\eta$ and $\partial \eta$.

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One has

\[ \square \log \frac{\omega_t^n}{\omega_{CY}^n} = |T|^2 \]
\[ \square \log \text{tr}_{\omega_{CY}} \omega_t \leq |T|^2 + C \text{tr}_{\omega_t} \omega_{CY}. \]
The $C^0$ estimates then follow from a long series of maximum principles:

1. $\log_2 \omega_n t \omega_n CY = |T|_2 \rightarrow \omega_n \geq C^{-1} \omega_n CY$.
2. $\log tr CY \omega_t + |\partial \eta|_2 \leq 0 \rightarrow tr \omega CY \omega_t \leq C e^{-C (f + t)}$.

Together these yield a priori estimates for the metric in terms of an upper bound for $f$. Recall $\partial f \partial t = \langle \partial g \partial t, \partial \eta + \partial \eta \rangle$. Here we use the favorable evolution equation for $|\nabla \Phi|^2$, and the fact that $\partial g \partial t = dJ d \Phi \sim \nabla^2 \Phi + T \ast \nabla \Phi$, to obtain $\square (\partial f \partial t + |\eta|^2 + A_1 |\partial \eta|^2 + A_2 |\nabla \Phi|^2) \leq 0$. 

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3. $\Box \log \text{tr}_{\omega_{CY}} \omega_t + |\partial \eta|^2 - Cf \leq C \longrightarrow \text{tr}_{\omega_{CY}} \omega_t \leq Ce^{C(f+t)}$. 

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Here we use the favorable evolution equation for |\nabla \Phi|^2, and the fact that \frac{\partial g}{\partial t} = dJd\Phi \sim \nabla^2 \Phi + T \ast \nabla \Phi,
The $C^0$ estimates then follow from a long series of maximum principles:

1. $\Box \log \frac{\omega^n_T}{\omega^n_{CY}} = |T|^2 \geq 0 \rightarrow \omega^n \geq C^{-1} \omega^n_{CY}$.

2. $\Box \log \frac{\omega^n_T}{\omega^n_{CY}} + |\partial \eta|^2 \leq 0 \rightarrow \omega^n \leq C \omega^n_{CY}$.

3. $\Box \log \text{tr}_{\omega_{CY}} \omega_t + |\partial \eta|^2 - Cf \leq C \rightarrow \text{tr}_{\omega_{CY}} \omega_t \leq Ce^{C(f+t)}$.

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Here we use the favorable evolution equation for $|\nabla \Phi|^2$, and the fact that $\frac{\partial g}{\partial t} = dJd\Phi \sim \nabla^2 \Phi + T \ast \nabla \Phi$, to obtain

$$\Box \left( \frac{\partial f}{\partial t} + |\eta|^2 + A_1 |\partial \eta|^2 + A_2 |\nabla \Phi|^2 \right) \leq 0.$$
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Naive argument for why $\lim_{i \to \infty} T_i < \infty$: If not, the metrics $g_{s_i T_i}$ on the one hand limit to a structure on the boundary of the $\epsilon$-ball around $g_0$. On the other hand, by the uniform decay estimate

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Convergence

\[ r^k(M, g) := \inf_x \min \left\{ r_h(x, g), \left( \sum_{i=0}^{k} |\nabla^i g \text{Rm}|^{\frac{1}{i+2}}(x) \right)^{-1}, \left( \sum_{i=0}^{k} |\nabla^i g H|^{\frac{1}{i+1}}(x) \right)^{-1} \right\}, \]

Proposition

There exists \( \epsilon(n) \) so that if \( (M_n, g_t, b_t, I, J_t) \) is a GKRF on \([-4, 0] \), satisfying

1. \( \sup_{M \times [-2, 0]} |\nabla \Phi| \leq \epsilon(n) \),
2. \( r^k(M, g_0) \geq 1 \).

Then for every \( t \in [-1, 0] \), we have \( r^k(M, g_t) > \frac{1}{2} \).

Proof

Argue by contradiction, assuming there exist counterexamples as \( \epsilon_i \to 0 \).

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