Singular sets of generalized Einstein metrics

Jeffrey Streets

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Choosing $n = 2$, and letting $x_1 \to x_2$, the Riemann curvature blows up, and the spaces converge to the orbifold $\mathbb{R}^4/\mathbb{Z}^2$.

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Theorem

(Cheeger-Naber 2014) Suppose $(M_i^n, g_i, p_i) \longrightarrow (X, d_X, p)$ *and furthermore*

1. $|\mathrm{Rc}^{g_i}| \leq n - 1$
2. $\mathrm{Vol}(B_1(p_i)) > v > 0$.

Then X *is a* $C^{1,\alpha}$ *Riemannian manifold away from a set* $\mathcal{S}$ *such that*

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$$\mathcal{F}(g, H, f) = \int_M \left(R - \frac{1}{12} |H|^2 + |\nabla f|^2 \right) e^{-f} dV_g,$$
$$\lambda(g, H) = \inf_{\left\{ f \mid \int_M e^{-f} dV_g = 1 \right\}} \mathcal{F}(g, H, f).$$

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A pair (g, H) is a critical point for λ if and only if

$$\mathrm{Rc}^g - \frac{1}{4} H^2 + \nabla^2 f = 0, \qquad d_g^* H + i_{\nabla f} H = 0,$$

where $H^2(X, Y) = \langle i_X H, i_Y H \rangle$.

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where $H^2(X, Y) = \langle i_X H, i_Y H \rangle$. In the special case that f is constant, we get the equations of a generalized Einstein metric:

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$$\begin{aligned} \nabla &= D + \tfrac{1}{2} g^{-1} H, \\ \mathrm{Rc}^{\nabla} &= \mathrm{Rc}^{g} - \tfrac{1}{4} H^2 - \tfrac{1}{2} d_g^* H. \end{aligned}$$

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We furthermore consider a very broad class of equations for a Riemannian metric g and a differential form $H \in \Lambda^*$:

$$\mathrm{Rc}^{g} - \tfrac{1}{4} H^2 = 0, \qquad dH + Q_1(H) = 0, \qquad d_g^* H + Q_2(H) = 0,$$

where again $H^2(X, Y) = \langle i_X H, i_Y H \rangle$, and Q_i denotes a quadratic expression in H obtained via wedge product, interior product, and metric contractions.

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These systems of equations also arise naturally in understanding non-Kähler Calabi-Yau geometries.

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Definition

Given a smooth manifold M, a generalized Kähler structure is a triple (g, I, J) consisting of integrable complex structures I and J, a Riemannian metric g compatible with both, yielding Kähler forms ω_I, ω_J which satisfy

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In dimension 4, in the locus where σ is invertible, the whole structure is recovered by the symplectic triple

$$\Omega, I\Omega, J\Omega, \qquad \Omega = \sigma^{-1}.$$

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If such a GK manifold admits a free, isometric, biholomorphic S^1 action, generated by X, we obtain a moment map μ via

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Furthermore, W satisfies a certain elliptic PDE, which can be used to define a principal connection η via $d\eta = \star_h dW + \ldots$ yielding finally

$$g = Wh + W^{-1}\eta^2.$$

Generalized Gibbons-Hawking Ansatz

For certain explicit choices of p and f this in fact yields solutions of the generalized Ricci soliton system

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Remark

In joint work with Y. Ustinovskiy we gave a classification of four-dimensional complete generalized Kähler-Ricci solitons using this ansatz.

Theorem

(Fu-Naber-S 2021) Suppose $(M_i^n, g_i, H_i, p_i) \longrightarrow (X, d_X, p)$ *and furthermore*

1. $\left|\mathrm{Rc}^{\nabla_i}\right| \leq n - 1$
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Theorem

(Fu-Naber-S 2021) There exists $C = C(n,v)$ *such that if* (M^n, g, H, p) *satisfies*

1. $\left|\mathrm{Rc}^{\nabla}\right| \leq n-1,$
2. $\mathrm{Vol}(B_1(p)) > v > 0.$

$$\int_{B_1(p)} |\mathrm{Rm}|^2 + |\nabla H|^2 + |H|^4 < C.$$

Corollary

There exists $C = C(v, D)$ *such that if* M^4 *satisfies* $|\mathrm{Rc}^{\nabla}| \leq 3$, $\mathrm{Vol}(B_1(p)) > v > 0$ *and* $\mathrm{diam}(M^n) \leq D$, *then* M^4 *can have one of at most* C *diffeomorphism types.*

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Corollary

Let (M^4, g, H) *be a complete manifold with* $H \in \Lambda^3$ *such that* $\mathrm{Rc}^{\nabla} \equiv 0$ *and* g *has maximal volume growth. Then* $H \equiv 0$ *and* g *is Ricci-flat.*

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Given (X, d_X) and $x \in X$, a tangent cone at x is a Gromov Hausdorff limit

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for some sequence $r_i \to \infty$.

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A metric space X is called *k-symmetric* if X is isometric to $\mathbb{R}^k \times C(Z)$ for some Z. Furthermore, we say X is *k-symmetric at $x \in X$* if there is an isometry of X with $\mathbb{R}^k \times C(Z)$ which carries x to a vertex of the cone $\mathbb{R}^k \times C(Z)$.

Codimension 4 Regularity - Tangent cones

We understand the limit space (X, d_X) through the structure of tangent cones:

Definition

Given (X, d_X) and $x \in X$, a tangent cone at x is a Gromov Hausdorff limit

$$(X, r_i^{-1} d_X, x) \to (X_x, d_\infty, x_\infty)$$

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Theorem

(Cheeger-Colding 1996) Suppose $(M_i^n, g_i, H_i, p_i) \longrightarrow (X, d_X, p)$ and furthermore

1. $\mathrm{Rc}^{g_i} \geq -(n-1)$
2. $\mathrm{Vol}(B_1(p_i)) > v > 0.$

Then every tangent cone at every $x \in X$ is a metric cone which is k-symmetric for some $k \geq 0$.

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$$\emptyset \subset \mathcal{S}^0 \subseteq \cdots \subseteq \mathcal{S}^{n-1} := \mathcal{S} \subseteq X^n ,$$

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Furthermore

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Proposition

Given a sequence of (M_i^n, g_i, H_i) *of Riemannian manifolds satisfying*

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Theorem

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3. If $\beta < 2\pi$, we refine the blowup sequence at the points of minimal harmonic radius to obtain a new limit which has $r_h = 1$, but has $\operatorname{Rc}^{\nabla} \equiv 0$. The rigidity proposition shows that then this space is flat, in fact $\mathbb{R}^n$, giving a contradiction.

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$$0 = \mathrm{Rc}^g(\partial_t, \partial_t) = \frac{1}{4} H^2(\partial_t, \partial_t) = \frac{1}{4} \left| i_{\partial_t} H \right|^2 .$$

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As the Ricci tensor of a cone is expressed $\mathrm{Rc}^g = \mathrm{Rc}^{g_\Sigma} - (n-2) g_\Sigma$, it follows easily that $L_{\partial_t} \mathrm{Rc}^g = 0$. Furthermore, since $L_{\partial_t} g = 2t g_\Sigma$, we have

$$0 = \left(L_{\partial_t} \mathrm{Rc}^{\nabla} \right)(X, X) = \left(L_{\partial_t} H^2 \right)(X, X) \sim t \left| i_X H \right|^2_{g_\Sigma} .$$

It follows that $i_X H \equiv 0$. Thus $H \equiv 0$, and then $\mathrm{Rc}^g \equiv 0$.

Theorem

Let (M_i^n, g_i, H_i, p_i) *satisfy* $|\operatorname{Rc}^{\nabla_i}| \to 0$, $\operatorname{Vol}(B_1(p_i)) > v > 0$ *and*

$$(M_i^n, d_i, p_i) \to \mathbb{R}^{n-3} \times C(Y),$$

Then $Y \cong S^2$ *and hence* $\mathbb{R}^{n-3} \times C(Y) = \mathbb{R}^n$.

Proof

1. By nonexistence of codimension 2 singularities, Y is smooth.

Codimension 3 singularities

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1. By nonexistence of codimension 2 singularities, Y is smooth.
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3. To rule out $\mathbb{R}P^2$, one notes that a slice of a splitting map will be a smooth 3-manifold with boundary $\mathbb{R}P^2$, impossible!