Deformations of G_2 instantons on nearly G_2 manifolds

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Nearly G_2 manifolds

A G_2 structure on M^7 is a 3-form φ which is *non-degenerate*.
G_2 structure $\leftrightsquigarrow$ "non-degenerate" 3-form $\varphi \rightsquigarrow g_\varphi$ and orientation nonlinearly.
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Definition

A G_2 structure φ on M^7 is *nearly parallel* if for some non-zero real constant τ_0

$$d\varphi = \tau_0 \psi.$$

In this case, the manifold (M, φ) is a nearly G_2 manifold.

More on nearly G_2 manifolds

- Introduced as manifolds with weak holonomy G_2 by Gray in 1971.
- Equivalently defined by the condition that the cone over a nearly G_2 manifold has exceptional holonomy inside $\mathrm{Spin}(7)$.
- Nearly G_2 manifolds are **positive Einstein** with $\mathrm{Ric}_{g_\varphi} = \frac{21}{8}\tau_0^2 g_\varphi$
- Under certain conditions nearly G_2 manifolds are compactification of M-theory
- Classification of homogeneous nearly G_2 manifolds: Friedrich–Kath–Moroianu–Semmelmann (1997)
- Infinitesimal deformations of nearly G_2 structures: Alexandrov–Semmelmann (2012)
- Obstructedness of the infinitesimal deformations:Dwivedi–S.(2020), Nagy–Semmelmann (2020).

Motivation from Nearly Kähler 6-manifolds

- The nearly G_2 manifolds are similar to the 6-dimensional nearly Kähler (NK) manifolds in many ways.
- NG_2 and NK 6-manifolds are the only 2 cases on which a unique totally skew-symmetric parallel metric connection exists [Cleyton–Swann].
- In 2016 Charbonneau–Harland studied the infinitesimal deformation space of instantons on 6-dim NK manifolds.
- The approach used in the NK case can be *partially* borrowed for the nearly G_2 instantons.

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τ_0 can be altered to any non-zero real number, we fix $\tau_0 = 4$.

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Instantons on nearly G_2 manifolds

$\mathcal{P} \to M$: principal G-bundle
A: connection 1-form on $\mathcal{P}$ with curvature F_A.

Definition

The connection A is a G_2 instanton on M if $F_A \cdot \eta = 0$ or equivalently $F_A \wedge \psi = 0$.

- Nearly G_2 instantons are **Yang–Mills** : $(d^A)^* F_A = 0$.

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$\epsilon \in \Gamma(\Lambda^1 \otimes \mathrm{Ad}_{\mathcal{P}})$ is an infinitesimal deformation of an instanton A if

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Goal: Describe the infinitesimal deformation space of the nearly G_2 instanton A

Infinitesimal deformations of instantons

(M, φ, η), nearly G_2 manifold.

family of connections ∇^t with totally skew-symmetric torsion [Agricola–Friedrich]

$$\boxed{\nabla^t_X Y = \nabla^g_X Y + \frac{t}{3}\varphi(X, Y, \cdot)}$$

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$D^{t,A}$: associated Dirac operators on $\not{S}M \otimes \mathrm{Ad}_{\mathcal{P}}$,

Proposition

ϵ is an infinitesimal deformation of a nearly G_2 instanton A if and only if

$$D^{t,A}(\epsilon \cdot \eta) = -\frac{t+5}{2}\epsilon \cdot \eta, \quad \forall t \in \mathbb{R}.$$

Infinitesimal deformations of nearly G_2 instantons

Schrödinger–Lichnerowicz formula (Agricola–Friedrich 2004)

$$(D^{t/3,A})^2\mu = (\nabla^{t,A})^*\nabla^{t,A}\mu + \frac{1}{4}\mathrm{Scal}_g\mu + \frac{t}{6}\mathrm{d}\varphi\cdot\mu - \frac{t^2}{18}\|\varphi\|^2\mu + F_A\cdot\mu.$$

- $\|\varphi\|^2 = 7, \mathrm{Scal}_g = 42,$
- $d\varphi\cdot\epsilon\cdot\eta = 4\psi\cdot\epsilon\cdot\eta = -4\epsilon\cdot\eta,$
- $F_A\cdot\epsilon\cdot\eta = -2\epsilon\lrcorner F_A\cdot\eta$

$$(D^{t/3,A})^2(\epsilon\cdot\eta) = (\nabla^{t,A})^*\nabla^{t,A}(\epsilon\cdot\eta) - \frac{1}{18}(7t^2+12t-189)\epsilon\cdot\eta - 2(\epsilon\lrcorner F_A)\cdot\eta.$$

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Theorem (Ball–Oliveira'19, S.'20)

Any G_2-instanton A on a principal G-bundle over a compact nearly G_2 manifold M is rigid if

(i) *the structure group G is abelian, or*

(ii) *the eigenvalues of the operator $\epsilon \mapsto -2\epsilon\lrcorner F_A$ are either all greater than $-\frac{28}{5}$ or all smaller than 6.*

Homogeneous nearly G_2 spaces

G/H: homogeneous nearly G_2 manifold
Classification: compact, simply connected
(Friedrich–Kath–Moroianu–Semmelmann 1997):

$(S^7, g_{round}) = \mathrm{Spin}(7)/\mathrm{G}_2,$	$(S^7,\, g_{squashed}) = \frac{\mathrm{Sp}(2)\times\mathrm{Sp}(1)}{\mathrm{Sp}(1)\times\mathrm{Sp}(1)}$
$\mathrm{SO}(5)/\mathrm{SO}(3),$	$M(3,2) = \frac{\mathrm{SU}(3)\times\mathrm{SU}(2)}{\mathrm{U}(1)\times\mathrm{SU}(2)}$
$N(k,l) = \mathrm{SU}(3)/S^1_{k,l}\ k,l \in \mathbb{Z},$	$Q(1,1,1) = \mathrm{SU}(2)^3/\mathrm{U}(1)^2.$

Here $S^1_{k,l} = \mathrm{diag}(e^{ik\theta}, e^{il\theta}, e^{-i(k+l)\theta})$

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all reductive $\implies \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$,
canonical homogeneous connection on G/H is an **instanton** on

- $G \times_{\mathrm{ad}} \mathfrak{m}$ with str group H
- $G \times_{\mathrm{ad}\circ\lambda} \mathfrak{m}$ with str group G_2. (λ, isotropy homomorphism)

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The first four are normal $\implies$ can find deformations in both the cases.

Deformations for homogeneous spaces

Theorem (S.'20)

The infinitesimal deformation space for the canonical connection on the four normal homogeneous nearly G_2 *spaces*

G/H	Structure group	
	H	G_2
$\mathrm{Spin}(7)/\mathrm{G}_2$	0	0
$\mathrm{SO}(5)/\mathrm{SO}(3)$	0	$\mathfrak{so}(5)$
$\dfrac{\mathrm{Sp}(2)\times\mathrm{Sp}(1)}{\mathrm{Sp}(1)\times\mathrm{Sp}(1)}$	$V_{\mathbb{R}}^{(0,1)}$	$\mathfrak{sp}(2)\oplus\mathfrak{sp}(1)\oplus V_{\mathbb{R}}^{(0,1)}$
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Remark: for the non-normal cases H is abelian so no deformation there but can't say anything about the G_2 case.

Idea of the proof

$$\left.\begin{aligned} d^A\epsilon\cdot\eta &= 0 \\ (d^A)^*\epsilon &= 0 \end{aligned}\right\} \iff D^{t,A}\epsilon\cdot\eta = -\left(\frac{t+5}{2}\right)\epsilon\cdot\eta$$

On G/H, $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{m}$, $TM = G\times_{\rho_m}\mathfrak{m}$
Set $E=\mathfrak{h}$ or $\mathfrak{g}_2$, ρ_L *left regular representation* on $G\times_{\rho_{\text{ad}}}(\mathfrak{m}^*\otimes E)$

Lemma

∇^{can} = *canonical connection*, $F\in\Lambda^2\mathfrak{m}^*\otimes\mathfrak{h}$, $\epsilon\in\mathfrak{m}^*\otimes E$,

$$\begin{aligned}(D^{-1/3,can})^2\epsilon\cdot\eta &= (\nabla^{-1,can})^*\nabla^{-1,can}\epsilon + \frac{97}{9}\epsilon - 2\epsilon\lrcorner F)\cdot\eta \\ &= (-\rho_L(\mathrm{Cas}_{\mathfrak{g}})\epsilon + \rho_E(\mathrm{Cas}_{\mathfrak{h}})\epsilon + \frac{49}{9}\epsilon)\cdot\eta.\end{aligned}$$

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$$= (\underbrace{-\rho_L(\mathrm{Cas}_{\mathfrak{g}})\epsilon + \rho_E(\mathrm{Cas}_{\mathfrak{h}})\epsilon}_{=0 \text{ for deformations}} + \frac{49}{9}\epsilon)\cdot\eta.$$

Can be solved using Frobenius reciprocity theorem

Idea of the proof

$$\ker\left((D^{-1/3,A})^2 - \frac{49}{9}\mathrm{id}\right) = \ker\left(D^{-1/3,A} + \frac{7}{3}\mathrm{id}\right) \oplus \ker\left(D^{-1/3,A} - \frac{7}{3}\mathrm{id}\right)$$
$$= \ker\left(D^{-1,A} + 2\mathrm{id}\right) \oplus \ker\left(D^{-1,A} - \frac{8}{3}\mathrm{id}\right)$$

Idea of the proof

$$\overbrace{\ker\left((D^{-1/3,A})^2 - \frac{49}{9}\mathrm{id}\right)}^{\text{previous lemma for canonical}} = \underbrace{\ker\left(D^{-1/3,A} + \frac{7}{3}\mathrm{id}\right)}_{\text{deformation space}} \oplus \ker\left(D^{-1/3,A} - \frac{7}{3}\mathrm{id}\right)$$

$$= \overbrace{\ker\left(D^{-1,A} + 2\mathrm{id}\right)} \oplus \ker\left(D^{-1,A} - \frac{8}{3}\mathrm{id}\right)$$

Step 1: Decompose LHS into irreps of G,

$$\ker((D^{-1/3,can})^2 - \frac{49}{9}\mathrm{id}) = \oplus V_i,$$

Step 2: Compute $D^{-1,can}$ on each V_i to see whether in -2 or $\frac{8}{3}$ eigenspace.

A neat shortcut in the NK case unavailable here:
In 6-dim NK manifolds the Killing spinors $\eta, \textbf{vol} \cdot \eta$ are independent

$$\ker((D^{-1,A})^2 - 4\mathrm{id}) = 2\underbrace{\ker(D^{-1,A} - 2\mathrm{id})}_{\text{deformation space in that case}}$$

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For $N\mathrm{G}_2$ there is no such relation between $\ker((D^{-1,A})^2 - 4\mathrm{id})$ and $\ker(D^{-1,A} + 2\mathrm{id})$ as we see from our computations for $E = \mathfrak{g}_2$,

	$\mathrm{Spin}(7)/\mathrm{G}_2$	$\mathrm{SO}(5)/\mathrm{SO}(3)$	$\frac{\mathrm{Sp}(2)\times\mathrm{Sp}(1)}{\mathrm{Sp}(1)\times\mathrm{Sp}(1)}$	$\frac{\mathrm{SU}(3)\times\mathrm{SU}(2)}{\mathrm{SU}(2)\times\mathrm{U}(1)}$
D^2	0	$\mathfrak{so}(5)$	$2\mathfrak{sp}(2)\oplus\mathfrak{sp}(1)\oplus V_{\mathbb{R}}^{(0,1)}$	$2\mathfrak{su}(2)\oplus 6\mathfrak{su}(3)$
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Integrability of the deformations

Proposition (Charbonneau–Harland)

On G/H for every element $\xi \in \Gamma(\Lambda^2_{14} \otimes \mathrm{Ad}_{\mathcal{P}})$ such that $\nabla^{-1,A}\xi = 0$,

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Observe: For A= canonical homogeneous connection on G/H

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Modulo gauge: for $K = H, \mathrm{G}_2$

Multiplicity of $\mathfrak{g}$ in the deformation space = trivial K-reps in $\mathfrak{g}_2 \otimes \mathit{Lie}(K)$ - 1

Integrability of deformations

The above accounts for all the deformations except:

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$\mathrm{SO}(5)/\mathrm{SO}(3)$	0	$\mathfrak{so}(5)$
$\frac{\mathrm{Sp}(2)\times\mathrm{Sp}(1)}{\mathrm{Sp}(1)\times\mathrm{Sp}(1)}$	$V_{\mathbb{R}}^{(0,1)}$	$\mathfrak{sp}(2)\oplus\mathfrak{sp}(1)\oplus\ V_{\mathbb{R}}^{(0,1)}$
$\frac{\mathrm{SU}(3)\times\mathrm{SU}(2)}{\mathrm{SU}(2)\times\mathrm{U}(1)}$	0	$2\mathfrak{su}(2)\oplus 2\mathfrak{su}(3)\oplus 2\mathfrak{su}(3)$

- $V_{\mathbb{R}}^{0,1}$: *lift of the deformation space of the standard ASD instanton on* S^4
- $2\mathfrak{su}(3)$: *Don't know yet!*

Integrability of deformations

The above accounts for all the deformations except:

G/H	Structure group	
	H	G_2
$\mathrm{Spin}(7)/\mathrm{G}_2$	0	0
$\mathrm{SO}(5)/\mathrm{SO}(3)$	0	$\mathfrak{so}(5)$
$\dfrac{\mathrm{Sp}(2)\times\mathrm{Sp}(1)}{\mathrm{Sp}(1)\times\mathrm{Sp}(1)}$	$V_{\mathbb{R}}^{(0,1)}$	$\mathfrak{sp}(2)\oplus\mathfrak{sp}(1)\oplus V_{\mathbb{R}}^{(0,1)}$
$\dfrac{\mathrm{SU}(3)\times\mathrm{SU}(2)}{\mathrm{SU}(2)\times\mathrm{U}(1)}$	0	$2\mathfrak{su}(2)\oplus 2\mathfrak{su}(3)\oplus 2\mathfrak{su}(3)$

- $V_{\mathbb{R}}^{0,1}$: *lift of the deformation space of the standard ASD instanton on S^4*
- $2\mathfrak{su}(3)$: *Don't know yet!*

Almost all the deformations are genuine.

Future directions

- It would be interesting to see if the deformations in the $\frac{\mathrm{SU}(3)\times\mathrm{SU}(2)}{\mathrm{SU}(2)\times\mathrm{U}(1)}$ case comes from the deformations of the nearly G_2 structure found by Alexandrov–Semmelmann.
- Sasakian, 3-Sasakian structures corresponds to the presence of $2, 3$ independent Killing spinors so a similar deformation problem can be studied there. The canonical connection is an instanton for these manifolds as well (Harland–Nölle, 2012).

Thanks a lot for your attention!!