G_2 instantons in twisted M-theory

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Simons Collaboration Meeting

Twisted holography

- Twist both sides of known holographic models with supersymmetry
- Pass to Q-cohomology and a protected subsector
- Match operator *algebras* exactly. Many examples
- Today: geometry and representation theory.

Twisting SUGRA: Formal definition [Costello,Li]

- SUSY is a local(gauge) symmetry in SUGRA
- Introduce (bosonic) ghosts $\Psi(\rightarrow \epsilon)$ for the gauge theory
- Twisted SUGRA: SUGRA with some components of Ψ, say ψ, non-zero.
- Recall: when twisting QFT, we pick a scalar Q and keep $\epsilon_Q \neq 0$.

Twisting SUGRA: Example of twisted M-theory [Eager,Hahner]

- Consider 11d background: $G_2 \times$HyperKahler.
- 11(=7+4)d Killing spinor

$$\Psi_{11} = \Psi_7 \otimes \Psi_4 = (\mathbf{1}_{\mathbf{G_2}} \oplus \mathbf{7}_{G_2}) \otimes (\mathbf{1}_{-\mathbf{1}} \oplus \mathbf{1}_{+\mathbf{1}} \oplus \mathbf{2}_{\mathbf{0}}),$$

  Reps under $G_2 \times SU(2)_- \times U(1)_+$, where $U(1)_+ \subset SU(2)_+$.
- Choose

$$Q = \mathbf{1}_{\mathbf{G_2}} \otimes \mathbf{1}_{-\mathbf{1}}.$$

- Get Q_m, $Q_{\dot\alpha}$ such that

$$\{Q, Q_m\} = P_m, \quad \{Q, Q_{\dot\alpha}\} = P_{-\dot\alpha}.$$

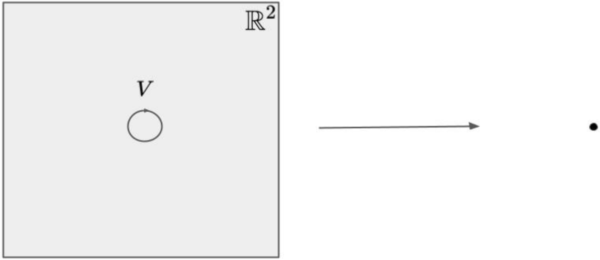
  Topological in 7 directions and Holomorphic in 4 directions.
- Turn on the ghost for Q and get twisted background.

$\Omega-$deformed twisted M-theory [Costello]

If there is $U(1)$ isometry, we can further deform Q into Q_V [Nekrasov]:

$$Q_V^2 = \mathcal{L}_V$$

Then the Q_V cohomology consists of a subset of Q-cohomology living on a fixed point set of $\mathcal{L}_V$.



Twisted M-theory: Simplest example

Background is specified by (g, C).

- Twisted and Ω deformed geometry

$$\mathcal{M} = \underbrace{\mathbb{C}_z \times \mathbb{C}_w}_{\mathcal{M}_4^{HK}} \times \underbrace{\mathbb{R}_t \times \mathbb{C}_{\epsilon_1} \times \mathrm{TN}_K}_{\mathcal{M}_7^{G_2}}.$$

with

$$\mathcal{M}_4^{HK} : \text{ holomorphic}$$
$$\mathcal{M}_7^{G_2} : \text{ topological with } \Omega_{\epsilon_1} \times \Omega_{\epsilon_2} \times \Omega_{\epsilon_3}$$

- 3-form field C

$$C = \epsilon_2 V^d d\bar{z} d\bar{w},$$

Here, V rotates $S^1_{TN} \subset TN_K$.

Twisted M-theory: IIA frame

- (Reduce $S^1_{TN} \to$) Twisted geometry

$$\underbrace{\mathbb{C}_z \times \mathbb{C}_w}_{\mathcal{M}_4^{Hol}} \times \underbrace{\mathbb{R}_t \times \mathbb{C}_{\epsilon_1} \times \mathbb{R}_{\epsilon_2} \times \mathbb{C}_{\epsilon_3}}_{\mathcal{M}_6^{Top}}$$

- K D6 branes(from TN_K) on

$$\underbrace{\mathbb{C}_z \times \mathbb{C}_w}_{\mathcal{M}_4^{Hol}} \times \underbrace{\mathbb{R}_t \times \mathbb{C}_{\epsilon_1}}_{\subset \mathcal{M}_6^{Top}}.$$

- B-field $\to$ non-commutativity on $\mathcal{M}_4^{Hol} = \mathbb{C}_z \times \mathbb{C}_w$.

$$B = \epsilon_2 d\bar{z} d\bar{w},$$

- Only open strings survive. Focus on D6 worldvolume theory.

Twisted M-theory: 5d Chern-Simons theory

- $(K$ D6$\to)$ 7d $U(K)$ SYM on $\overbrace{\mathbb{C}_z \times \mathbb{C}_w}^{H} \times \overbrace{\mathbb{R}_t \times \mathbb{C}_{\epsilon_1}}^{T}$ localizes on 5d $U(K)$ Chern-Simons theory on $\overbrace{\mathbb{C}_z \times \mathbb{C}_w}^{H} \times \overbrace{\mathbb{R}_t}^{T}$.

$$\frac{1}{\epsilon_1}\int_{\mathbb{C}_z\times\mathbb{C}_w\times\mathbb{R}_t} dz \wedge dw \wedge \left(AdA + \frac{2}{3}A \star_{\epsilon_2} A \star_{\epsilon_2} A\right),$$

$$f \star_{\epsilon_2} g = fg + \epsilon_2 \frac{1}{2}\epsilon_{ij}\frac{\partial}{\partial_{z_i}} f \frac{\partial}{\partial_{z_j}} g + \dots.$$

The 5d gauge field has only three components.

$$A = A_t dt + A_{\bar{z}} d\bar{z} + A_{\bar{w}} d\bar{w},$$

Twisted M-theory: Operator algebra

- Equation of motion

$$F_A = 0$$

  Only positive ghost number modes survive.
- Gauge symmetry

$$\mathfrak{g} = \mathfrak{gl}_K \to \mathfrak{g}' = \mathrm{Diff}_{\epsilon_2}\mathbb{C}_z \otimes \mathfrak{gl}_K$$

- Classical algebra of operators: Lie algebra cohomology of $\mathfrak{g}'$:

$$C^*(\mathfrak{g}').$$

- At the quantum level, gets a deformation $\sim \mathcal{O}(\epsilon_1)$
- Koszul duality(Twisted holgraphic duality) [Costello]

$$C^*(\mathrm{Diff}_{\epsilon_2}\mathbb{C}_z \otimes \mathfrak{gl}_K) \leftrightarrow U(\mathrm{Diff}_{\epsilon_2}\mathbb{C}_z \otimes \mathfrak{gl}_K)$$

Twisted M-theory: General G_2 background

- CY_3-like background:

$$\mathcal{M}_7^{G_2} = \mathbb{R}_t \times CY_3^{m|n}.$$

Here, $CY_3^{m|n}$ is a generalized conifold:

$$xy = z^m w^n$$

Relevant 5d CS algebra:

$$U_{\epsilon_1}(\mathrm{Diff}_{\epsilon_2}\mathbb{C}_z \otimes \mathfrak{gl}_{m|n}) \leftrightarrow CY_3^{m|n}.$$

- Genuinely G_2 background [Main focus]

$$\mathcal{M}_7^{G_2}[N,K] = \text{A cone on } \mathrm{WCP}^3_{N,N,K,K}.$$

Relevant 5d CS algebra: [Oh,Zhou]

$$U_{\epsilon_1}(\mathrm{Diff}_{\epsilon_2}\mathbb{C}_z \otimes \mathfrak{gl}_N) \otimes U_{\epsilon_1'}(\mathrm{Diff}_{\epsilon_2}\mathbb{C}_z \otimes \mathfrak{gl}_K) \leftrightarrow \mathcal{M}_7^{G_2}[N,K].$$

5d CS and enumerative geometry of CY_3

Replace $\mathcal{M}_4^{HK} = \mathbb{C}_z \times \mathbb{C}_w \to TN_1$.

- 11d geometry: $TN_1 \times \mathbb{R}_t \times \mathbb{C}_1 \times TN_K$.
  - Reduce on $S^1 \subset TN_K$:

    $$\text{5d U(K) CS on } TN_1 \times \mathbb{R}_t \text{ with } \mathcal{A}_K = U_{\epsilon_1}(\mathrm{Diff}_{\epsilon_2}\mathbb{C} \otimes \mathfrak{gl}_K).$$

  - Reduce $S_1^1 \subset \mathrm{TN}_1$:

    $$\text{1 D6 brane on } \mathbb{R}_t \times \mathbb{C}_1 \times \mathrm{TN}_K$$

    D6 partition function = rank 1 DT partition function Z_K.
- Theorem [Costello]: Z_K forms a character of a module of $\mathcal{A}_K$.
- "9-11" flip→ one IIA description sheds light on another.

The G_2 example and the 5d CS

Consider $\text{TN}_1 \times \mathcal{M}_7^{G_2}$($=$ A cone on $\text{WCP}^3_{N,N,K,K}$)

- Reduce $S^1_G \subset \mathcal{M}_7^{G_2}$

  N and K stacks of intersecting D6 brane on $TN_1 \times \left(\mathcal{R}^3_1 \cup \mathcal{R}^3_2\right)$.

  Two copies of 5d CS on $TN_1 \times \mathcal{R}^3_1/\Omega$ and $TN_1 \times \mathcal{R}^3_2/\Omega$:

$$\mathcal{A}_N \otimes \mathcal{A}_K = U_{\epsilon_1}(\text{Diff}_{\epsilon_2}\mathbb{C} \otimes \mathfrak{gl}_N) \otimes U_{\epsilon_1'}(\text{Diff}_{\epsilon_2}\mathbb{C} \otimes \mathfrak{gl}_K).$$

- Reduce $S^1_{TN} \subset \text{TN}_1$

$$1 \text{ D6 brane on } \mathcal{M}_7^{G_2}.$$

  D6 partition function $=$ rank 1 DT partition function $Z_{N,K}$.
  Hard to study: G_2 instanton is elusive.

Conjecture: Starting point

G_2 Donaldson-Thomas partition function $\mathcal{Z}_{N,K}$ is a character of a non-trivial module of the algebra $\mathcal{A}_N \otimes \mathcal{A}_K$.

String theory will give an answer for G_2 instantons and an evidence for the conjecture.

D6 partition function on G_2 manifold

- G_2 manifold $\mathcal{M}_7$ is equipped with associative 3-form $\varphi (d\varphi = d\star\varphi = 0)$.
- The partition function of D6(7d SYM) on G_2:

$$\mathcal{Z} = \int \left[\prod d\phi_i\right] e^{-\frac{1}{g^2}\int_{G_2} YM[A]+\cdots},$$

- Localizes around i G_2 instantons [Tian] :

$$\mathcal{M}_i = \left\{ A \mid \mathcal{F}_A^+ = 0, \quad \int \mathcal{F}\wedge\mathcal{F} = i \right\},$$

G_2 instanton

- Moduli space of G_2 instanton is a solution to $\star\mathcal{F} = -\mathcal{F} \wedge \varphi$.
- 2-forms on G_2 are organized in G_2 representations. [Joyce]:

$$\wedge^2 T^* \mathcal{M}_7^{G_2} = \wedge^2_{\mathbf{7}} \oplus \wedge^2_{\mathbf{14}}, \quad H^2(\mathcal{M}_7^{G_2}) = H^2_{\mathbf{7}}(\mathcal{M}_7^{G_2}) \oplus H^2_{14}(\mathcal{M}_7^{G_2}).$$

- Yang-Mills action is also decomposed:

$$S_{YM}(\mathcal{F}) = 3S_{YM}(\mathcal{F}_7) - \frac{\alpha}{g^2}\kappa(\mathcal{F}), \text{ where } \kappa(\mathcal{F}) = \int_{\mathcal{M}_7} \varphi \wedge \mathrm{ch}_2(\mathcal{F}).$$

- 'Self-dual condition': $S_{YM}(\mathcal{F}_7) = 0 \;\Rightarrow\; S_{YM}(\mathcal{F}) = -\frac{\alpha}{g^2}\kappa(\mathcal{F})$.

Localization of path integral

- The path integral localizes on G_2 instanton saddle:

$$\mathcal{Z} \sim \int \left[\prod d\phi\right] e^{-\frac{1}{g^2}\int_{G_2} \kappa(E)} = \int \left[\prod d\phi\right] e^{-\frac{1}{g^2}\int_{G_2} \mathrm{Tr}\, \mathcal{F}_A^2 \wedge \varphi}.$$

- G_2 instantons wrap the associative 3-cycles:

$$\mathcal{Z}^{inst}_{N,K} = \sum_{\alpha \in H_3(G_2)} \sum_{i=0} q_\alpha^i \int_{\mathcal{M}_{\alpha,i}} 1, \quad \text{where } q_\alpha = e^{-\frac{1}{g^2}\int_\alpha \varphi}.$$

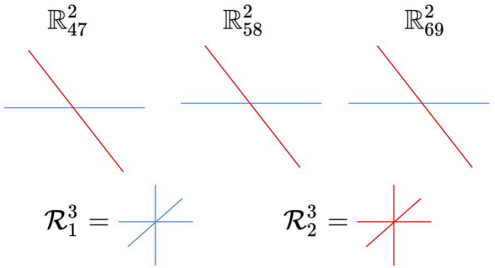
- $\mathcal{M}_{\alpha,i}$ is not yet fully known. Will rely on D-branes to solve.

How to deal with G_2 instantons I: G_2 to D6 stacks

Back to twisted M-theory on $TN_1 \times G_2$ reduce $S^1_G \subset G_2$.
Advatage: S^1_G reduction is Q-exact in twisted M-theory.

[Berkooz,Douglas,Leigh]

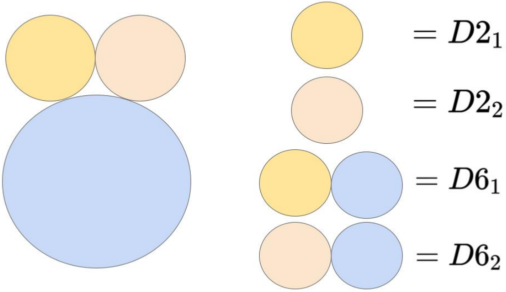
	0	1	2	3	4	5	6	7	8	9
Geometry	$\mathbb{C}_z$		$\mathbb{C}_w$		$\mathbb{R}^3_1$			$\mathbb{R}^3_2$		
N $D6_1$	+	+	+	+	+	+	+			
K $D6_2$	+	+	+	+	/	/	/	—	—	—



How to deal with G_2 instantons II: D2 instanton stacks

G_2 instantons are M2 branes wrapping $\alpha \in H_3(G_2)$.
M2 branes map to D2 branes in type IIA.

	0	1	2	3	4	5	6	7	8	9
Geometry	$\mathbb{C}_z$		$\mathbb{C}_w$		$\mathbb{R}^3_1$			$\mathbb{R}^3_2$		
N $D6_1$	+	+	+	+	+	+	+			
K $D6_2$	+	+	+	+	/	/	/	—	—	—
M_1 $D2_1$					+	+	+			
M_2 $D2_2$					/	/	/	—	—	—



How to deal with G_2 instantons III: Idea

- No mystery remains:
  1. Hard G_2 geometry becomes N, K D6 branes stacks.
  2. Hard G_2 instantons become M_1, M_2 D2 branes stacks.
- Can analyze $\mathcal{M}^{N,K}_{M_1,M_2}$ by string quantization.
- Can compute $\int_{\mathcal{M}^{N,K}_{M_1,M_2}} 1$ using Witten index(Physics) / equivariant K-theoretic index(Math) :

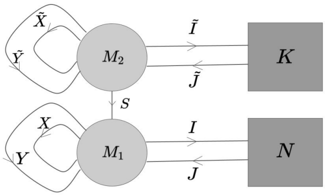
$$\mathcal{Z}^{N,K} = 1 + \sum_{M_1=1} \sum_{M_2=1} q_{\alpha_1}^{M_1} q_{\alpha_2}^{M_2} Z^{N,K}_{M_1,M_2},$$

$$\left(\mathcal{Z}^{N,K}_{M_1,M_2} = \int_{\mathcal{M}^{N,K}_{M_1,M_2}} 1 \right)$$

G_2 instanton moduli space I: String quantization

$$
\begin{aligned}
D2_1 - D2_1 \text{ strings} &\equiv \text{3d } \mathcal{N}=4 \text{ vector and adjoint hyper } (X, Y) \\
D2_2 - D2_2 \text{ strings} &\equiv \text{3d } \mathcal{N}=4 \text{ vector and adjoint hyper } (\tilde{X}, \tilde{Y}) \\
D2_1 - D6_1 \text{ strings} &\equiv \text{3d } \mathcal{N}=4 \text{ fundamental hyper } (I, J) \\
D2_2 - D6_2 \text{ strings} &\equiv \text{3d } \mathcal{N}=4 \text{ fundamental hyper } (\tilde{I}, \tilde{J}) \\
D2_1 - D2_2 \text{ strings} &\equiv \text{3d } \mathcal{N}=2 \text{ bifundamental chiral } S \text{ } \textbf{[New]} \\
D2_1 - D6_2 \text{ strings} &\equiv \text{3d } \mathcal{N}=1 \text{ massive scalar } \Phi \text{ } \textbf{[X]} \\
D2_2 - D6_1 \text{ strings} &\equiv \text{3d } \mathcal{N}=1 \text{ massive scalar } \Phi \text{ } \textbf{[X]} \\
D6_1 - D6_1 \text{ strings} &\equiv \text{7d } U(N) \text{ vector } \textbf{[X]} \\
D6_2 - D6_2 \text{ strings} &\equiv \text{7d } U(K) \text{ vector } \textbf{[X]} \\
D6_1 - D6_2 \text{ strings} &\equiv \text{4d } \mathcal{N}=1 \text{ bifundamental chiral } \textbf{[X]}
\end{aligned}
$$

G_2 instanton moduli space II: Double quiver



Two ADHM quivers satisfy their own complex moment equations:

$$[X, Y] + IJ = 0,$$
$$\left[\tilde{X}, \tilde{Y}\right] + \tilde{I}\tilde{J} = 0.$$

and the modified real moment map equations:

$$\left[X, X^\dagger\right] + \left[Y, Y^\dagger\right] + II^\dagger - JJ^\dagger - S^\dagger S = \xi \cdot \mathrm{I}_{M_1 \times M_1},$$
$$\left[\tilde{X}, \tilde{X}^\dagger\right] + \left[\tilde{Y}, \tilde{Y}^\dagger\right] + \tilde{I}\tilde{I}^\dagger - \tilde{J}\tilde{J}^\dagger + SS^\dagger = \xi \cdot \mathrm{I}_{M_2 \times M_2}.$$

G_2 instanton partition function via Witten index I

Physics approach: Study Witten index of D0 brane SUSY QM
T-dualize 56 directions: Convert "angles" to B-field background.

	0	1	2	3	4	5	6	7	8	9
Background					A	B	B	A	B	B
$(D2_1 \to)D0$					$+$					
$(D6_1 \to)D4_1$	$+$	$+$	$+$	$+$	$+$					
$(D2_2 \to)D4_2$					$-$	$+$	$+$	$\vert$	$+$	$+$
$(D6_2 \to)D8$	$+$	$+$	$+$	$+$	$-$	$+$	$+$	$\vert$	$+$	$+$

Table: [Balasubramanian, Leigh] A: angles, B: B-field.

Focus on $D0 - D0$, $D0 - D4_1$, $D0 - D4_2$ strings.

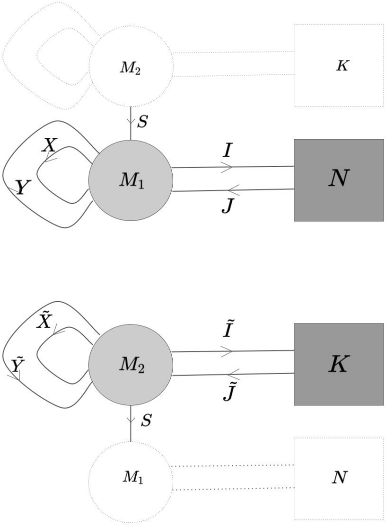
G_2 instanton partition function via Witten index II

strings	Type	fields	$SU(2)_\mp \times SU(2)^R_\mp$	$U(M_1) \times U(N) \times U(M_2)$
D0-D0	vector	gauge	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	**(Adj,1,1)**
		scalar	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	
		fermions	$(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{2})$	
	Fermi	fermions	$(\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1})$	
	t-hyper	scalars	$(\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2})$	
		fermions	$(\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1})$	
	hyper	scalars	$(\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1})$	
		fermions	$(\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{2})$	
D0-D4_1	hyper	scalars	$(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})$	$(\mathbf{M_1},\mathbf{N},\mathbf{1})$
		fermions	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})$	
	Fermi	fermions	$(\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1})$	
D0-D4_2	t-hyper	scalar	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})$	$(\mathbf{M_1},\mathbf{1},\mathbf{M_2})$
		fermions	$(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})$	

Table: The field content of the D0 $U(M_1)$ gauged quantum mechanics with $D4_1, D4_2$. [Kim,Kim,Lee; Hwang,Kim,Kim,Park], [Kim]

G_2 instanton partition function via Witten index III

Subquivers(it can only see the “messenger” S from the other.)



Limitation of Witten index

- Only part of the entire quiver so far.
- No T-duality frame where both $D2_1$, $D2_2$ map to $D0$ branes.
  $\leftrightarrow$(Not a conventional quiver gauge theory.)
- Need more powerful framework for entire quiver.
  $\rightarrow$ Equivariant K-theoretic index of the quiver.

Equivariant K-theoretic index

- Virtual tangent bundle $\mathbb{T}$ of moduli space = Cohomology of
  $\mathrm{End}(\mathcal{V}_1) \oplus \mathrm{End}(\mathcal{V}_2) \to \mathrm{End}(\mathcal{V}_1)^{\oplus 2} \oplus \mathrm{End}(\mathcal{V}_2)^{\oplus 2} \oplus \mathcal{V}_1^{\oplus N} \oplus \mathcal{V}_1^{*\oplus N}$
  $\oplus \mathcal{V}_2^{\oplus K} \oplus \mathcal{V}_2^{*\oplus K} \oplus \mathrm{Hom}(\mathcal{V}_1, \mathcal{V}_2) \to \mathrm{End}(\mathcal{V}_1) \oplus \mathrm{End}(\mathcal{V}_2)$.
- Define 5-torus $\mathbf{T}_i$ actions:
  $(r_1, r_2) \in \mathbf{T}_1 : (X, Y, I, J) \mapsto (r_1 X, r_2 Y, I, r_1 r_2 J)$,
  $(s_1, s_2) \in \mathbf{T}_2 : (\tilde{X}, \tilde{Y}, \tilde{I}, \tilde{J}) \mapsto (s_1 \tilde{X}, s_2 \tilde{Y}, \tilde{I}, s_1 s_2 \tilde{J})$, $\quad t \in \mathbf{T}_3 : S \mapsto tS$
- $\mathbf{T}_i$-equivariant K-theory class of $\mathbb{T}$:

$$
\begin{aligned}
T^{vir} =& \mathcal{V}_1 + \mathcal{V}_1^* r_1 r_2 - \mathcal{V}_1 \mathcal{V}_1^* (1 - r_1)(1 - r_2) + \mathcal{V}_2 + \mathcal{V}_2^* s_1 s_2 \\
& - \mathcal{V}_2 \mathcal{V}_2^* (1 - s_1)(1 - s_2) + t \mathcal{V}_2 \mathcal{V}_1^*
\end{aligned}
$$

- Theorem: Torus fixed point $(Y^{(1)}, Y_0^{(2)}, \cdots, Y_{M_1}^{(2)})$– $M_1 + 2$ Young diagrams– of the moduli space $\mathcal{M}_{M_1, M_2}^{N,K}$ is compact.

$$
\text{Index: } \chi(\mathcal{M}^{1,1}(M_1, M_2), \mathcal{O}) = \sum_{Y^{(1)}, Y_0^{(2)}, Y_{(i,j)}^{(2)}} S^{\bullet}(T^{\mathrm{vir}})^*
$$

Sample computation $M_1 = M_2 = N = K = 1$

$$Y_0^{(1)} = \square,\ Y_0^{(2)} = \square,\ Y_1^{(2)} = \emptyset, \text{ and}$$
$$Y_0^{(1)} = \square,\ Y_0^{(2)} = \emptyset,\ Y_1^{(2)} = \square,$$

and the virtual tangent characters at these fixed points are

$$T^{\mathrm{vir}} = r_1 + r_2 + s_1 + s_2 + t, \text{ and}$$
$$T^{\mathrm{vir}} = r_1 + r_2 + s_1 + s_2 + (t-1)s_1 s_2 + t^{-1}.$$

$\chi(\mathcal{M}^{1,1}(1,1), \mathcal{O})$ is P.E. of above:

$$\left(\frac{1}{(1-r_1)(1-r_2)(1-s_1)(1-s_2)} \left(\frac{1}{1-t} + \frac{1 - s_1 s_2}{(1 - t s_1 s_2)(1 - t^{-1})} \right) \right)^*$$
$$= \frac{r_1 r_2 s_1^2 s_2^2 t}{(1-r_1)(1-r_2)(1-s_1)(1-s_2)(t s_1 s_2 - 1)}$$

Special limit: Factorization(to connect with 5d CS)

Open subvariety:

$$\mathring{\mathcal{M}}^{1,1}(M_1, M_2) = \mathcal{M}^1(M_1) \times \mathcal{M}^1(M_2).$$

In the limit $r_1 r_2 \to 1, s_1 s_2 \to 1$, the equivariant K-theory partition function of $\mathcal{M}^{1,1}$ factorizes as

$$Z_{1,1} = \sum_{\underline{\lambda}} t^{|\underline{\lambda}|} \mathcal{Z}_1^{\underline{\lambda}}(r_1, r_2, q_{\alpha_1})^*_{r_1 r_2 \to 1} \mathcal{Z}_1^{\underline{\lambda}}(s_1, s_2, q_{\alpha_2})_{s_1 s_2 \to 1}$$

Here $\mathcal{Z}_1^{\underline{\lambda}}(r_1, r_2, q_{\alpha_1})$ is the equivariant K-theory partition function for the ADHM quiver with S:

$$\mathcal{Z}_1^{\underline{\lambda}}(r_1, r_2, q_{\alpha_1}) = \sum_{M_1} \chi(\mathcal{M}^1(M_1), S^{\underline{\lambda}}(\mathcal{V})) q_{\alpha_1}^{M_1} \sim \text{Witten index.}$$

Hint of 5d CS

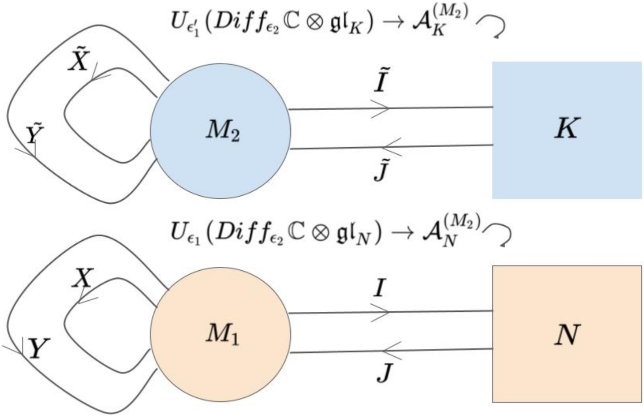


Figure: There is no S!

Summary so far

- $\mathcal{M}^{N,K}(M_1, M_2)$ is given by the connected double quiver.
- We compute $\int_{\mathcal{M}^{1,1}(M_1,M_2)} 1$ as equivariant K-theoretic index / Witten index.
- The G_2 instanton partition function:

$$\begin{aligned}\mathcal{Z}_{1,1} =& 1 + \sum_{M_1=1} \sum_{M_2=1} q_{\alpha_1}^{M_1} q_{\alpha_2}^{M_2} \int_{\mathcal{M}^{1,1}(M_1,M_2)} 1 \\ =& 1 + \sum_{M_1=1} \sum_{M_2=1} q_{\alpha_1}^{M_1} q_{\alpha_2}^{M_2} \ \chi(\mathcal{M}^{1,1}(M_1, M_2), \mathcal{O})\end{aligned}$$

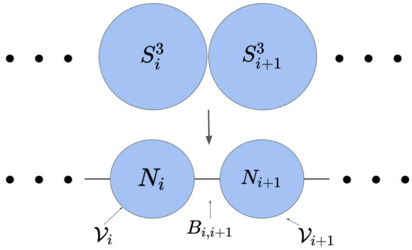
- As expected, $\mathcal{Z}_{1,1}$ admits an action of 5d CS algebra $U_{\epsilon_1}(\mathrm{Diff}_{\epsilon_2}\mathbb{C} \otimes \mathfrak{gl}_1) \times U_{\epsilon_1'}(\mathrm{Diff}_{\epsilon_2}\mathbb{C} \otimes \mathfrak{gl}_1)$.

Big picture emerges from 3 dual frames

Goal: Want to generalize the story presented so far.

- N D3 branes at tip of a local toric CY_3 cone X: 4d $\mathcal{N}=1$ quiver SCFT $\mathcal{T}_{X,N}$. (Use Mirror symmetry 'as T-duality' ($X \leftrightarrow X^\vee$) and go to type IIA. [Feng, He, Kennaway, Vafa])

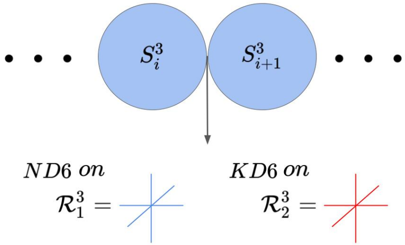
- N D6 branes wrapping special Lagrangian 3-cycles $\mathcal{C}_i \in H_3(X^\vee, \mathbb{Z})$ in $X^\vee$.
  D6_i on S_i^3 gives 4d $\mathcal{N}=1$ vector $\mathcal{V}_i$. At $S_i^3 \cap S_j^3$, there is 4d $\mathcal{N}=1$ Bi-fund chiral B_{ij}.



Big picture emerges from 3 dual frames

Can uplift to M-theory on G_2 manifold $\mathcal{M}_{X^\vee,N}$.

- Intersecting D6 stacks become hyperKahler fibration on associative 3-cycles $\mathcal{C}_i \equiv G_2$ manifold $\mathcal{M}_{X^\vee,N}$.
  This is $U(1)$ fibration over $X^\vee$, where $U(1) = S^1$ degenerates along $\mathcal{C}_i$(=local $\mathbb{C}^2/\mathbb{Z}_{N_i}$ singularity).
- Geometric engineering: $T_{X,N} \equiv T_{\mathcal{M}_{X^\vee,N}}$.
- Make contact with twisted M-theory example:
  Zoom in the intersection



Open questions

- Can we compute metric of $\mathcal{M}_{X^\vee,N}$? [Bryant, Salamon]
- Can we compute full G_2 Donaldson-Thomas partition function, not just instanton part? [Donaldson-Thomas]
- What is the relation between our story and [Joyce, 2016]?
- Is there a set of fundamental building blocks of 4d $\mathcal{N} = 1$ SCFT via G_2 geometric engineering? [Work in progress with Michele Del Zotto and Yehao Zhou.]

Open question: Another G_2 example and its quiver

We have seen 'edge'(bi-fundamental). Missing component to engineer a general quiver: a trivalent vertex, possibly comes from [Acharya-Witten, Atiyah-Witten]:

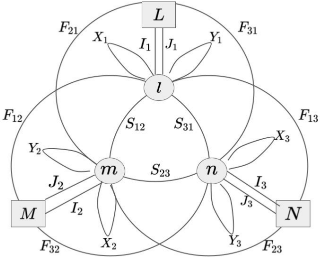


Figure: $G_2 = \wedge^{2,-} WCP^2_{N_1,N_2,N_3}$ and three stacks of D6/D2's.

[Work in progress with Michele Del Zotto and Yehao Zhou.]

Thank You!