Some progress in unification of enumerative and analytic geometry

and gauge and string theory

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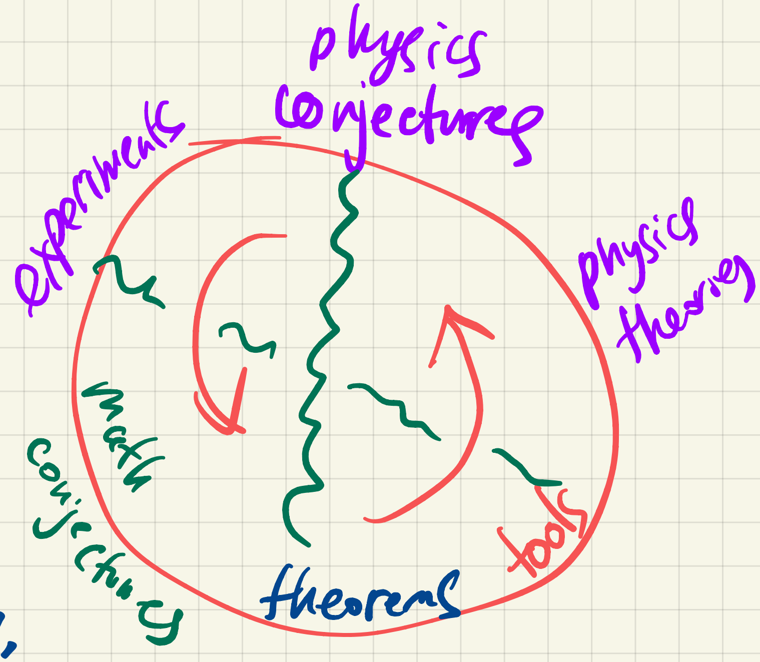
Some progress ....



Special holonomy

Supersymmetry

exact results
starting point of
expansion, perturbation theory,
approximations..

Solve Einstein equations

hyper-Kähler

CY_3

$Sp(2)$

Calabi-Yau's

CY_4

4, 6, 8, 10

CY_5

7

G_2

$Spin(7)$

Relation Between CY_3 and HK

$$\begin{array}{ccl} & \nearrow & \mathcal{M}_V \quad \text{special Kähler} \\ CY_3 & & \\ & \searrow & \mathcal{M}_H \quad \text{quat Kähler} \end{array}$$

$$CY_3 \rightsquigarrow CY_2$$

IIA on X

$\mathcal{N}=2$ $d=4$ sugra +
- → vector multiplets $U(1)^r$ gauge fields
- → hypermultiplets

dynamics of v.m.

$$\mathcal{F}_0(t) \qquad t = [\omega + i B] \in H^2(X, \mathbb{C})$$

$$\mathcal{F}_0(t) = \frac{1}{3!}\int_X t \wedge t \wedge t \; + \sum_{\beta \in H_2(X,\mathbb{Z})} e^{-\int_\beta t}\, GW_{0,\beta}(X)$$

$\mathbb{C}$-volume X

Kähler form

$\mathrm{Re}\, t \longleftrightarrow [\omega]$

$\mathrm{Im}\, t \longleftrightarrow$ [B-field]

enumerative (topology) generating function

at the same time: analytic object complex geometry of mirror CY_3

$$X^\vee$$

Mirror $X^\vee$

$c_1(X^\vee) = 0$

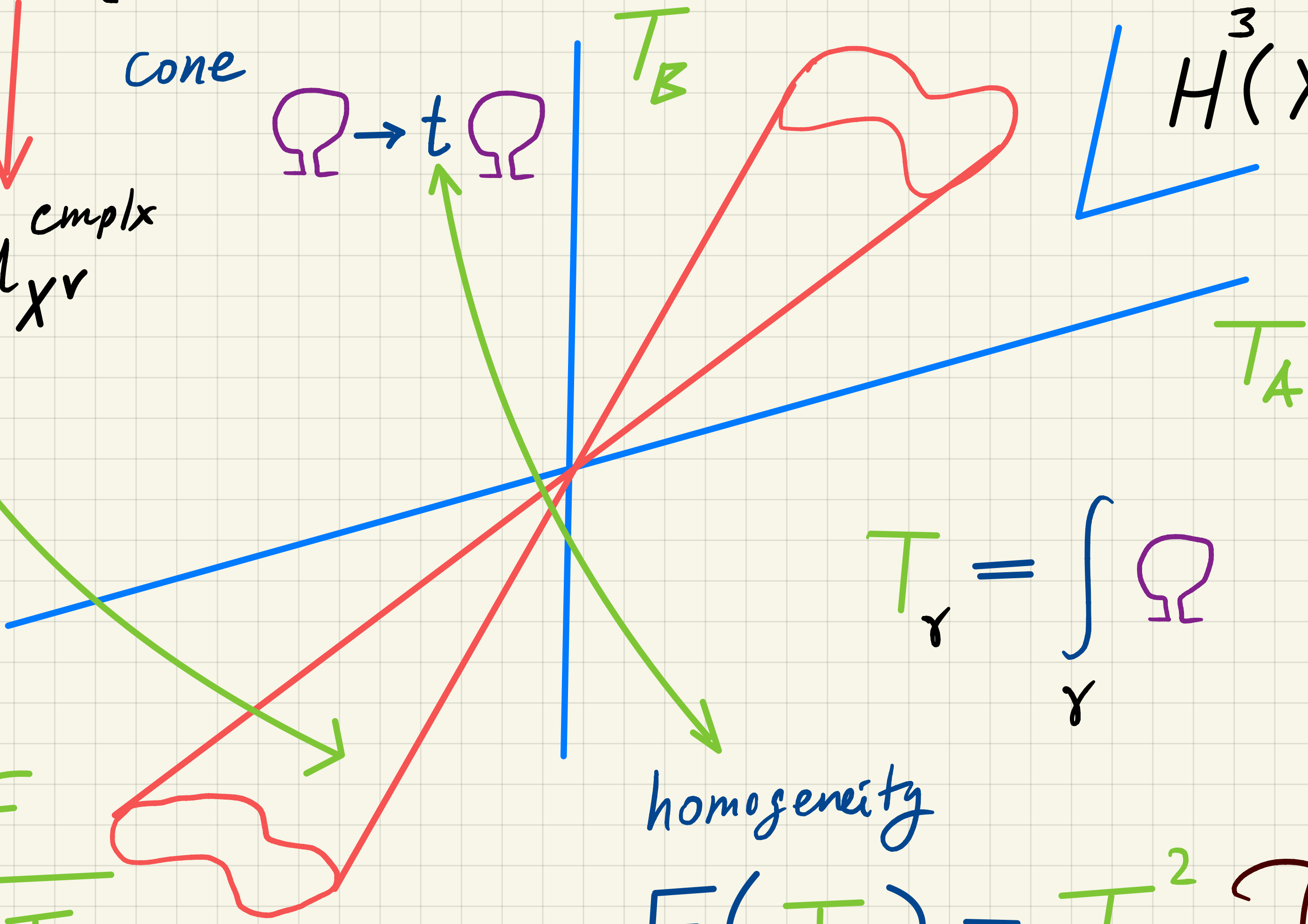
$$\widetilde{\mathcal{M}}_{X^\vee} = \left\{ (X^\vee, \Omega) \mid \Omega \in H^0(K_{X^\vee}) \right\}$$

$\mathcal{M}^{cmplx}_{X^\vee}$

cone

$\Omega \to t\,\Omega$

$H^3(X^\vee, \mathbb{C})$



$$T_\gamma = \int_\gamma \Omega \qquad \gamma \in H_3(X^\vee, \mathbb{Z})$$

$$t = \frac{T_a}{T_0} \qquad a = 1, \dots, h_{2,1}$$

$$T_B = \frac{\partial F}{\partial T_A}$$

homogeneity

$$F(T) = T_0^2\, \mathcal{F}_0(t)$$

Local mirror symmetry and its refinements

$$e^{\sum_{g=0}^{\infty} \mathcal{F}_g^X \hbar^{2g-2}} = e^{\sum_{\beta, g\geq 0} e^{-\int_\beta t} GW^X_{g,\beta}\, \hbar^{2g-2}}$$

$$\|$$

$$Z_X$$

$\uparrow$ enumeration of all curves, possibly disconnected

for non-compact X admits (several) refinements $\mathcal{E}_a$'s

Suppose X is toric $\Longrightarrow$

$[g_{\mu\nu}]$ $\leftarrow$ 10d metric

in addition to scalars in 4d describing the moduli of X

produce scalars and vectors $\longleftrightarrow$ Symmetries of X

Kaluza-Klein

Einstein theory on $\left[\begin{array}{c}\text{U(1) bundle over}\\ B\end{array}\right]$

$\|$

Einstein(B) coupled to Maxwell$(U(1))$

coupled to a scalar field

(size of fibers)

10d (super)gravity on toric X — non-compact CY$_3$

e.g. $\mathcal{O}(-1)\oplus\mathcal{O}(-1) \to \mathbb{P}^1$

superpartners of A^a

$\Sigma^a \in \mathbb{C}$

(y^m) (x^μ)

$X \times \mathbb{R}^4$

isometry of X

V_a^m

$h^X_{mn}\, dy^m dy^n + \eta_{\mu\nu}\, dx^\mu dx^\nu$

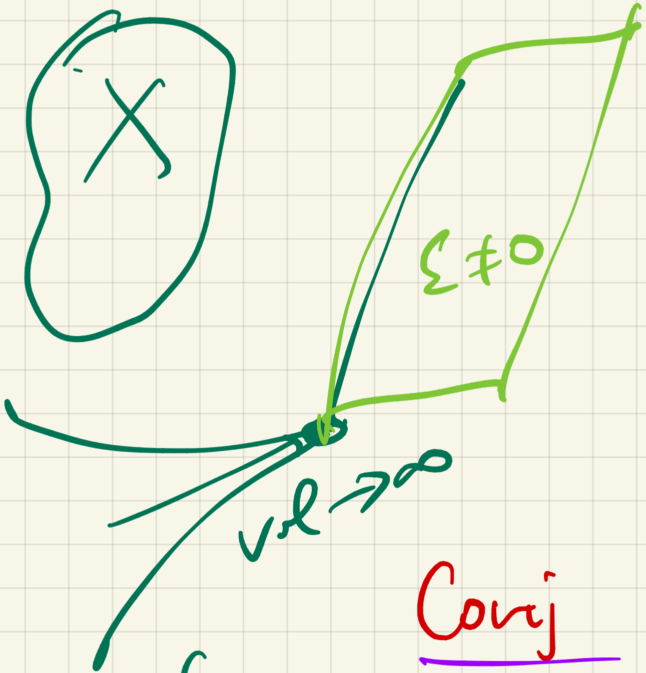
$\downarrow$

$a = 1, 2$

$h^X_{mn}\left(dy^m + V^m_a A^a_\mu dx^\mu\right)\left(dy^n + V^n_b A^b_\nu dx^\nu\right) + \eta_{\mu\nu} dx^\mu dx^\nu$

$\rightsquigarrow$ gauge fields in $\mathbb{R}^4$

Turning on ϵ^a's is allowed without breaking Poincaré$(\mathbb{R}^4)$



refinement

Conj

local mirror symmetry for finite perfs

$$F_0(t) = \int_X \frac{t\wedge t\wedge t}{6} \xrightarrow{\text{turn on } \epsilon} \int_X e^{t+\epsilon\cdot h}$$

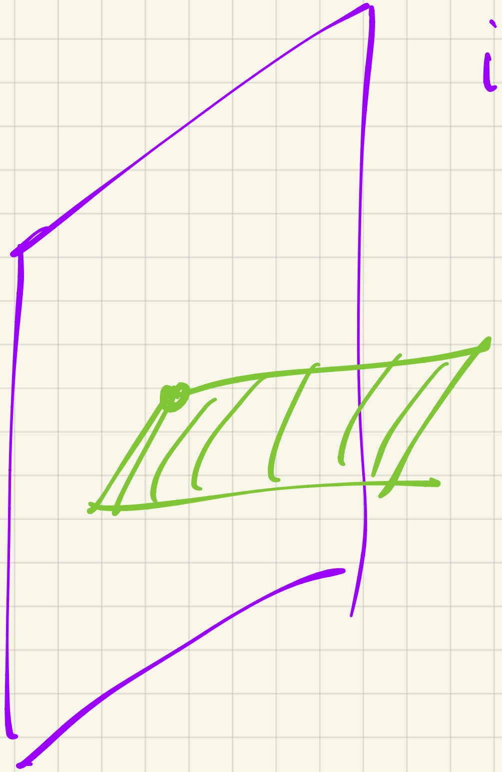
finite

$$= \infty$$

for non-comp X

Another point

when X is non-compact



inside the 10d theory

4d $\mathcal{N}=2$ supersymmetric theory

diff. geometry is enumerative

$\mathcal{M}_V \times \mathcal{M}_H$

$\mathbb{R}^4 \to S^1 \times \mathbb{R}^3$

$\widetilde{\mathcal{M}}_H \times \mathcal{M}_H$

rigid special Kähler

Hyper kähler

HK

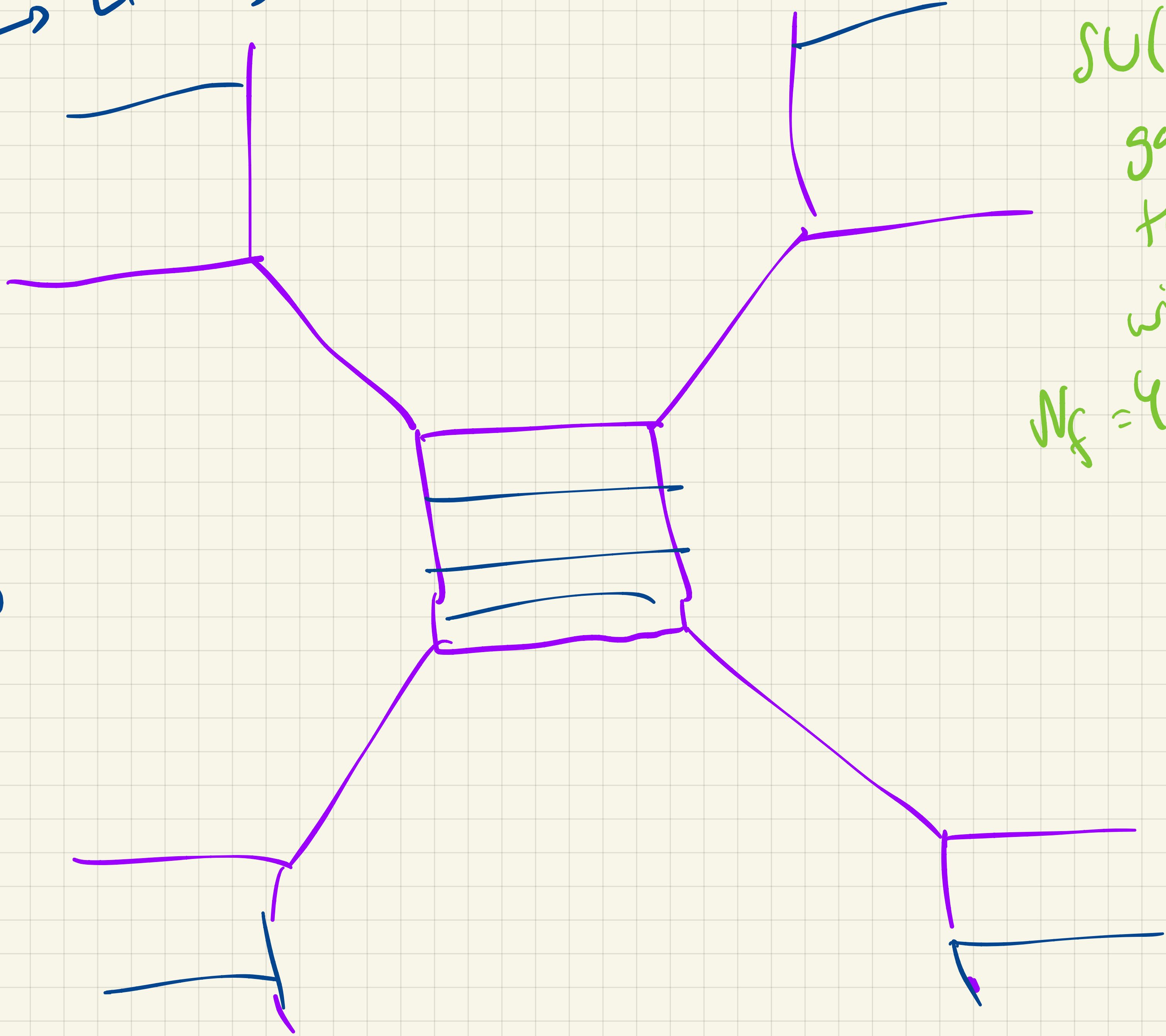
base of an $\boxed{\text{algebraic integrable system}}$

E.g.

$X \to \mathrm{Lie}(\mathbb{T})^* = \mathbb{R}^2$

CY_3

toric $T^3 = U(1)^3 \supset \mathbb{T} = U(1)^2$



$SU(2)$ gauge theory with $N_f = 4$ hypers analytic

geometric meaning

$$Z^{SU(N),\ N_f=2N}_{S^1\times\mathbb{R}^4} = \sum_{k=0}^{\infty} q^k \int_{K} \prod_{i=1}^{2N} \mathrm{Euler}(m_i, \ker \not{D})$$

$$= \exp \sum_{g=0}^{\infty} \hbar^{2g-2} F_g(a, m, \tau; \varepsilon)$$

K-DT (X)

$$\mathcal{M}^{\text{framed}}_{SU(N),k} = \{A \mid F_A^+ = 0\}$$

$$(a, m_i ; q = e^{2\pi i \tau} ; \varepsilon_1, \varepsilon_2)$$

$i = 1, \dots, 4$

$\Uparrow$

$U(1)$

$\mathrm{Lie}(\hat{SU}(N)) \otimes \mathbb{C}$

$SU(2)$

rotations of Euclidean $\mathbb{R}^4$

$$S^1_t \times \underbrace{CY \times \mathbb{R}^4}$$

~~BPS/CFT~~

defect

Conj.

$$\Psi = \sum q^{k} y^{n} \int^{K} \prod_{i=1}^{4} \mathrm{Euler}(m_i)$$

$\mathcal{M}^{\text{framed, parabolic}}$

full HK

$\mathbb{R}^4 = \mathbb{C}^2 \supset \mathbb{C} = \mathbb{R}^2$

$\epsilon_2 \to 0$

classical geometry of $\tilde{\mathcal{M}}_H$

I

I

ϵ_2/ϵ_1

obeys a Knizhnik-Zamolodchikov equation

Conj: q-KZ-ell for 5d/6d $\simeq$ quantum version of $\mathbb{C}$-symplectic geometry $\tilde{\mathcal{M}}_H$

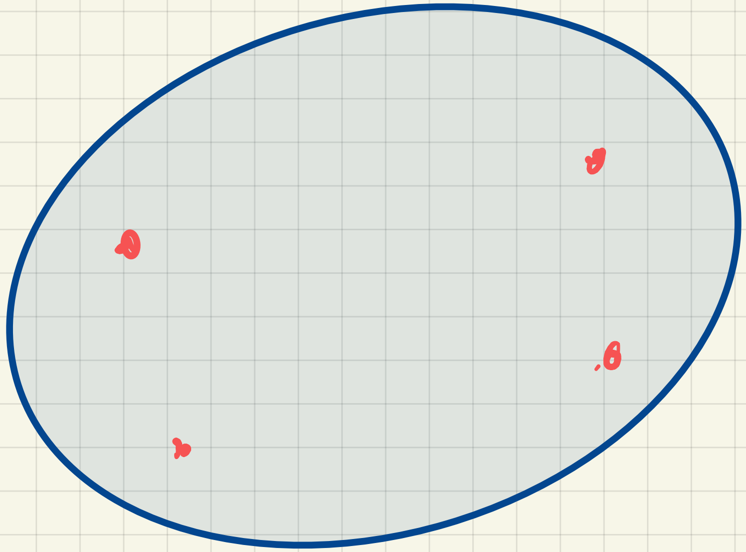
For SU(N) gauge in 4d

$\overline{\mathcal{M}}_K$

analogen Hitchin moduli space for

Nahm

moduli space of SU(2) monopoles with 2N singularities on $\mathbb{R}^2 \times S^1$



SU(N) Higgs pairs

S^2, 4pt

SL(N)

$(\bar{A}, \phi)$

$$D_{\bar{A}}\phi = \sum J_i \delta^{(2)}(z_i)$$

(m_i)