Counting surfaces on Calabi-Yau 4-folds

joint with

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Curves on smooth projective 3-fold X $/\mathbb{C}$

- stable maps $f : C \to X$
- 1-dimensional closed subschemes $Z \subset X$
- PT pairs (F, s):
  1-dimensional sheaf F, no subsheaves of dimension ≤ 0
  $s \in H^0(X, F)$, $\mathrm{coker}(s)$ of dimension ≤ 0

$\rightsquigarrow$ moduli spaces $\overline{M}_{g,n}(X, \beta)$ Kontsevich
$I_{\beta,n}(X)$ Grothendieck, $P_{\beta,n}(X)$ Le Potier, Pandharipande-Thomas

$$\mathrm{ch}(\mathcal{O}_Z) = \mathrm{ch}(F) = (0, 0, \beta, n - \tfrac{1}{2}\beta c_1(X)) \in H^*(X, \mathbb{Q})$$

Theorem (Stoppa-Thomas)

$\exists$ projective scheme $\mathcal{N}$, reductive group $G \curvearrowright \mathcal{N}$, G-linearizations $\mathcal{L}_{-1}, \mathcal{L}_0$ such that there are no strictly semistables and

$$\mathcal{N}/\!/_{\mathcal{L}_{-1}} G = I_{\beta,n}(X), \quad \mathcal{N}/\!/_{\mathcal{L}_0} G = P_{\beta,n}(X)$$

Surfaces on smooth projective 4-fold X

- 2-dimensional closed subschemes $Z \subset X$
- for $i = 0, 1$: PT_i pairs (F, s):
  2-dimensional sheaf F, no subsheaves of dimension $\leq i$
  $s \in H^0(X, F)$, $\mathrm{coker}(s)$ of dimension $\leq i$

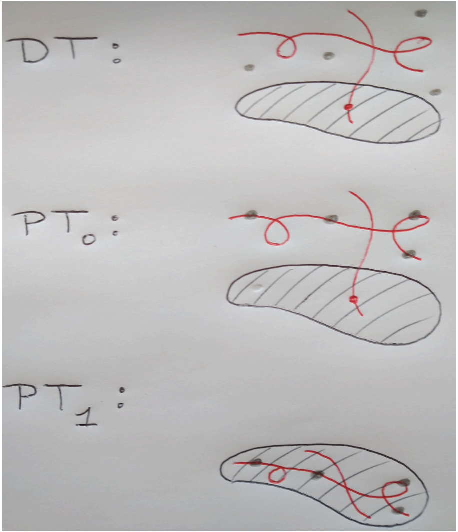
$$\rightsquigarrow I_{\gamma,\beta,n}(X),\ P^{(i)}_{\gamma,\beta,n}(X), \quad i = 0, 1$$

$$\mathrm{ch}(\mathcal{O}_Z) = \mathrm{ch}(F) = (0, 0, \gamma, \beta, n - \tfrac{1}{2}\beta c_1(X) - \gamma \mathrm{td}_2(X)) \in H^*(X, \mathbb{Q})$$

Theorem (Bae-K-Park)

$\exists$ projective scheme $\mathcal{N}$, reductive algebraic group $G \curvearrowright \mathcal{N}$, G-linearizations $\mathcal{L}_{-1}, \mathcal{L}_0, \mathcal{L}_1$ such that there are no strictly semistables and

$$\mathcal{N} /\!/_{\mathcal{L}_{-1}} G = I_{\gamma,\beta,n}(X), \quad \mathcal{N} /\!/_{\mathcal{L}_i} G = P^{(i)}_{\gamma,\beta,n}(X), \quad i = 0, 1$$


DT:
PT_0:
PT_1:

For X Calabi-Yau 3-fold, let $M = I_{\beta,n}(X), P_{\beta,n}(X)$

element $[(F,s)] \in M$ gives complex

$$I^\bullet = [\mathcal{O}_X \xrightarrow{s} F] \in D^b(X)$$

where $\mathrm{Hom}(I^\bullet, I^\bullet) = \mathbb{C}$, $\mathrm{Ext}^{<0}(I^\bullet, I^\bullet) = 0$, $\quad \det(I^\bullet) \cong \mathcal{O}_X$

$\rightsquigarrow \exists$ 2-term perfect obstruction theory $\underbrace{\Omega_M^{\mathrm{vir}}}_{(R\mathrm{Hom}_X(I^\bullet, I^\bullet)_0[1])^\vee} \to \mathbb{L}_M$ Thomas, PT

symmetric $\Omega_M^{\mathrm{vir}} \cong (\Omega_M^{\mathrm{vir}})^\vee[1]$

virtual dimension $\mathrm{vd} := \mathrm{rk}(\Omega_M^{\mathrm{vir}}) = 0$

Local structure M Brav-Bussi-Joyce, using Pantev-Toën-Vaquié-Vezzosi

let $\Phi : A \to \mathbb{C}$, $\quad$ A smooth variety, $\quad$ $s := d\Phi \in H^0(A, \Omega_A)$

let $M = Z(s) \subset A$

$$\begin{array}{ccccc} T_A|_M & \xrightarrow{d\circ s^*} & \Omega_A|_M & \qquad & \Omega_M^{\mathrm{vir}} \\ \Big\downarrow{\scriptstyle s^*} & & \Big\| & & \Big\downarrow \\ I_{M/A}/I_{M/A}^2 & \xrightarrow{d} & \Omega_A|_M & \qquad & \mathbb{L}_M \end{array}$$

[true globally for $I_{0,n}(\mathbb{C}^3) = \mathrm{Hilb}^n(\mathbb{C}^3)$ Behrend-Bryan-Szendrői (...)]

For X Calabi-Yau 4-fold, let $M = I_{\gamma,\beta,n}(X), P^{(i)}_{\gamma,\beta,n}(X)$

element $[(F,s)] \in M$ gives complex

$$I^\bullet = [\mathcal{O}_X \xrightarrow{s} F] \in D^b(X)$$

where $\mathrm{Hom}(I^\bullet, I^\bullet) = \mathbb{C}$, $\mathrm{Ext}^{<0}(I^\bullet, I^\bullet) = 0$, $\quad \det(I^\bullet) \cong \mathcal{O}_X$

$\rightsquigarrow \exists$ 3-term obstruction theory $\underbrace{\Omega^{\mathrm{vir}}_M}_{(R\mathrm{Hom}_X(I^\bullet, I^\bullet)_0[1])^\vee} \to \mathbb{L}_M$ Bae-K–Park

also: Gholampour-Jiang–Lo for $\mathrm{DT}/\mathrm{PT}_1$ pairs

$\gamma = 0$ (curves): Cao+(...)

symmetric $\Omega^{\mathrm{vir}}_M \cong (\Omega^{\mathrm{vir}}_M)^\vee[2] \quad (\Rightarrow \det(\Omega^{\mathrm{vir}}_M)^{\otimes 2} \cong \mathcal{O}_M)$

virtual dimension $\mathrm{vd}_{\mathbb{R}} := \mathrm{rk}(\Omega^{\mathrm{vir}}_M) = 2n - \gamma^2$

Local model M BBJ, using PTVV

- A smooth variety, $V \to A$ vector bundle
- $M = Z(s) \subset A$, $s \in H^0(A, V)$
- non-degenerate symmetric bilinear form $q : V \otimes V \to \mathcal{O}_A$
- s isotropic

$$\begin{array}{ccccccc} T_A|_M & \xrightarrow{(d\circ s^*)^*} & V|_M \overset{q}{\cong} V^*|_M & \xrightarrow{d\circ s^*} & \Omega_A|_M & & \Omega_M^{\mathrm{vir}} \\ & & \downarrow{\scriptstyle s^*} & & \| & & \downarrow \\ & & I_{M/A}/I_{M/A}^2 & \xrightarrow{d} & \Omega_A|_M & & \mathbb{L}_M \end{array}$$

$$\mathrm{vd}_{\mathbb{R}} = \mathrm{rk}(\Omega_M^{\mathrm{vir}}) = 2\dim(A) - \mathrm{rk}(V)$$

[true globally for $I_{0,0,n}(\mathbb{C}^4) = \mathrm{Hilb}^n(\mathbb{C}^4)$ K-Rennemo]

CY3: Behrend-Fantechi, Li-Tian $\Rightarrow [M]^{\mathrm{vir}} \in H_0(M)$

Behrend-Fantechi, Kontsevich, Nekrasov-Okounkov $\Rightarrow \widehat{\mathcal{O}}_M^{\mathrm{vir}} \in K_0(M)$

CY4: Borisov-Joyce, Oh-Thomas $\Rightarrow [M]^{\mathrm{vir}} \in H_{\mathrm{vd}_{\mathbb{R}}}(M)$

Oh-Thomas $\Rightarrow \widehat{\mathcal{O}}_M^{\mathrm{vir}} \in K_0(M, \mathbb{Z}[\frac{1}{2}])$

requires orientation: square root $\det(\Omega_M^{\mathrm{vir}})^{\otimes 2} \cong \mathcal{O}_M$

locally: e.g. choice of maximal isotropic $0 \to \Lambda \to V \to \Lambda^* \to 0$

globally: existence shown by Cao-Gross-Joyce
using Joyce-Tanaka-Upmeier

$$\int_{[M]^{\mathrm{vir}}} \alpha, \quad \alpha \in H^*(M, \mathbb{Q}), \quad \chi(M, \widehat{\mathcal{O}}_M^{\mathrm{vir}} \otimes E), \quad E \in K^0(M)$$

Three reasons why the invariant is trivial...

Example. $X \subset \mathbb{P}^5$ sextic (then $h^{3,1}(X) = 426$)

suppose $\exists \mathbb{P}^2 \subset X$, $\gamma = [\mathbb{P}^2]$

Issue 1+2: $\mathrm{vd}_{\mathbb{R}} = 2\chi(\mathcal{O}_{\mathbb{P}^2}) - \gamma^2 = -19$ odd and negative

Issue 3: take smooth family $\mathcal{X} \to (B, 0)$ of sextics, B contractible

$\mathrm{Hdg}_\gamma := \{t \in B : \widetilde{\gamma}_t \in H^4(\mathcal{X}_t, \mathbb{Q}) \text{ has type } (2,2)\} \subset B$

has codimension 19

Solution. Combine reducing procedure of Kiem-Park and K-Thomas

$\xi \in H^1(X, T_X) \overset{\text{CY4}}{\cong} H^1(X, \Omega_X^3)$ gives cosection

$$\mathrm{Ext}^2(I^\bullet, I^\bullet) \xrightarrow{\mathrm{sr}} H^3(X, \Omega_X) \xrightarrow{\cup\xi} H^4(X, \Omega_X^4) \cong \mathbb{C}$$

sr semi-regularity map Buchweitz-Flenner

for $\gamma \in H^4(X, \mathbb{Z})$ of type $(2, 2)$:

$$B_\gamma : H^1(X, T_X) \otimes H^1(X, T_X) \longrightarrow \mathbb{C}, \quad \xi_1 \otimes \xi_2 \mapsto \int_X \iota_{\xi_1}\iota_{\xi_2}\gamma \cup \Omega$$

symmetric and of rank ρ_γ $\quad (= \mathrm{codim}\, T_0\mathrm{Hdg}_\gamma$, if KS_0 iso)

Carlson-Green-Griffiths

Theorem (Bae-K-Park)

For X CY4 and $M = I_{\gamma,\beta,n}(X), P^{(i)}_{\gamma,\beta,n}(X)$, there exists

$$[M]^{\mathrm{red}} \in H_{\mathrm{rvd}_\mathbb{R}}(M, \mathbb{Z}[\tfrac{1}{2}]), \quad \mathrm{rvd}_\mathbb{R} := 2n - \gamma^2 + \rho_\gamma$$

$$H^1(X, T_X) \xrightarrow{\ \mathrm{ob} := \iota\_\mathrm{At}(I^\bullet)\ } \mathrm{Ext}^2(I^\bullet, I^\bullet)$$

$$H^1(X, T_X) \xrightarrow{\ \iota\_\gamma\ } H^3(X, \Omega_X) \xleftarrow{\ \mathrm{sr}\ } \mathrm{Ext}^2(I^\bullet, I^\bullet)$$

$$H^3(X, \Omega_X) \xrightarrow[\cong]{\ \mathrm{SD}\ } H^1(X, T_X)^*$$

$$H^1(X, T_X) \xrightarrow{\ B_\gamma\ } H^1(X, T_X)^* \xleftarrow{\ \mathrm{ob}^\vee\ } \mathrm{Ext}^2(I^\bullet, I^\bullet)$$

Pick: $V \hookrightarrow H^1(X, T_X) \twoheadrightarrow \dfrac{H^1(X, T_X)}{\ker(B_\gamma)}$, with the composite $V \xrightarrow{\cong} \dfrac{H^1(X, T_X)}{\ker(B_\gamma)}$

Get: $V \hookrightarrow \mathrm{Ext}^2(I^\bullet, I^\bullet) \quad \Rightarrow \quad \mathrm{Ext}^2(I^\bullet, I^\bullet) \twoheadrightarrow V^* \cong \mathbb{C}^{\rho_\gamma}$

Theorem

For any $I^\bullet \in M$: $I^\bullet$ is semiregular $\Leftrightarrow$ M is smooth at $I^\bullet$ of dimension $\mathrm{rvd}_{\mathbb{R}} = \underbrace{2n - \gamma^2 + \rho_\gamma \geq 0}_{\textit{tadpole conjecture F-theory?}}$

Fix:

1. X CY4, $\gamma \in H^4(X,\mathbb{Z})$ of type (2,2)
2. $M = I_{\gamma,\beta,n}(X), P^{(i)}_{\gamma,\beta,n}(X)$
3. smooth family $\mathcal{X} \to (B,0)$, B smooth algebraic, $\mathcal{X}_0 = X$
4. horizontal section $\{\widetilde{\gamma}_t \in H^4(\mathcal{X}_t,\mathbb{Z})\}_{t\in B}$, $\widetilde{\gamma}_0 = \gamma$

Family of moduli $\mathcal{M} \to (B,0)$, $\mathcal{M}_0 = M$,

$\rightsquigarrow \int_{[\mathcal{M}_t]^{\mathrm{red}}} \alpha$, $\quad \alpha$ descendent insertion

Theorem

$\int_{[\mathcal{M}_t]^{\mathrm{red}}} \alpha$ *is invariant over* Hdg_γ

$\int_{[\mathcal{M}_0]^{\mathrm{red}}} \alpha \neq 0 \Rightarrow \forall t \in \mathrm{Hdg}_\gamma\ \exists [I^\bullet] \in \mathcal{M}_t\colon\ \widetilde{\gamma}_t = \mathrm{ch}_2(I^\bullet) \in H^{2,2}(\mathcal{X}_t)$

$\int_{[\mathcal{M}_0]^{\mathrm{red}}} \alpha \neq 0 \Rightarrow$ variational Hodge conjecture for $\mathcal{X} \to (B,0)$, γ

Corollary

$S \subset X$ semiregular, lci, rigid $\Rightarrow \mathrm{rvd}_{\mathbb{R}} = 0$, *i.e.* $\rho_{[S]} = h^1(N_{S/X})$ *and* $\int_{[\mathcal{M}_0]^{\mathrm{red}}} 1 \neq 0$

Example.

- $X \subset \mathbb{P}^5$ sextic: any ci surface $S \subset \mathbb{P}^5$ st. $S \subset X$ is semiregular STEENBRINK, rigid for types

$$(1,1,1),(1,1,2),(1,1,3),(1,2,2)$$
$$(1,2,3),(2,2,2),(2,2,3),(1,1,4)$$

- $S \cong \mathbb{P}^2$ on any CICY4
- "low degree" S on any CICY4

Example. For $X \subset \mathbb{P}^5$ sextic

1. if $\mathbb{P}^2 \subset X$ and $\gamma = 2[\mathbb{P}^2]$, then $\mathrm{rvd}_{\mathbb{R}} < 0$
2. if $\mathbb{P}^2 \cup_{\mathrm{pt}} \mathbb{P}^2 \subset X$ and $\gamma = [\mathbb{P}^2 \cup_{\mathrm{pt}} \mathbb{P}^2]$, then $\mathrm{rvd}_{\mathbb{R}} < 0$

... more cosections?

Take X CY4 and $M = I_{\gamma,\beta,n}(X), P^{(i)}_{\gamma,\beta,n}(X)$

Universal pair $\mathbb{I}^\bullet = \{\mathcal{O} \to \mathbb{F}\}$

Tautological complex $L^{[n]} := R\pi_{M*}\mathbb{F} \boxtimes L, \quad L \in \mathrm{Pic}(X)$

Nekrasov genus $\langle\!\langle L \rangle\!\rangle_{X,\gamma,\beta,n} := \chi(M, \widehat{\mathcal{O}}^{\mathrm{vir}} \otimes \widehat{\Lambda}^\bullet(L^{[n]} \otimes y^{-1}))$

where $\widehat{\Lambda}^\bullet E := \sum_i (-1)^i \frac{\Lambda^i E}{\sqrt{\det(E)}}, \quad y$ formal parameter

Conjecture (A)

There exist orientations such that

$$\frac{\sum_n \langle\!\langle L \rangle\!\rangle^{\mathrm{DT}}_{X,\gamma,\beta,n} q^n}{\sum_n \langle\!\langle L \rangle\!\rangle^{\mathrm{DT}}_{X,0,0,n} q^n} = \sum_n \langle\!\langle L \rangle\!\rangle^{\mathrm{PT}_0}_{X,\gamma,\beta,n} q^n$$

$\gamma = 0$ (curves): conjectured by Cao-K-Monavari

Take X toric CY4, $\quad T = \{t_1 t_2 t_3 t_4 = 1\} \subset (\mathbb{C}^*)^4 \curvearrowright X$

Fix $Z \subset X = \mathbb{C}^4$ 2-dimensional, T-invariant, no embedded points

$\mathcal{O}_Z|_{(\mathbb{C}^*)^2 \times \mathbb{C}^2} \cong \mathcal{O}_A[x_1^{\pm 1}, x_2^{\pm 1}], \quad A \subset \mathbb{C}^2$ 0-dimensional
$\mathcal{O}_Z|_{\mathbb{C}^* \times \mathbb{C}^3} \cong \mathcal{O}_B[x_1^{\pm 1}], \quad B \subset \mathbb{C}^3$ 1-dimensional

$\Rightarrow \boldsymbol{\lambda} = \{\lambda_{ij}\}_{1 \leq i < j \leq 4}$ finite 2D partitions
$\Rightarrow \boldsymbol{\mu} = \{\mu_i\}_{i=1}^4$ compatible 3D partitions

$\rightsquigarrow$ topological vertex $\mathsf{V}^{\mathrm{DT}}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(q,t,y)$, $\mathsf{V}^{\mathrm{PT}_0}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(q,t,y)$

Conjecture (B)

Fix $\boldsymbol{\lambda}, \boldsymbol{\mu}$ such that no moduli on PT_0 side. Then $\exists$ signs such that

$$\frac{\mathsf{V}^{\mathrm{DT}}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(q,t,y)}{\mathsf{V}^{\mathrm{DT}}_{\varnothing\varnothing\varnothing\varnothing\varnothing\varnothing\varnothing\varnothing\varnothing\varnothing}(q,t,y)} = \mathsf{V}^{\mathrm{PT}_0}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(q,t,y)$$

Roughly: by Oh-Thomas localization: (B) $\Rightarrow$ (A) for X toric CY4

Magnificent Four NEKRASOV

SUSY Yang-Mills on $\mathbb{C}^4 \rightsquigarrow$ measure on 4D partitions π

$$\mathsf{V}^{\mathrm{DT}}_{\varnothing\varnothing\varnothing\varnothing\varnothing\varnothing\varnothing\varnothing\varnothing\varnothing}(q,t,y) = \mathrm{Exp}\left(\frac{[t_1t_2][t_1t_3][t_2t_3][y]}{[t_1][t_2][t_3][t_4][y^{\frac{1}{2}}q][y^{\frac{1}{2}}q^{-1}]}\right)$$

$[x] := x^{\frac{1}{2}} - x^{-\frac{1}{2}}$, equality proved by K-RENNEMO

on $\mathrm{Hilb}^n(\mathbb{C}^4)$ local structure holds globally and explicitly!

derivation NEKRASOV-PIAZZALUNGA sign $(-1)^{|\pi|+|\{(i,i,i,j)\in\pi \,:\, i<j)\}|}$

Compact case:

BOJKO using JOYCE's (conjectural?) vertex algebra wall-crossing

Special case $\boldsymbol{\lambda}$ general $\boldsymbol{\mu} = \boldsymbol{\mu}_{\min}$ minimal, i.e. $\mathrm{PT}_0 = \mathrm{PT}_1$:

$$\mathsf{V}^{\mathrm{PT}_0}_{\boldsymbol{\lambda},\boldsymbol{\mu}_{\min}}(q,t,y)$$

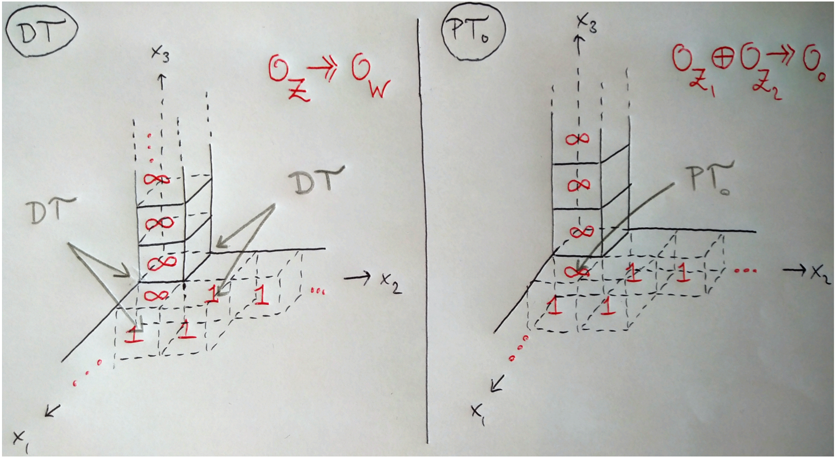
polynomial of deg $\ell(\mathcal{E}xt^3(\mathcal{O}_Z,\mathcal{O}_X)) < \infty$, $\quad Z$ pure 2-dim $\leftrightarrow \boldsymbol{\lambda}$

$\ell(\mathcal{E}xt^3(\mathcal{O}_Z,\mathcal{O}_X)) = 0 \Leftrightarrow Z$ Cohen-Macaulay

For $Z = \mathbb{C}^2 \cup_{\mathrm{pt}} \mathbb{C}^2$, (normalized) topological vertex is

$$\mathsf{V}^{\mathrm{PT}_0}_{(1)\varnothing\varnothing\varnothing\varnothing(1)\boldsymbol{\mu}_{\min}} = 1 + \frac{[t_1t_2][y]}{[t_1t_3][t_2t_3]}q$$

Question. Related to Nekrasov's gauge origami?


DT
x_3
$\mathcal{O}_Z \twoheadrightarrow \mathcal{O}_W$
DT
DT
x_2
x_1
PT$_0$
x_3
$\mathcal{O}_{Z_1} \oplus \mathcal{O}_{Z_2} \twoheadrightarrow \mathcal{O}_0$
PT$_0$
x_2
x_1

Let X be compact CY4 and $Y \subset X$ smooth connected divisor

Let $M_X := I_{\gamma,\beta,n}(X), P^{(i)}_{\gamma,\beta,n}(X), \quad M_Y := I_{\gamma,\beta,n}(Y), P^{(i)}_{\gamma,\beta,n}(Y)$

"Twist away divisor": M_Y relates to DT/PT moduli spaces on Y

Theorem

Assume:

- $\mathcal{O}_X(Y)^{[n]}$ *is a vector bundle concentrated in deg 0*
- $(R\mathcal{H}om_{\pi_Y}(\mathbb{I}^\bullet_Y, \mathbb{I}^\bullet_Y)_0)^\vee \to \mathbb{L}_{M_Y}$ *2-term obstruction theory*

Then, for any orientation on M_X, there exists a sign $\pm$ for each connected component $M_Y^e \subset M_Y$ such that

$$e(\mathcal{O}_X(Y)^{[n]}) \cap [M_X]^{\mathrm{vir}} = \sum_e \pm \iota_*[M_Y^e]^{\mathrm{vir}}$$

By Park's virtual pull-back formula, $\rightsquigarrow$ special cases Conj. (A)

Let $X \xrightarrow{p} B$ be Weierstraß CY4, discriminant $\Delta \subset B$, f fibre class

Conjecture (C)

For $\beta \in H_2(B)$ effective, $\gamma := p^*\beta$, $n := \frac{\beta \cdot \Delta}{12}$

$$\frac{\sum_d Q^d \int_{[P^{(0)}_{\gamma,df,n}(X)]^{\mathrm{vir}}} \prod_i \tau_{k_i}(\sigma_i)}{\sum_d Q^d \int_{[P_{df,0}(X)]^{\mathrm{vir}}} 1} = \sum_d Q^d \int_{[P^{(1)}_{\gamma,df,n}(X)]^{\mathrm{vir}}} \prod_i \tau_{k_i}(\sigma_i)$$

for all $\sigma_i \in H^*(X)$ such that $p_*\sigma_i \in H^{\geq 2}(B)$

proved when all $C \subset B$ with $[C] = \beta$ irreducible Gorenstein and:

- **$K_B < 0$ and X toric**: via stationary correspondence on B
  OBLOMKOV-OKOUNKOV-PANDHARIPANDE
- **$K_B = 0$**: via DT–PT correspondence on B
  PANDHARIPANDE-THOMAS, BRIDGELAND, TODA

Let X be compact CY4, $v = (0, 0, \gamma, \beta, n - \frac{1}{12}\gamma c_2(X)) \in H^*(X, \mathbb{Q})$

Theorem

Assume:

- *pure surfaces $S \subset X$ with $[S] = \gamma$ are CM with constant $\mathrm{ch}(\mathcal{O}_S) =: v(\gamma)$.*
- *$\beta - v(\gamma)_3$ is irreducible effective*
- *$\forall$ pure 1-dim sheaf G with $\mathrm{ch}(G) = v - v(\gamma)$ we have*

$$\mathrm{Ext}^2_X(I_{S/X}, G) = 0 \quad \forall [S] \in I_{v(\gamma)}(X)$$

Consider $p : P_v^{(0)}(X) \to M_{v-v(\gamma)}(X) \times I_{v(\gamma)}(X)$

$$(F, s) \mapsto (\text{torsion subsheaf } F, \text{support } F)$$

Then $[P_v^{(0)}(X)]^{\mathrm{vir}} = p^!\Big([M_{v-v(\gamma)}(X)]^{\mathrm{vir}} \times [I_{v(\gamma)}(X)]^{\mathrm{vir}}\Big)$

$\rightsquigarrow$ special case PT_0–PT_1 correspondence