# **Hyperkähler implosion**

Frances Kirwan
Oxford University

(based on joint work with Andrew Dancer and Andrew Swann)

Recall SYMPLECTIC REDUCTION:
$(X, \omega)$ symplectic manifold (over $\mathbb{R}$)
$K$ compact Lie group with Lie algebra $\mathfrak{k}$ acting on $(X, \omega)$
$\mu : X \to \mathfrak{k}^*$ **moment map** satisfies

$$d\mu_x(\xi).a = \omega_x(\xi, a_x) \qquad \forall x \in X, \xi \in T_x X, a \in \mathfrak{k}$$

and $\mu$ is $K$-equivariant (for the coadjoint action on $\mathfrak{k}^*$).

[**Special case**: $(X, \omega)$ is compact **Kähler** and $K$ acts holomorphically; then the action extends to $K_{\mathbb{C}} =$ complexification of $K$.]

Let $\zeta \in \mathfrak{k}^*$ be a regular value of $\mu : X \to \mathfrak{k}^*$.
Let $K_\zeta$ be its stabiliser for the coadjoint action.

Then the **Marsden-Weinstein reduction** at $\zeta$
$\mu^{-1}(\zeta)/K_\zeta$ is a symplectic orbifold.

Often we take $\zeta = 0$: $X/\!/K = \mu^{-1}(0)/K$ **'symplectic quotient'**

Kähler case: grad$\mu(x).a = i\, a_x \; \forall a \in \mathfrak{k}$ and $\mu^{-1}(0)$ is a *slice* for exp(i$\mathfrak{k}$), where $K_{\mathbb{C}} = K\,$exp(i$\mathfrak{k}$). Under reasonable conditions (eg $X$ compact, 0 regular value of $\mu$), there is an induced Kähler structure on $\mu^{-1}(0)/K =$ (open subset of $X$)$/G$.

When 0 is *not* a regular value of $\mu$, then $X/\!/K = \mu^{-1}(0)/K$ has a stratified symplectic/Kähler structure with more serious singularities.

If $X$ is projective, $L \to X$ is a very ample line bundle, and $K$ acts linearly on $X$ preserving a Fubini-Study form, then the symplectic quotient $X/\!/K$ can be identified with the GIT quotient $X/\!/K_{\mathbb{C}} := \mathrm{Proj}(\mathcal{O}_L(X)^{K_{\mathbb{C}}}) \cong X^{ss}/\!\sim$ of $X$ by $K_{\mathbb{C}}$.

## Hyperkähler quotients

$X$ hyperkähler manifold, cx structures $i, j, k$ with $ij = k = -ji$, $i^2 = j^2 = k^2 = -1$ etc, metric $g$, Kähler forms $\omega_1$, $\omega_2$, $\omega_3$; compact group $K$ acting on $X$ preserving $i, j, k, g$.
Hyperkähler moment map

$$\mu = (\mu_1, \mu_2, \mu_3) : X \to \mathfrak{k}^* \otimes \mathbb{R}^3 \cong i\mathfrak{k}^* \oplus j\mathfrak{k}^* \oplus k\mathfrak{k}^*$$

Often fix the cx structure $i$ and write $\mu = \mu_{\mathbb{R}} \oplus \mu_{\mathbb{C}} : X \to \mathfrak{k}^* \oplus \mathfrak{k}^*_{\mathbb{C}}$ with $\mu_{\mathbb{R}} = \mu_1$ and $\mu_{\mathbb{C}} = \mu_2 + i\mu_3$; then $\mu_{\mathbb{C}}$ is holomorphic wrt $i$.
Hyperkähler quotient

$$X/\!/\!/K = \mu^{-1}(0)/K = \mu_{\mathbb{C}}^{-1}(0)/\!/K$$

(Hitchin, Karlhede, Lindström, Rocek)

**Examples:** moduli spaces of Higgs bundles, $T^*K_{\mathbb{C}}$, (closures of) coadjoint orbits in $\mathfrak{k}^*_{\mathbb{C}}$, quiver varieties ...
(Hitchin, Kronheimer, Kovalev, Kobak–Swann, Nakajima ... )

## SYMPLECTIC IMPLOSION (Guillemin, Jeffrey, Sjamaar 2001)

**Ingredients**: $(X, \omega)$ symplectic manifold

Hamiltonian action of compact connected group $K$

$\mu : X \to \mathfrak{k}^*$ moment map

$T$ maximal torus of $K$, Lie algebra $\mathfrak{t} \subseteq \mathfrak{k}$

Weyl group $W = N_T/T$ acts on $\mathfrak{t}$ and $\mathfrak{t}^*$ which decompose into Weyl chambers.

$\mathfrak{t}^*_+ =$ positive Weyl chamber $\cong \mathfrak{t}^*/W \cong \mathfrak{k}^*/K$.

Recall $K_\zeta = \{k \in K | (Ad^*k)\zeta = \zeta\}$. Its commutator subgroup $[K_\zeta, K_\zeta]$ is generated by the commutators $khk^{-1}h^{-1}$ for $k, h \in K_\zeta$.

The **imploded cross-section** of $X$ is $X_{impl} = \mu^{-1}(\mathfrak{t}^*_+)/\sim$ where $x \sim y \Leftrightarrow x = ky$ for some $k \in [K_\zeta, K_\zeta]$ with

$$\zeta = \mu(x) = \mu(y) \in \mathfrak{t}^*_+.$$

**Examples:** (1) $K = SU(2)$ so $\mathfrak{t}^*_+ = [0, \infty) = \{0\} \sqcup (0, \infty)$, and

$$X_{impl} = \frac{\mu^{-1}(0)}{SU(2)} \sqcup \mu^{-1}((0, \infty))$$

(2) $K = SU(3)$. Over the interior points of $\mathfrak{t}^*_+$ no collapsing occurs since $[K_\zeta, K_\zeta] = [T, T]$ is trivial. Over nonzero boundary points of $\mathfrak{t}^*_+$ we have $K_\zeta \cong U(2)$ and $[K_\zeta, K_\zeta] \cong SU(2)$.
Over $0 \in \mathfrak{t}^*_+$ we have $K_\zeta = SU(3) = [K_\zeta, K_\zeta]$.

$X_{impl}$ inherits a symplectic structure and $T$-action with moment map $X_{impl} \to \mathfrak{t}^*_+ \subseteq \mathfrak{t}^*$ induced by the restriction of $\mu$.

$K$ acts on itself by left translation and hence on $T^*K \cong \mathfrak{k}^* \times K$ with moment map

$$\mu(p,q).a = p \cdot a_q \ \ \forall a \in \mathfrak{k},\, q \in K,\, p \in T^*_q K = \mathfrak{k}^*.$$

$(T^*K)_{impl}$ '**universal imploded cross-section**' is an affine algebraic variety over $\mathbb{C}$. In general

$$X_{impl} \cong (X \times (T^*K)_{impl})//K$$

which is an algebraic variety if $X$ is algebraic.

**Example:** $K = SU(2) \cong S^3 \subseteq \mathbb{C}^2$

$$(T^*SU(2))_{impl} = \frac{\mu^{-1}(0)}{SU(2)} \sqcup \mu^{-1}((0,\infty))$$

$$\cong \{\text{point}\} \sqcup (\mathbb{C}^2 \setminus \{0\}) \cong \mathbb{C}^2$$

with induced $T$-action multiplication by $t^{-1}$.

**Link with Kähler/algebraic geometry**:

$G = K_{\mathbb{C}}$ complexification of $K$;

$B$ Borel subgroup of $G$ (maximal soluble subgp) such that $G = KB$ and $K \cap B = T$.

$N \subseteq B$ maximal unipotent subgroup of $G$;

$B = T_{\mathbb{C}} N$ with $T_{\mathbb{C}}$ complex torus.

**FACT:** $K_{\mathbb{C}}/N$ is a quasi-affine variety whose algebra of regular functions $\mathcal{O}(K_{\mathbb{C}}/N) = \mathcal{O}(K_{\mathbb{C}})^N$ is finitely generated, so that $K_{\mathbb{C}}/N$ has a canonical affine completion

$$K_{\mathbb{C}} /\!\!/ N = Spec(\mathcal{O}(K_{\mathbb{C}})^N).$$

**Thm** (GJS): $K_{\mathbb{C}} /\!\!/ N$ has a $K$-invariant Kähler structure which is symplectically iso to the universal implosion $(T^*K)_{impl}$.

**Cor**: $X$ affine or projective variety acted on linearly by $K_{\mathbb{C}} \Rightarrow$

$$X_{impl} \cong (X \times (K_{\mathbb{C}} /\!\!/ N)) /\!\!/ K_{\mathbb{C}} \cong X /\!\!/ N.$$

**GIT for non-reductive group $G$ acting linearly on affine $X$**

**Problem 1:** $\mathcal{O}(X)^G$ is not necessarily finitely generated.

**Problem 2:** *If* $\mathcal{O}(X)^G$ is finitely generated then $X/\!/G = \mathrm{Spec}(\mathcal{O}(X)^G)$ fits into a similar diagram to the reductive case

$$\begin{array}{rccc} & X & \longrightarrow & X/\!/G \\ & \cup & & \cup \quad \text{open} \\ \text{stable} & X^s & \longrightarrow & X^s/G \end{array}$$

**but** $X \to X/\!/G$ is **not necessarily onto**.

Example: The inclusion $K_{\mathbb{C}}/N \to K_{\mathbb{C}}/\!/N$ is not onto (though its complement has codimension $\geqslant 2$), so neither is $K_{\mathbb{C}} \to K_{\mathbb{C}}/\!/N$.

There is a similar picture for projective (or projective-over-affine) $X$ with ample line bundle $L$, involving $X^{ss} \to X/\!/G = \mathrm{Proj}(\mathcal{O}_L(X)^G)$, provided that $\mathcal{O}_L(X)^G$ is finitely generated.

**Generalised symplectic implosion**:
$G = K_{\mathbb{C}}$ complexification of $K$;
$P$ parabolic subgroup of $K_{\mathbb{C}}$;
$U_P$ unipotent radical of $P$.
**FACT:** $K_{\mathbb{C}}/U_P$ is a quasi-affine variety whose algebra of regular functions $\mathcal{O}(K_{\mathbb{C}}/U_P) = \mathcal{O}(K_{\mathbb{C}})^{U_P}$ is finitely generated, so that $K_{\mathbb{C}}/U_P$ has a canonical affine completion

$$K_{\mathbb{C}} /\!\!/ U_P = Spec(\mathcal{O}(K_{\mathbb{C}})^{U_P}).$$

**Thm**: $K_{\mathbb{C}} /\!\!/ U_P$ has a $K$-invariant Kähler structure such that it can be described symplectically as a generalised universal implosion $(T^*K)^{(P)}_{impl}$.
**Cor**: $X$ affine or projective variety acted on linearly by $K_{\mathbb{C}} \Rightarrow$ its $U_P$-invariants are finitely generated and

$$X /\!\!/ U_P \cong (X \times (K_{\mathbb{C}} /\!\!/ U_P)) /\!\!/ K_{\mathbb{C}} \cong X^{(P)}_{impl}.$$

To show that $(T^*K)_{impl} \cong K_{\mathbb{C}} /\!\!/ N$, GJS reduce to the case when $K$ is semisimple, connected and simply connected. Let $\{V_\varpi : \varpi \in \Pi\}$ be the set of fundamental representations of $K$, and let $v_\varpi$ be a highest weight vector in $V_\varpi$, fixed by $N$.

$\bigoplus_{\varpi\in\Pi} V_\varpi$ is a representation of $K \times T$, where $T$ acts on $V_\varpi$ as multiplication by the weight $-\varpi$. Then

$$\mathcal{T} := \overline{T_{\mathbb{C}}(\sum_{\varpi\in\Pi} v_\varpi)} = \bigoplus_{\varpi\in\Pi} \mathbb{C}v_\varpi \subseteq \bigoplus_{\varpi\in\Pi} V_\varpi$$

is the toric variety associated to $-\mathfrak{t}^*_+$. Recall $K_{\mathbb{C}} = KAN$ where $A \subseteq T_{\mathbb{C}}$, so $K_{\mathbb{C}}\mathcal{T} = K\mathcal{T}$ is closed in $\bigoplus_{\varpi\in\Pi} V_\varpi$. GJS show that

$$K_{\mathbb{C}}\mathcal{T} = \overline{K_{\mathbb{C}}(\sum_{\varpi\in\Pi} v_\varpi)} = \overline{K_{\mathbb{C}}/N} \cong K_{\mathbb{C}} /\!\!/ N.$$

$K_{\mathbb{C}} /\!\!/ N$ is usually singular, but has a 'partial desingularisation'

$$\widetilde{K_{\mathbb{C}} /\!\!/ N} = K \times_T \mathcal{T} \cong K_{\mathbb{C}} \times_B \mathcal{T}.$$

$K_{\mathbb{C}} /\!\!/ N$ can be constructed as the affinisation of $\widetilde{K_{\mathbb{C}} /\!\!/ N}$.

When $K = SU(n)$, $K_{\mathbb{C}} /\!\!/ N$ can be constructed using quiver representations $0 \to \mathbb{C} \to \mathbb{C}^2 \to \cdots \to \mathbb{C}^n$. If

$$M = \bigoplus_{i=1}^{n-1} \mathbb{C}^{i(i+1)} = \bigoplus_{i=1}^{n-1} \mathrm{Hom}(\mathbb{C}^i, \mathbb{C}^{i+1})$$

then $K_{\mathbb{C}} /\!\!/ N \cong M/\!\!/\!\!/(SU(1) \times SU(2) \times \cdots \times SU(n-1))$.

$K_{\mathbb{C}} /\!\!/ N$ can also be constructed using the 'baby Nahm equations' on $[0,1]$ and $(0,1)$.

## Towards hyperkähler implosion

Suppose given $X$ hyperkähler manifold, with complex structures $i, j, k$, metric $g$, Kähler forms $\omega_1$, $\omega_2$, $\omega_3$;
compact group $K$ acting on $X$ preserving $i, j, k, g$;
hyperkähler moment map $\mu = (\mu_1, \mu_2, \mu_3) : X \to \mathfrak{k}^* \otimes \mathbb{R}^3$.

The hyperkähler implosion $X_{\mathsf{hkimpl}}$ should be stratified hyperkähler with an induced $T$-action.

Look for the **universal hyperkähler implosion**

$$(T^* K_{\mathbb{C}})_{\mathsf{hkimpl}}$$

with an induced hyperkähler action of $T \times K$ and then define

$$X_{\mathsf{hkimpl}} = (X \times (T^* K_{\mathbb{C}})_{\mathsf{hkimpl}}) /\!/\!/ K.$$

Recall the **symplectic case**: $(T^*K)_{\mathrm{impl}} = K_{\mathbb{C}} /\!\!/ N$
(where $K_{\mathbb{C}} \cong T^*K \cong K \times \mathfrak{k}^*$) is stratified Kähler,
(real) dimension $\dim K + \dim T$, residual action of $K \times T$.
Also $(T^*K)_{\mathrm{impl}} = (K \times \mathfrak{t}^*_+)/\sim$
with $(k, \zeta) \sim (k', \zeta')$ iff $\zeta = \zeta'$, $k'k^{-1} \in [K_\zeta, K_\zeta]$,
so $(T^*K)_{\mathrm{impl}}/\!/_\zeta T = K$-coadjoint orbit of $\zeta$.

Fixing a cx structure, the universal hyperkähler implosion $(T^*K_{\mathbb{C}})_{\mathrm{hkimpl}}$ should be the holomorphic symplectic quotient of $T^*K_{\mathbb{C}} \cong K_{\mathbb{C}} \times \mathfrak{k}^*_{\mathbb{C}}$ by the maximal unipotent subgroup $N$ of $K_{\mathbb{C}}$ in the sense of non-reductive GIT (geometric invariant theory):

$$(K_{\mathbb{C}} \times (\mathfrak{k}_{\mathbb{C}}/\mathrm{Lie}N)^*) /\!\!/ N = (K_{\mathbb{C}} \times \mathfrak{n}^0) /\!\!/ N.$$

Well-defined, meaning $\mathcal{O}(K_{\mathbb{C}} \times \mathfrak{n}^0)^N$ is finitely generated? (Yes: Ginzburg–Riche). Hyperkähler? Geometry? $K_{\mathbb{C}}$-adjoint orbits as $T$-hyperkähler quotients?

For $K = SU(n)$, co-adjoint orbits of $K_{\mathbb{C}} = SL(n;\mathbb{C})$ appear in **quiver varieties**: $0 \rightleftarrows \mathbb{C} \rightleftarrows \mathbb{C}^2 \rightleftarrows \cdots \rightleftarrows \mathbb{C}^n$.
Consider the flat hyperkähler manifold

$$M = \bigoplus_{i=1}^{n-1} \mathbb{H}^{i(i+1)} = \bigoplus_{i=1}^{n-1} \mathrm{Hom}(\mathbb{C}^i, \mathbb{C}^{i+1}) \oplus \mathrm{Hom}(\mathbb{C}^{i+1}, \mathbb{C}^i).$$

Then $M///(U(1)\times U(2)\times \cdots \times U(n-1))$ can be identified with the **nilpotent cone** $\mathcal{N}$ in $\mathfrak{k}_{\mathbb{C}}$, which is the closure of the generic nilpotent coadjoint orbit in $\mathfrak{k}_{\mathbb{C}}$. If we shift the moment map by a suitable constant we get other coadjoint orbits in $\mathfrak{k}_{\mathbb{C}}$. So consider

$$Q = M///(SU(1) \times SU(2) \times \cdots \times SU(n-1)).$$

$Q$ is stratified hyperkähler with dimension $2(\dim K + \dim T)$ and a residual action of $(S^1)^{n-1} \times SU(n) \cong T \times K$ which preserves the hyperkähler structure, and an action of $SU(2)$ which rotates the complex structures.

The **universal hyperkähler implosion** $Q$ for $K = SU(n)$ has a resolution of singularities $\tilde{Q}$ which is a holomorphic symplectic quotient of $T^*K_{\mathbb{C}} \times \mathcal{T}_{\mathbb{H}}$ by $B = T_{\mathbb{C}}N$.
Here $\mathcal{T}_{\mathbb{H}}$ is the hypertoric variety for $T$ associated to the hyperplane arrangement given by the root planes in $\mathfrak{t}$.

The algebra of invariants $\mathcal{O}(K_{\mathbb{C}} \times \mathfrak{n}^0)^N$ is finitely generated, and for any complex structure $Q$ is the **holomorphic symplectic quotient** (in the sense of non-reductive GIT) Spec$(\mathcal{O}(K_{\mathbb{C}} \times \mathfrak{n}^0)^N)$ of $T^*K_{\mathbb{C}} = K_{\mathbb{C}} \times \mathfrak{k}^*_{\mathbb{C}}$ by the maximal unipotent subgroup $N$ of $K_{\mathbb{C}}$.

**Simple example:** $K = \mathrm{S}U(2)$, $G = \mathrm{S}L(2)$ acting on $X = \mathbb{H}^k$,

$$N = \mathbb{C}^+ = \{\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} : a \in \mathbb{C}\}.$$

The universal hyperkähler implosion is $Q = (T^*K_{\mathbb{C}})_{\mathsf{hkimpl}} = \mathbb{H}^2$, and $X_{\mathsf{hkimpl}} = \mu_{\mathbb{C}}^{-1}(\mathfrak{n}^0) /\!\!/ N = (\mathbb{H}^2 \times X)/\!/\!/\mathrm{S}U(2)$.

$K =$ connected, simply connected, semisimple compact Lie group.
$\{V_\varpi : \varpi \in \Pi\}$ = set of fundamental representations of $K$.
$\bigoplus_{\varpi\in\Pi} V_\varpi$ is a repn of $K \times T$, with $T$ acting on $V_\varpi$ via weight $-\varpi$.
$v_\varpi$ = highest weight vector in $V_\varpi$, fixed by $N$.
$w_0$ = longest element of the Weyl group $W$ of $K$, so $K_\mathbb{C} = \overline{Bw_0B}$.

Identify $V_\varpi \otimes_\mathbb{C} \mathbb{H}$ with $V_\varpi \oplus V_\varpi^{(w_0)}$ where $V_\varpi^{(w_0)} = V_\varpi$ with the action conjugated by $w_0$. Then

$$\mathcal{T}_\mathbb{H} := \bigoplus_{\varpi\in\Pi} v_\varpi \mathbb{H} \subseteq \bigoplus_{\varpi\in\Pi} V_\varpi \otimes_\mathbb{C} \mathbb{H}$$

is the hypertoric variety for $T$ associated to the hyperplane arrangement given by the root planes in $\mathfrak{t}$, and the closure $\overline{K_\mathbb{C}\mathcal{T}_\mathbb{H}}$ in $\bigoplus_{\varpi\in\Pi} V_\varpi \otimes_\mathbb{C} \mathbb{H}$ is a natural candidate for $(T^*K_\mathbb{C})_{hkimpl}$.

There are other possible constructions using the Nahm equations and using quiver varieties (at least in some cases).

**Final question**: can we find hyperkähler structures on more general holomorphic symplectic quotients by non-reductive groups?

**Eg**: a quiver variety with dimension vector $\mathbf{d}$ and multiplicity vector $\mathbf{m}$ is a holomorphic symplectic quotient of a flat space $\mathbb{H}^N$ by a product over the vertices $\prod_v \mathsf{Aut}_{\mathbb{C}[z]/(z^{m_v})}(\mathbb{C}^{d_v} \otimes_{\mathbb{C}} \mathbb{C}[z]/(z^{m_v}))$ $\cong \prod_v \{g_0 + g_1 z + \ldots + g_{m_v - 1} z^{m_v - 1} : g_0 \in \mathsf{GL}(d_v), g_1, \ldots, \in \mathfrak{gl}(d_v)\}$.

This group acts via upper triangular block matrices of the form

$$\begin{pmatrix} g_0 & g_1 & g_2 & \cdots & g_{m_v-1} \\ & g_0 & g_1 & \cdots & g_{m_v-2} \\ & & g_0 & \cdots & g_{m_v-3} \\ & & & \ddots & \\ & & & & g_0 \end{pmatrix}.$$