Hyperkahler implosions and symplectic duality

Oxford : January 2022

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arXiv:2004.09620

arXiv:2103.05458

Coulomb branch / Higgs branch duality

Complex-symplectic (hyperkahler) spaces

deformation parameters
$\leftrightarrow$
Cartan algebra of symmetry group

Abelian case (hypertoric varieties)

Hyperkahler quotients of $\mathbb{H}^d$ by torus $N \leq T^d$

$$1 \to N \ \to T^d \to T^n \to 1$$

gives hyperkahler quotient of
real dimension $4n$.

T^n action

dim $N = d - n$ deformation parameters

$\mathfrak{n} =$ Lie N defined by $\ker \beta : e_i \mapsto u_i$.

d vectors $u_1, \ldots, u_d$ in $\mathbb{R}^n$.

Duality swaps roles of N and T^n (so $d-n \leftrightarrow n$).

On one side

$$d - n \text{ deformation parameters}$$

$$T^n \text{ symmetry}$$

Dualise:

$$n \text{ deformation parameters}$$

$$\widehat{N} \cong T^{d-n} \text{ symmetry}$$

Combinatorial interpretation–Gale duality

Nonabelian case – quiver varieties

Nakajima viewpoint:

Higgs branch : hyperkahler quotient of linear space M by K compact.

Coulomb branch: birational to $T^*(\hat{T}_{\mathbb{C}})/W$

$$\dim_{\mathbb{R}}(\text{Coulomb branch}) = 4 \text{ rank } K$$

Hyperkahler implosion for K

Complex-symplectic space with $K_{\mathbb{C}} \times T_{\mathbb{C}}$ action

Dimension is

$$\dim K_{\mathbb{C}} + \text{rank } K_{\mathbb{C}}$$

Complex-symplectic reduction by $T_{\mathbb{C}}$ gives Kostant varieties, of dimension

$$\dim K_{\mathbb{C}} - \text{rank } K_{\mathbb{C}}$$

Reduction at zero gives nilpotent cone.

Can view implosion as nonreductive quotient

$$(K_{\mathbb{C}} \times \mathfrak{u}^0)//U$$

Complex-symplectic quotient of $T^*K_{\mathbb{C}}$ by maximal unipotent U.

In $SU(n)$ case, we have quiver description of implosion

Hyperkahler quotient of a linear space
at level 0.

Should fit into duality picture.

What might dual look like?

Balance conditions for unitary quivers.

We hyperkahler reduce by unitary groups $U(N)$ at gauge nodes labelled by integers N

$$\bigcirc_N$$

Balance of node is

$$-2N + \sum_j N_j$$

where sum is taken over nodes adjacent to the given node.

A node is balanced if balance $= 0$.

Balanced nodes should give Dynkin diagram of semisimple part of (subgroup of) symmetry group of dual variety.

Dual of implosions?

For $K = SU(n)$, we need a $SU(n) \times T^{n-1}$ symmetry.

Consider a Bouquet quiver:

Take quiver for nilpotent cone of $SU(n)$, excise n-dim flavour node at end, replace with n 1-dim nodes attached to the $(n-1)$-dim node to keep it balanced.

Balanced nodes form A_{n-1} diagram $---\>$ $SU(n)$ symmetry

Unbalanced nodes $--\>$ T^{n-1} symmetry after decoupling

Example: $n = 3$

Now *all* nodes are balanced–affine D_4 diagram

Suggests $SO(8)$ symmetry in implosion

Correct! the $SU(3)$ implosion is the Swann bundle of the quaternionic Kähler Wolf space

$$\tilde{\mathrm{Gr}}_4(\mathbb{R}^8) = SO(8)/SO(4) \times SO(4)$$

Computational checks using the monopole formula (Cremonesi-Hanany-Zaffaroni) to find Hilbert series, dimension of dual.

Bouquet quiver is effective quotient by

$$U(1)^{n-1} \times \prod_{i=1}^{n-1} U(i)$$

$$\begin{aligned} \text{rank} &= \frac{1}{2}n(n-1) + (n-1) \\ &= \frac{1}{2}(n^2 - 1 + n - 1) \end{aligned}$$

$$\begin{aligned} \dim_{\mathbb{C}}(\text{implosion}) &= \dim SL(n,\mathbb{C}) + \text{rank}\, SL(n,\mathbb{C}) \\ &= n^2 - 1 + n - 1 \end{aligned}$$

consistent with Nakajima

Other groups?

Orthosymplectic quivers : nodes alternate between SO and Sp

D_n case : excise $SO(2n)$ flavour node from nilpotent quiver, replace with bouquet of n $SO(2)$ nodes

Get correct dimension, symmetry group etc

Also B_n case – excise $Sp(n)$ node, bouquet of $U(1)$ nodes