On SCFTs at canonical singularities

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Connections between String Theory and Special Holonomy workshop
Oxford, 11 January 2021

based on 2007.15600, 2111.13564 with S. Schäfer-Nameki, Y. Wang
2012.12827 with S. Giacomelli, S. Schäfer-Nameki, Y. Wang

Canonical singularities

“Compactification” on Calabi-Yau threefolds

Let us start at the begining.

On a manifold $\mathbf{X}$, reduced holonomy implies the existence of Killing spinors, which imply “space-time supersymmetry” for string theory compactified on $\mathbf{X}$.

Let $\mathbf{X}$ be a Calabi-Yau threefold ($\mathcal{K}_{\mathbf{X}} \cong 0$).

“Compactification” on Calabi-Yau threefolds

In Type II string theory, we consider:

$$\mathbb{R}^4 \times \mathbf{X} \qquad (4+6=10) \qquad \to \qquad \text{4d } \mathcal{N}=2 \text{ supersymmetry}$$

In M-theory, we consider:

$$\mathbb{R}^5 \times \mathbf{X} \qquad (5+6=11) \qquad \to \qquad \text{5d } \mathcal{N}=1 \text{ supersymmetry}$$

The two setup are simply related:

$$\text{M-theory on } \mathbb{R}^4 \times S^1 \times \mathbf{X} \qquad \longleftrightarrow \qquad \text{Type IIA on } \mathbb{R}^4 \times \mathbf{X}$$

Non-compact threefolds and singularities

If $\mathbf{X}$ is compact, the space-time theory is a quantum gravity theory with:

$$G_N \sim \frac{1}{\mathrm{vol}(\mathbf{X})}$$

At large volume, and assuming $\mathbf{X}$ is smooth, the supergravity approximation is valid.

Instead, we are interested in **$\mathbf{X}$ non-compact.** Then, in the strict IR limit, we expect to have a supersymmetric quantum field theory (SQFT) on $\mathbb{R}^d$ ($d = 4$ or $d = 5$):

$$\mathbf{X} \;\mapsto\; \mathrm{SQFT}_d[\mathbf{X}]$$

The SQFT description is an *effective low-energy description* which breaks down at:

$$\mu > \frac{1}{\mathrm{vol}(\mathcal{C}_n)} \;, \qquad \mathcal{C}_n \subset \mathbf{X} \;,$$

and its UV completion, *in general*, is the full string theory in 10d (or M-theory in 11d).

Non-compact threefolds and singularities

The *'interesting'* SQFTs are the well-defined ones – *i.e.* UV complete ones. By the heuristic discussion above, we obtain such theories in a limit when all n-cycles inside $\mathbf{X}$ obtain vanishing volumes, if such a limit exists (by tuning moduli).

We then need to consider singularities.

Example: $\mathbf{X}$ is a cone over a smooth Sasaki-Einstein five-manifold L_5, with the conical Ricci-flat metric:

$$ds^2(\mathbf{X}) = dr^2 + r^2 ds^2(L_5)$$

Note: In the rest of this talk, we ignore important questions about the metric, and give a purely algebraic description of $\mathbf{X}$.

Canonical singularities

Mathematically, our main object of study will be (threefold) canonical singularities.

Definitions: [Reid, 1980]

A canonical singularity is a threefold $\mathbf{X}$ with a single isolated singularity such that:

(i) $\mathcal{K}_{\mathbf{X}}$ is Cartier;

(ii) for any resolution $\pi : \tilde{\mathbf{X}} \to \mathbf{X}$, with exceptional divisors E_i, we have:

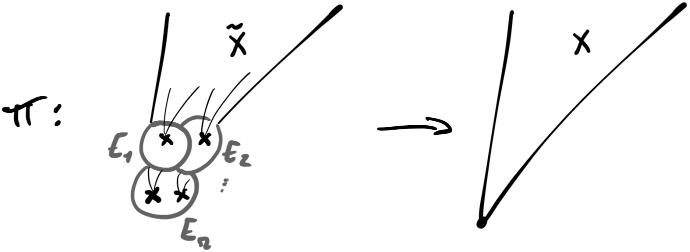
$$\mathcal{K}_{\tilde{\mathbf{X}}} = \pi^*(\mathcal{K}_{\mathbf{X}}) + \sum_i a_i E_i \qquad \text{with} \quad a_i \geq 0 \;, \forall i \;.$$

A terminal singularity is a canonical singularity with $a_i > 0$, $\forall i$.

If $a_i = 0$, $\forall i$, we say that the canonical singularity 'admits a crepant resolution', or that it is smoothable. The resulting smooth threefold $\tilde{\mathbf{X}}$ is a local Calabi-Yau.

The rank r of a canonical singularity $\mathbf{X}$

Generally, we have $a_i = 0$ for only some of the E_i's. The canonical singularity $\mathbf{X}$ then admit a partial crepant resolution with residual terminal singularities.



We denote by r the number of crepant exceptional divisors. That number is independent of the choice of resolution. We will call this 'the rank of $\mathbf{X}$'.

Isolated hypersurface singularities

For simplicity, we will focus on a particularly simple class of canonical singularities: the quasi-homogeneous isolated hypersurface singularities (IHS):

$$\mathbf{X} = \{F(x_1, \cdots, x_n) = 0 \mid (x_1, \cdots, x_n) \in \mathbb{C}^n\} \ ,$$

with an isolated singularity at the origin. By the quasi-homogeneity condition:

$$F(\lambda^q x) = \lambda F(x) \ , \qquad \lambda \in \mathbb{C}^*$$

the singularity $\mathbf{X}$ is essentially determined by the scaling weights $q_i \in \mathbb{Q}_{>0}$. We will call two singularities $F_1(x) = 0$ and $F_2(x) = 0$ 'physically equivalent' if their ordered scaling weights are the same.

The IHS is a canonical singularity if and only if:

$$\sum_{i=1}^{n} q_i > \frac{n-2}{2}$$

Isolated hypersurface singularities

For twofolds ($n = 3$), we have the well-known ADE (a.k.a. Du Val) surface singularities:

$$A_n \ : \ x^2 + y^2 + z^{n+1} \ , \qquad E_6 \ : \ x^2 + y^3 + z^4$$
$$D_n \ : \ x^2 + y^2 z + z^{n-1} \ , \qquad E_7 \ : \ x^2 + y^3 + yz^3 \qquad E_8 \ : \ x^2 + y^3 + z^5$$

For threefolds ($n = 4$), there is a full classification of quasi-homogeneous polynomials [Kreuzer, Skarke , 1992], which can be divided into 19 ‘Types’.

[Yau, Yu, 2005; Xie, Yau, 2015; Davenport, Melnikov, 2016; CC, Schäfer-Nameki, Wang, 2021].

Example: The simplest type is:

$$\text{Type I:} \qquad F = x_1^a + x_2^b + x_3^c + x_4^d \ , \qquad (q_i) = \left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}\right) \ .$$

For a canonical singularity, we then need:

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} > 1$$

Resolutions: divisors, curves, and three-cycles

Crepant resolutions give a (partial) smoothing of $\mathbf{X}$. In string theory, we view the resolution as a continuous process by tuning complexified Kähler structure moduli:

$$t_{\mathcal{C}} = \int_{\mathcal{C}} (B + i\omega)$$

We can take a basis of divisors and curves of $\mathbf{X}$ to span the homology groups:

$$[\mathcal{C}] \in H_2(\tilde{\mathbf{X}}, \mathbb{Z}) \cong \mathbb{Z}^{r+f} \ , \qquad [E] \in H_4(\tilde{\mathbf{X}}, \mathbb{Z}) \cong \mathbb{Z}^{r} \ .$$

Here, f will be called the flavor rank. It is the number of 'unpaired' compact 2-cycles inside $\tilde{\mathbf{X}}$. It is also the rank of the divisor class group of $\mathbf{X}$,

$$f = \rho(\mathbf{X})$$

Interestingly, the resolution $\tilde{\mathbf{X}}$ also generally has non-trivial 3-cycles: [Caibar, 2005]

$$H_3(\tilde{\mathbf{X}}, \mathbb{Z}) \cong \mathbb{Z}^{b_3} \ , \qquad b_3 = 2\sum_{i=1}^{r} g_i \ ,$$

which come about because the exceptional divisors E_i are ruled surfaces over genus-g_i curves.

Deformation and Milnor ring

Another way to 'smoothen' a singularity is to deform it. For an IHS, this means:

$$\widehat{\mathbf{X}} = \left\{ \widehat{F}(x) \equiv F(x) + \sum_{l=1}^{\mu} t_l x^{\mathfrak{m}_l} = 0 \right\} ,$$

with the monomials $x^{\mathfrak{m}} = x_1^{\mathfrak{m}_1} \cdots x_4^{\mathfrak{m}_4}$ a basis of the Milnor ring:

$$\mathcal{M}(F) = \mathbb{C}[x_1, x_2, x_3, x_4]/\mathcal{J} , \qquad \mathcal{J} \equiv (\partial_{x_1} F, \cdots, \partial_{x_4} F) .$$

We assign a scaling weight $q[t_l]$ to these monomial in the natural way:

$$Q_l = q(\mathfrak{m}_l) \equiv \sum_{i=1}^{4} q_i \mathfrak{m}_{l,i} .$$

and define:

$$\ell_l = Q_l + \sum_{i=1}^{4} q_i - 1 ,$$

Deformation and singularity spectrum

The spectrum of the singularity $\mathbf{X}$ is the ordered set of μ rational numbers:

$$\mathcal{S}_{\mathbf{X}} = \{\ell_l\}_{l=1}^{\mu} \ , \qquad \ell_1 \leq \ell_2 \leq \cdots \leq \ell_\mu \ ,$$

with $\mu \equiv \dim \mathcal{M}(F)$ the Milnor number.

Some non-trivial facts, provided that $\mathbf{X}$ is canonical: [Arnold *et al.*, 1981; Caibar, 2005]

- $\mathcal{S}_{\mathbf{X}} \subset (0, 2)$.
- the spectral numbers such that $\ell_l \neq 1$ come in pairs:

$$\ell_l + \ell_{\mu-l+1} = 2 \qquad \text{if} \qquad \ell_l < 1$$

- the number of unpaired spectral numbers is the flavor rank:

$$\#\{\ell_l | \ell_l = 1\} = f = \rho(\mathbf{X}) \ .$$

We then write the Milnor number as:

$$\mu = 2\widehat{r} + f \ .$$

Here, $\widehat{r}$ will be called 'the 4d rank'.

The homotopy type of $\widehat{\mathbf{X}}$ is that of a bouquet of 3-spheres, and:

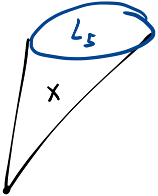
$$H_3(\widehat{\mathbf{X}}, \mathbb{Z}) \cong \mathbb{Z}^{\mu}$$

The link of the singularity

The link of an IHS is a smooth five-manifold defined as:

$$L_5(\mathbf{X}) = \left\{ x \in \mathbb{C}^4 \;\middle|\; F(x) = 0 \;, \quad \sum_{i=1}^{4} |x_i|^2 = \varepsilon \right\} .$$

We can think of it as the five-manifold 'at the boundary'.



The link manifold can have interesting topology, which is determined directly by the scaling weights:

$$\begin{aligned} H_0(L_5,\mathbb{Z}) &\cong \quad H_5(L_5,\mathbb{Z}) \cong \mathbb{Z} \;, & H_2(L_5,\mathbb{Z}) &\cong \quad \mathbb{Z}^f \oplus \mathfrak{t}_2 \\ H_1(L_5,\mathbb{Z}) &\cong \quad H_4(L_5,\mathbb{Z}) \cong 0 \;, & H_3(L_5,\mathbb{Z}) &\cong \quad \mathbb{Z}^f \;, \end{aligned}$$

Note that the torsion part, $\mathfrak{t}_2$, can be non-trivial. We compute it from (q_i) using a conjecture from [Orlik, 1972]. (Proven in some cases.)

The flavor rank f appears as the non-trivial Betti number.

4d SCFTs (Type IIB)

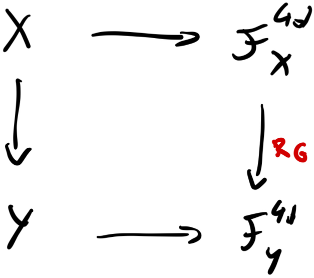
4d SCFTs from $\mathbf{X}$

Consider a canonical singularity $\mathbf{X}$. We claim (or 'we expect') that there exists a map:

$$\mathrm{IIB} \, : \{\text{canonical singularity}\} \to \{\text{4d } \mathcal{N} = 2 \text{ SCFT}\} \quad : \quad \mathbf{X} \quad \mapsto \quad \mathscr{T}_{\mathbf{X}}^{\mathrm{4d}}$$

This was studied in many string theory papers, starting with [Shapere, Vafa, 1999].

This map should be 'functorial'. (What this means precisely is still being understood...)



4d SCFTs from $\mathbf{X}$

To have a more concrete handle on the SCFT $\mathscr{T}_{\mathbf{X}}^{\rm 4d}$, we go onto its moduli space:

- Coulomb branch $\leftrightarrow$ deformed singularity $\widehat{\mathbf{X}}$;
- Higgs branch $\leftrightarrow$ resolved singularity $\tilde{\mathbf{X}}$.

The most basic data is the dimension of the moduli space:

$$\widehat{r} = \dim_{\mathbb{C}}(\mathrm{CB}) = \mathrm{rank}(\mathscr{T}_{\mathbf{X}}^{\rm 4d}) \ ,$$

$$\widehat{d}_H = r + f = \dim_{\mathbb{H}}(\mathrm{HB}) \ .$$

Moreover, the flavor rank is the flavor rank (the number of conserved currents):

$$f = \mathrm{rank}(G_F^{\rm 4d})$$

The CB spectrum

The family of deformed singularities $\widehat{\mathbf{X}}$ gives the extended Coulomb branch (the Seiberg-Witten geometry). Recall we have:

$$\mu = 2\widehat{r} + f$$

Moduli:

$$\varphi_\alpha = \int_{S^3_\alpha} \Omega$$

- $\widehat{r}$ Coulomb branch parameters;
- f mass parameters.

The dimensions of the CB operators are given by the spectrum of $\mathbf{X}$, properly normalised:

$$\Delta_l = \frac{\sum_{i=1}^4 q_i - \ell_l}{\sum_{i=1}^4 q_i - 1} \qquad \text{for} \quad \ell_l < 1 \ (\Delta_l > 1).$$

The normalisation is such that the holomorphic 3-form:

$$\Omega = \frac{dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4}{d\widehat{F}}$$

has dimension $\Delta = 1$.

The Higgs branch

The moduli space of the SCFT $\mathscr{T}_{\mathbf{X}}^{4\mathrm{d}}$ has two main branches: Coulomb and Higgs.

The Higgs branch is an hyper-Kähler cone of quaternionic dimension

$$\widehat{d}_H = r + f$$

This equals the real dimension of the extended Kähler cone Y of $\mathbf{X}$. Roughly, the HB takes the form:

$$(\mathbb{C}^*)^{\widehat{d}_H} \to \mathsf{HB} \to Y_{\mathbb{C}}$$

The physics near the origin of the HB is strongly corrected by D-brane instantons – hard problem.

Note: this HB is *not necessarily* an hyper-Kähler quotient. (It is if $\mathscr{T}_{\mathbf{X}}^{4\mathrm{d}}$ is Lagrangian.)

Some very interesting qualitative feature of the HB physics can be gleaned from the classical resolution $\tilde{\mathbf{X}}$, however.

The Higgs branch

In general, given a (partial) crepant resolution $\tilde{\mathbf{X}}$ of $\mathbf{X}$, we have:

- b_3 3-cycles;
- a set of residual terminal singularities, $\mathbf{X}_{\rm IR}$.

This corresponds to non-trivial physics on the Higgs branch:

- $\frac{1}{2}b_3$ free vector multiplets;
- a set of non-trivial residual SCFTs $\mathscr{T}^{\rm 4d}_{\mathbf{X}_{\rm IR}}$

Anomaly matching for the $U(1)_r$ symmetry on the HB allows us to relate the resolution data to the SCFT data:

$$\boxed{A_{\mathbf{X}} = r + f - \frac{1}{2}b_3 + A_{\mathbf{X}_{\rm IR}}}$$

The conformal anomalies a, c, for a given $\mathbf{X}$, are fully determined by the spectrum of the singularity. In particular:

$$A_{\mathbf{X}} \equiv 24(c-a)|_{\mathbf{X}} = \mu\Delta_{\rm max} - \widehat{r} - 6\sum_{\Delta_l > 1}(\Delta_l - 1)$$

We checked this identity it in many cases. Mathematical proof?

One-form symmetries

4d SCFTs can have non-trivial one-form symmetries under which line operators are charged.

In the Type IIB description of $\mathscr{T}^{4\mathrm{d}}_{\mathbf{X}}$, line operators (on the CB) arise as D3-branes wrapping relative 3-cycles. We then have the 'defect group':

$$\mathbf{D}^{(1)}_{\mathscr{T}^{4\mathrm{d}}_{\mathbf{X}}} = \mathrm{Tor}\left[H_3(\widehat{\mathbf{X}}, L_5, \mathbb{Z})/H_3(\widehat{\mathbf{X}}, \mathbb{Z})\right] \cong \mathbb{Z}^{\mu}/\mathcal{M}_{3,3}\mathbb{Z}^{\mu} \cong \mathfrak{t}_2$$

where we used a long exact sequence in homology. [1]

The one-form symmetry of $\mathscr{T}^{4\mathrm{d}}_{\mathbf{X}}$ is determined by a choice of polarization of the defect group:

$$\Lambda^{(1)}_{\mathscr{T}^{4\mathrm{d}}_{\mathbf{X}}} \subset \mathbf{D}^{(1)}_{\mathscr{T}^{4\mathrm{d}}_{\mathbf{X}}} \;, \qquad \widehat{\Lambda}^{(1)}_{\mathscr{T}^{4\mathrm{d}}_{\mathbf{X}}} = \mathrm{Hom}\left(\Lambda^{(1)}_{\mathscr{T}^{4\mathrm{d}}_{\mathbf{X}}}, U(1)\right) \;.$$

Note that we always have:

$$\mathfrak{t}_2 \cong \mathfrak{f} \oplus \mathfrak{f} \;,$$

[1] $\cdots \to H_k(L_5) \xrightarrow{h_k} H_k(\widehat{\mathbf{X}}) \xrightarrow{f_k} H_k(\widehat{\mathbf{X}}, L_5) \xrightarrow{g_k} H_{k-1}(L_5) \xrightarrow{h_{k-1}} H_{k-1}(\widehat{\mathbf{X}}) \to \cdots$

Examples for $\mathscr{T}^{\rm 4d}_{\mathbf{X}}$

Example 1:

$$\mathbf{X} \;:\; \text{Type VII}(3,3,3,2) : \quad F = x_1^3 + x_2^3 + x_2 x_3^3 + x_3 x_4^2$$

This gives:

$$r = 1 \;, \quad f = 2 \;, \quad \widehat{r} = 10 \;, \quad b_3 = 0 \;, \quad \mathfrak{t}_2 = 0$$

with CB spectrum:

$$\Delta = \left\{ \frac{6}{5}, \frac{6}{5}, \frac{8}{5}, \frac{8}{5}, 2, \frac{11}{5}, \frac{12}{5}, \frac{12}{5}, \frac{14}{5}, \frac{18}{5} \right\}$$

The resolved singularity has proper transform:

$$x_1^3 + x_2^3 + \delta_1 x_2 x_3^3 + x_3 x_4^2 = 0$$

and a single residual terminal singularity of Type I$(2,2,2,3)$. The latter corresponds to:

$$\mathbf{X}_{\rm IR} \quad : \quad r = f = 0 \;, \quad \widehat{r} = 1 \;, \quad \Delta = \left\{ \frac{6}{5} \right\} \;.$$

We then have:

$$A_{\mathbf{X}} = 24(c-a) = \frac{16}{5} = r + f + A_{\mathbf{X}_{\rm IR}} = 3 + \frac{1}{5} \;.$$

Examples for $\mathscr{T}_{\mathbf{X}}^{\rm 4d}$

Example 2:

$$\mathbf{X} \ : \ \text{Type I}(3,3,3,3) : \qquad F = x_1^3 + x_2^3 + x_3^3 + x_4^3$$

This gives:

$$r = 1 \ , \quad f = 6 \ , \quad \widehat{r} = 5 \ , \quad b_3 = 0 \ , \quad \mathfrak{t}_2 = 0$$

with CB spectrum:

$$\Delta = \{2, 2, 2, 2, 3\}$$

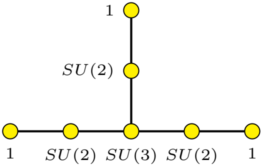
The resolved singularity is smooth. It is simply:

$$\tilde{\mathbf{X}} \cong \mathrm{Tot}\,(K \to dP_6)$$

We then have:

$$A_{\mathbf{X}} = 24(c-a) = 7 = r + f = \widehat{d}_H = 24 - 3 \times 3 - 8 \ .$$

In this case, the SCFT $\mathscr{T}_{\mathbf{X}}^{\rm 4d}$ is a Lagrangian theory:



Examples for $\mathscr{T}_{\mathbf{X}}^{4\text{d}}$

Example 3:

$$\mathbf{X} \;:\; \text{Type I}(3,3,3,3) : \quad F = x_1^3 + x_2^3 + x_3^3 + x_4^4$$

This gives:

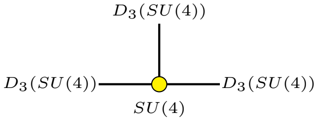
$$r = 1\,, \quad f = 0\,, \quad \widehat{r} = 12\,, \quad b_3 = 2\,, \quad \mathfrak{t}_2 = \mathbb{Z}_4^2$$

with CB spectrum:

$$\Delta = \left\{ \frac{4}{3}, \frac{4}{3}, \frac{4}{3}, \frac{5}{3}, \frac{5}{3}, \frac{5}{3}, 2, \frac{8}{3}, \frac{8}{3}, \frac{8}{3}, 3, 4 \right\}$$

The resolved singularity is smooth but the exceptional divisor itself is singular. Note that, here, we have both 3-cycles and a one-form symmetry.

The SCFT is a 'trinion': [CC, Giacomelli, Schäfer-Nameki, Wang, 2020]



Its $\mathbb{Z}_4$ one-form symmetry is carried by the gauge group. (This is a general rule for such IHS models [Buican, Jiang, 2021].)

Some conjectures

From observation of many models, we formulate the following: [CC, Schäfer-Nameki, Wang, 2021]

Conjecture 1: *For any smoothable model $\mathbf{X}$ with smooth exceptional divisors, the associated 4d SCFT $\mathscr{T}_{\mathbf{X}}^{\rm 4d}$ has an integral CB spectrum.*

Conjecture 2: *Any model $\mathbf{X}$ with an integral CB spectrum is smoothable (not necessarily with smooth divisors).*

We can consider this at any fixed rank r. In other words, we have the inclusions of sets:

$$\{\text{'fully smooth' }\} \subset \{\text{integral CB spectrum}\} \subset \{\text{smoothable}\} \subset \{\text{rank-}r \text{ models}\}$$

Some conjectures

These conjectures relate the data of the deformation theory of $\mathbf{X}$ to the smoothness of its partial crepant resolution.

Let us pause here to consider how remarkable that is.

(Or is it? You tell me.)

Examples

Here, by 'model', we mean the set of all canonical IHS with the same spectrum.

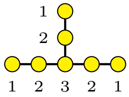
At any fixed r, the number of smoothable models appears to be finite.

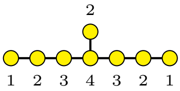
We counted them by brute force (by generating a large dataset):

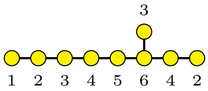
r	1	2	3	4	5	6	7	8	9	10
# in our dataset	1167	649	693	666	506	549	486	495	491	461
# smoothable	12	24	32	23	32	39	23	41	?	?
# of $\mathbb{Z}$CB	3	6	10	9	12	18	7	9	12	12
# 'fully smooth'	3	5	10	7	9	14	5	8	7	7

Side note: The $r = 1$ 'fully smooth' models are sort-of famous:

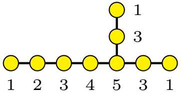
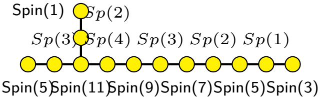
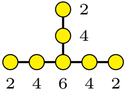
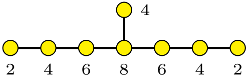
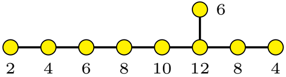
$$E_6 \ : x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0 \ , \quad E_7 \ : x_1^2 + x_2^4 + x_3^4 + x_4^4 = 0 \ , \quad E_8 \ : x_1^2 + x_2^3 + x_3^6 + x_4^6 = 0 \ .$$







More examples: Rank-2 fully-smooth models

F	r	f	$\widehat{r}$	$\widehat{d_H}$	b_3	$\mathfrak{t}_2$	Lagrangian $\mathscr{T}^{4\mathrm{d}}_{\mathbf{X}}$
$x_1^5 + x_2^2 x_1 + x_3^5 + x_3 x_4^2$	2	8	14	10	0	0	
$x_1^2 + x_2^5 + x_3^5 + x_4^5$	2	0	32	2	12	$\mathbb{Z}_2^{12}$	
$x_1^3 + x_2^3 + x_3^3 + x_4^6$	2	6	17	8	2	$\mathbb{Z}_2^2$	
$x_1^2 + x_2^4 + x_3^4 + x_4^8$	2	7	28	9	2	$\mathbb{Z}_2^2$	
$x_1^2 + +x_2^3 + x_3^6 + x_4^{12}$	2	8	51	10	2	$\mathbb{Z}_2^2$	

5d SCFTs (M-theory)

5d SCFTs from $\mathbf{X}$

Consider a canonical singularity $\mathbf{X}$. We claim (or 'we expect') that there exists a map:

$$\mathrm{M-theory}\ :\{\text{canonical singularity}\} \to \{\text{5d SCFT}\}\quad :\quad \mathbf{X} \quad \mapsto \quad \mathcal{T}_{\mathbf{X}}^{\mathrm{5d}}$$

This was studied in many string theory papers, starting with [Morrison, Seiberg, 1996].

Again, this map should be 'functorial'.

5d SCFTs from $\mathbf{X}$

To have a more concrete handle on the SCFT $\mathcal{T}_{\mathbf{X}}^{\rm 5d}$, we go onto its moduli space:

- Coulomb branch $\leftrightarrow$ resolved singularity $\tilde{\mathbf{X}}$;
- Higgs branch $\leftrightarrow$ deformed singularity $\widehat{\mathbf{X}}$.

The most basic data is the dimension of the moduli space:

$$r = \dim_{\mathbb{R}}(\mathrm{CB}) = \mathrm{rank}(\mathcal{T}_{\mathbf{X}}^{\rm 5d}) \ ,$$

$$d_H = \widehat{r} + f = \dim_{\mathbb{H}}(\mathrm{HB}) \ .$$

Moreover, the flavor rank is the flavor rank (the number of conserved currents):

$$f = \mathrm{rank}(G_F^{\rm 5d})$$

The 5d Coulomb branch

Low-energy physics on the CB of $\mathcal{T}_{\mathbf{X}}^{\rm 5d}$ is well-understood if $\tilde{\mathbf{X}}$ is 'fully smooth'. (11d SUGRA valid.) Abelian gauge theory in IR. The 5d prepotential is determined by triple intersection numbers of divisors. [Morrison, Intriligator, Seiberg, 1996]

If $b_3 > 0$, we have $\frac{1}{2}b_3$ free hypermultiplets at a generic point on the CB.

If residual terminal singularity $\mathbf{X}_{\rm IR}$, interpretation more subtle.

- Conservative interpretation: 'fully smooth mode' $\oplus$ 'free sector'.
- Full story: *work in progress...*

The 5d Higgs branch

We can count hypermultiplets, and thus find the quaternionic dimension of the 5d Higgs branch, by looking at the periods:

$$h_\alpha \sim \int_{S^3_\alpha} \Omega$$

in $\widehat{\mathbf{X}}$. And so we find:

$$d_H = \widehat{r} + f$$

M2-brane instanton make the classical picture unreliable. To know more about the structure of the hyper-Kähler cone HB$[\mathcal{T}_{\mathbf{X}}^{\rm 5d}]$, we need extra tricks.

Magnetic quiverines

Reductions to 3d

Consider the dimensional reduction of our SCFTs to 3d:

$$\mathsf{EQ}^{(4)} \cong \mathscr{T}_{\mathbf{X}}^{\mathrm{4d}} \text{ on } \mathbb{R}^3 \times S^1 \qquad \leftrightarrow \qquad \text{IIB on } \mathbb{R}^3 \times S^1_\beta \times \mathbf{X}$$

$$\mathsf{EQ}^{(5)} \cong \mathcal{T}_{\mathbf{X}}^{\mathrm{5d}} \text{ on } \mathbb{R}^3 \times T^2 \qquad \leftrightarrow \qquad \text{IIA on } \mathbb{R}^3 \times S^1_\beta \times \mathbf{X}$$

Under favourable circumstances, the very low-energy physics can be described by a 3d $\mathcal{N} = 4$ SCFT, which we call an electric quiverine.

The HB stays the same under such dimensional reduction, so:

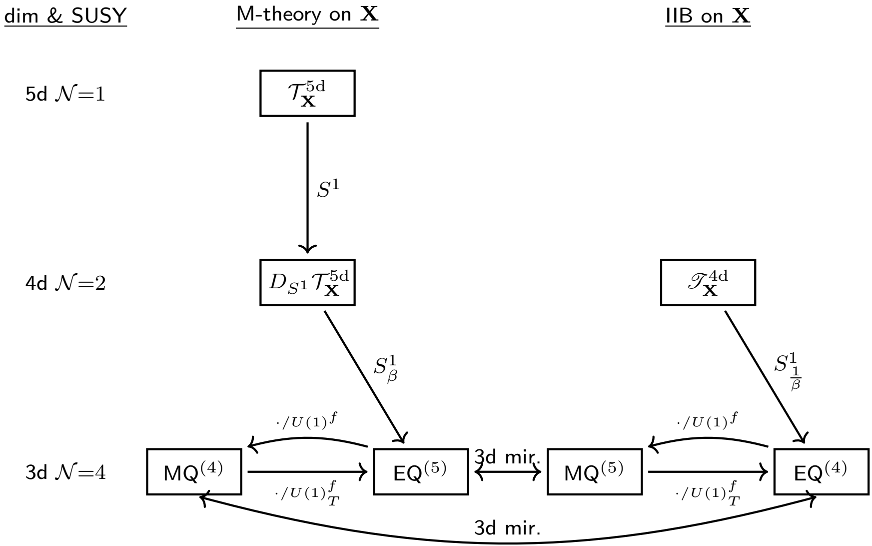
$$\mathrm{HB}[\mathscr{T}_{\mathbf{X}}^{\mathrm{4d}}] = \mathrm{HB}[\mathsf{EQ}^{(4)}] , \qquad \mathrm{HB}[\mathcal{T}_{\mathbf{X}}^{\mathrm{5d}}] = \mathrm{HB}[\mathsf{EQ}^{(5)}]$$

Then, we define the magnetic quiverine to be the **3d mirror** of the electric quiverines. Thus, by definition:

$$\mathrm{HB}[\mathscr{T}_{\mathbf{X}}^{\mathrm{4d}}] = \mathrm{CB}[\mathsf{MQ}^{(4)}] , \qquad \mathrm{HB}[\mathcal{T}_{\mathbf{X}}^{\mathrm{5d}}] = \mathrm{CB}[\mathsf{MQ}^{(5)}]$$

In some cases, we can show that the magnetic quiver is an ordinary quiver. In general, this is just abstract nonsense.

Roadmap


dim & SUSY
M-theory on $\mathbf{X}$
IIB on $\mathbf{X}$
5d $\mathcal{N}=1$
$\mathcal{T}^{5d}_{\mathbf{X}}$
S^1
4d $\mathcal{N}=2$
$D_{S^1}\mathcal{T}^{5d}_{\mathbf{X}}$
$\mathscr{T}^{4d}_{\mathbf{X}}$
S^1_β
$S^1_{\frac{1}{\beta}}$
3d $\mathcal{N}=4$
$\cdot/U(1)^f$
$\cdot/U(1)^f_T$
MQ$^{(4)}$
EQ$^{(5)}$
3d mir.
MQ$^{(5)}$
EQ$^{(4)}$
$\cdot/U(1)^f$
$\cdot/U(1)^f_T$
3d mir.

$U(1)^f$ gauging and 3d mirrors

For both $\mathscr{T}_{\mathbf{X}}^{4\text{d}}$ and $\mathcal{T}_{\mathbf{X}}^{5\text{d}}$, we have an abelian flavour symmetry:

$$T_F = U(1)^f \subset G_F$$

which is preserved on the CB. We then have background vector multiplets for T_F that arise from:

- f ‘unpaired 3-cycles inside $\widehat{\mathbf{X}}$ in IIB;
- f ‘unpaired’ 2-cycles inside $\tilde{\mathbf{X}}$ in M-theory.

The magnetic quiverines (the 3d mirrors) can be obtained as a gauging of T_F:

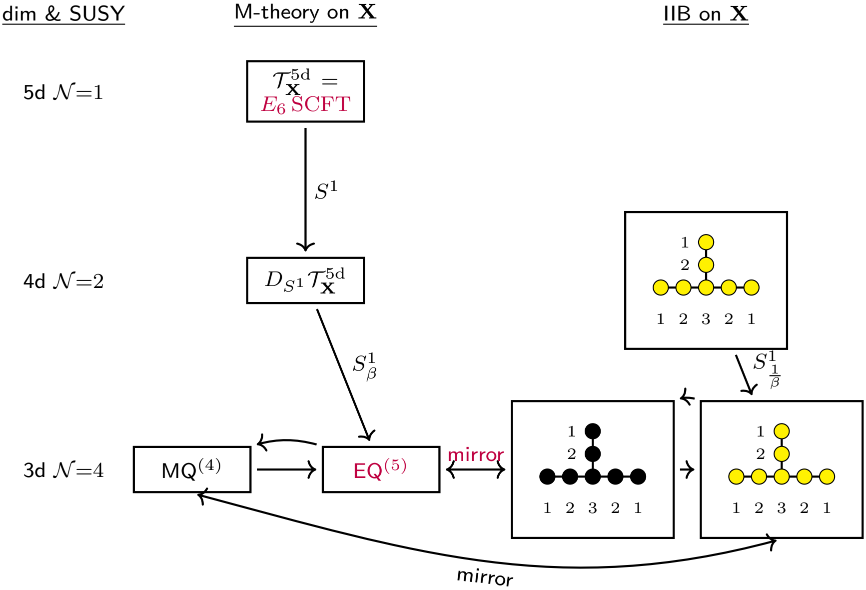
$$\mathsf{MQ}^{(4)} = \mathsf{EQ}^{(5)}/U(1)^f \ , \qquad \mathsf{MQ}^{(5)} = \mathsf{EQ}^{(4)}/U(1)^f \ .$$

This operation is reversible: [Witten, 2003]

$$\mathsf{EQ}^{(5)} = \mathsf{MQ}^{(4)}/U(1)^f_T \ , \qquad \mathsf{EQ}^{(4)} = \mathsf{MQ}^{(5)}/U(1)^f_T \ .$$

In string theory, this is T-duality.

Examples of MQ$^{(5)}$: $\mathbf{X} = \{x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0\}$

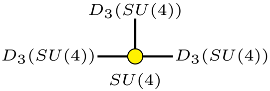


Other examples: D_p^b trinions

If the 4d SCFT $\mathscr{T}_{\mathbf{X}}^{\mathrm{4d}}$ is Lagrangian, this works nicely. In general, we need to work harder.

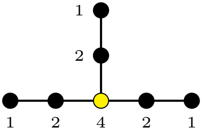
In a number of 'simple' cases so far, we can work out the '3d mirror' of $\mathscr{T}_{\mathbf{X}}^{\mathrm{4d}}$.

Example: $F = x_1^3 + x_2^3 + x_3^3 + x_4^4$ with $\mathscr{T}_{\mathbf{X}}^{\mathrm{4d}} =$

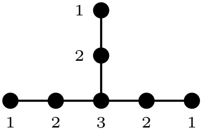


This has an electric quiver description in 3d, from which we obtain MQ$^{(5)}$ (here, $f = 0$):

MQ$^{(5)}[E_6^{(1)}] \cong$



$\cong$



$\oplus \mathbb{H}$

There are subtleties in this last step...

Outlook

Summary and outlook

Summary:

- We revisited the geometric engineering of 4d *and* 5d SCFTs at canonical singularities.
- Unifying concept: 3d quiverines.
- Leads to interesting interplays between physics and mathematics – much of it still to be understood.

Outlook:

- In IIB/M-theory engineering, the CB is classical, and the HB is highly quantum – 'hypermultiplet moduli space in Type II'. Direct methods to study the hyper-K"ahler CB in string theory must be developed.
- We focussed on canonical singularities that are hypersurfaces. General case should be addressed systematically, too.
- 'Rank-zero' ($r = 0$) 5d SCFTs (associated to terminal singularities) deserve further study.