Special Lagrangian submanifolds in K3-fibered Calabi-Yau 3-folds

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Def

(X, ω, Ω) is a Calabi-Yau manifold if

- X is a complex manifold with $\dim_{\mathbb{C}} X = n$
- ω is a Kähler form on X
- $\Omega \in H^0(K_X)$ is nowhere-vanishing
- $\frac{1}{n!}\omega^n = (-1)^{n(n-1)/2}\left(\frac{\sqrt{-1}}{2}\right)^n \Omega \wedge \overline{\Omega}$

$(\Rightarrow \mathrm{Ric}(\omega) = 0, \quad \mathrm{Hol}(X,\omega) \subset SU(n))$

__Def__ (Harvey – Lawson)

Fix a CY mfd (X, ω, Ω).

A real n-dim'l submanifold $L \subset X$ is special Lagrangian with phase $\theta \in \mathbb{R}$ if

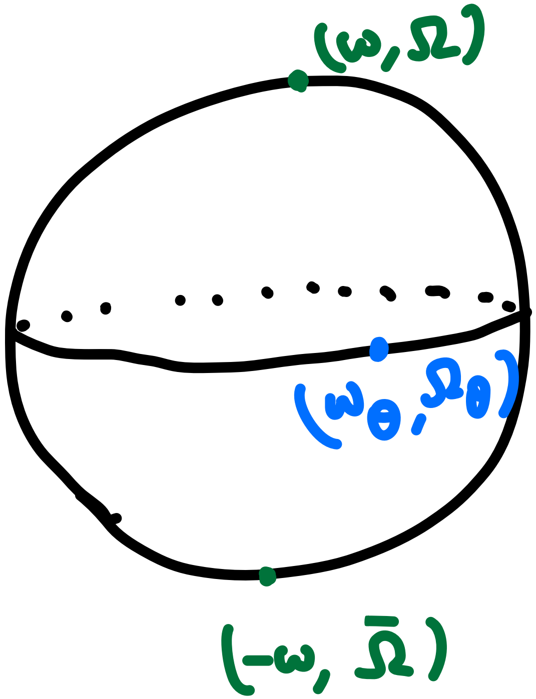
- $\mathrm{Re}(e^{-i\theta}\Omega)\big|_L = d\mathrm{vol}_L$ ($\Rightarrow$ volume minimizing, minimal)

Equivalently,

- $\mathrm{Im}(e^{-i\theta}\Omega)\big|_L = 0$ and
- $\omega\big|_L = 0$

Examples

- geodesics in CY 1-folds
- hyperkähler rotations of holomorphic curves in hyperkähler 4-manifolds:



$$\omega_\theta = \mathrm{Re}(e^{-i\theta}\Omega)$$

$$\Omega_\theta = \omega + i\,\mathrm{Im}(e^{-i\theta}\Omega)$$

E.g. $X = K3$, $[C]\cdot[\omega] = 0$,
$[C]^2 = -2$

$\Rightarrow \exists!$ holo. curve C in $[C]$ diffeo. to S^2 (Riemann-Roch)

$\Rightarrow$ C is SL in X_θ

Examples (cont'd)

- $\iota : X \to X$, $\iota^2 = \mathrm{Id}$, $\iota^*\Omega = \bar{\Omega}$, $\iota^*\omega = -\omega$

  $L = \mathrm{Fix}(\iota)$ is SL (with $\theta = 0$)

- E.g. Bryant constructed SL tori in CY hypersurfaces

- $X = \{ z_1^2 + z_2^2 + \dots + z_{n+1}^2 = 1 \} \subset \mathbb{C}^{n+1}$

  ω_{st} = Stenzel metric

  ι = complex conjugation

  $\iota^*\omega_{st} = -\omega_{st}$ since ω_{st} is $SO(n+1)$-invariant

  $L = \mathrm{Fix}(\iota) \simeq S^n \hookrightarrow T^*S^n$

Examples (cont'd)

- Harvey - Lawson T^2 - cone
- SL cones with higher genus (Haskins - Kapouleas)
- Lawlor necks, examples of Joyce - Lee - Tsui
- Deformation of SL (McLean):
  Infinitesimal deformations of cpt SL L are unobstructed and are parametrized by $H^1(L)$
- Desingularizations of conically singular SL (Joyce, D. Lee, Y.-I. Lee)

Q How to find compact SLs ?

- Lagrangian mean curvature flow :

$L \subset X, \quad \omega|_L = 0$

$\Omega|_L = e^{i\theta} dvol_L, \quad e^{i\theta} : L \to S^1$

$H = J \nabla \theta .$

$\frac{dL_t}{dt} = H(L_t)$ preserves Lagrangian condition

(Smoczyk)

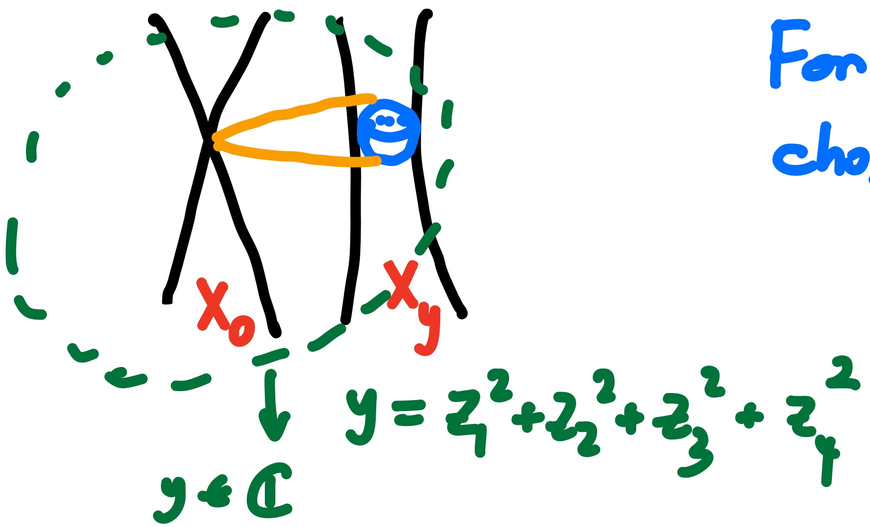
Problem : singularities

Big picture : Thomas - Yau conj. , Joyce's program

- "Gluing methods":
  Under complex / metric degeneration of (X, ω, Ω), geometry becomes semi-explicit

- SYZ fibration in generic region in Fermat quintic near large complex structure limit (Y. Li)

- CY with smoothable ODP singularities (Hein-Sun)



For $0 < |y| \ll 1$, vanishing S^3 can be chosen to be SL.

(See also Y.-M. Chan, Biquard-Rollin, Spotti)

CY 3-folds with Lefschetz fibration

(X, ω_X, Ω) : CY3

$Y = \mathbb{P}^1$, ω_Y = Fubini-Study metric

$\pi : X \to Y$ holo. submersion s.t.

at each singular point P of π,

$\exists$ coordinates around P and $\pi(P)$ s.t.

$$y = \pi(z_1, z_2, z_3) = z_1^2 + z_2^2 + z_3^2$$

$X_y = \pi^{-1}(y)$ is K3 by adjunction

Theorem (Y. Li)

Consider CY metrics $\hat{\omega}_t \in [\omega_X + \frac{1}{t}\omega_Y]$, $0 < t \ll 1$.
Then at singular point P, $\exists$ nbd U of P,
$F_t : F_t^{-1}(U) \subset \mathbb{C}^3 \longrightarrow U$ s.t.

$$\left(\frac{2A_0}{t}\right)^{\frac{1}{3}} F_t^* \hat{\omega}_t \longrightarrow \omega_{\mathbb{C}^3} \quad \text{in} \quad C^0_{loc}(\omega_{\mathbb{C}^3})$$

Here the model metric $\omega_{\mathbb{C}^3}$ is the complete CY metric on $\omega_{\mathbb{C}^3}$ asymptotic to $\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$ constructed independently by Y. Li, Conlon-Rochon, and Székelyhidi

<u>Note</u> Let $S \subset Y$ be the discriminant locus of π.

- Tosatti showed

  $t\tilde{\omega}_t \to \pi^* \tilde{\omega}_Y$ as currents

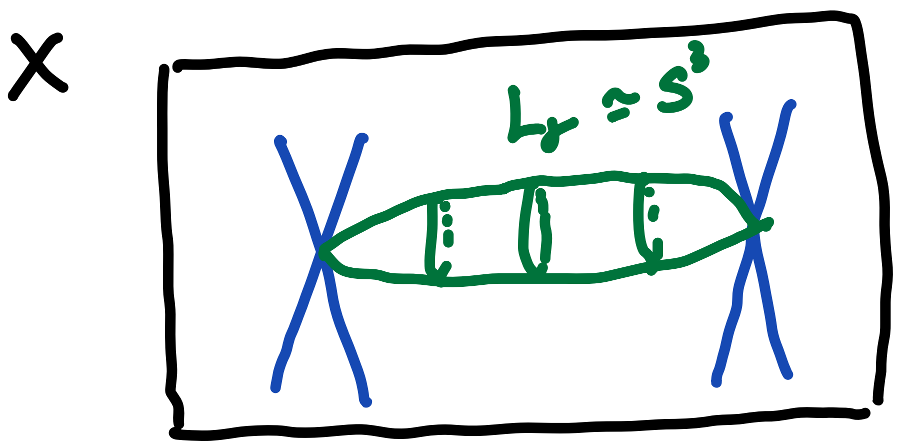
  $t\tilde{\omega}_t \to \pi^* \tilde{\omega}_Y$ in $C^{1,\beta}_{loc}$ on $X \setminus f^{-1}(S)$

  Here $\tilde{\omega}_Y = \pi_*(i\Omega \wedge \bar{\Omega})$ is a generalized KE metric s.t. $Ric(\tilde{\omega}_Y) =$ Weil-Peterson metric of the fibration

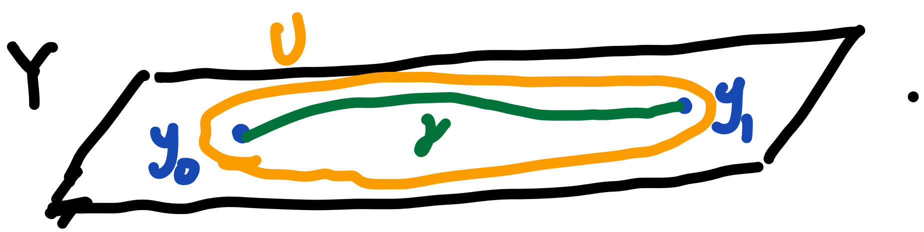
- Hein-Tosatti obtained smooth asymptotics to semi-Ricci flat metric ω_{SRF} away from $f^{-1}(S)$

- Y. Li's gluing construction is optimal near $f^{-1}(S)$, but needs to be improved away from $f^{-1}(S)$.

New special Lagrangians



Each vanishing cycle can be chosen as a SL S^2.



$$L_\gamma = \bigcup_{y \in \gamma} L_y$$

Assume

- Vanishing cycles L_y of y_0 and y_1 match up to sign in X_y, $y \in U$

$\alpha = \int_{L_y} \Omega$ locally well-def.

$\phi = \alpha \otimes \alpha$ well-def. quad. diff.

- Assume γ is a geodesic wrt ϕ ($\Rightarrow \operatorname{Arg} \phi(\gamma', \gamma') = 2\theta = \text{const.}$)

Theorem (C.–Lin)

For $0 < t \ll 1$, $\exists$ SL S^3 $\tilde{L}_{\gamma,t} \subset (X, \tilde{\omega}_t, \Omega)$ such that

$$\tilde{L}_{\gamma,t} \subset (X, t\tilde{\omega}_t) \xrightarrow[GH]{} \gamma \subset (Y, \tilde{\omega}_Y)$$

Remarks

- Examples can be constructed using algebraic geometry and quadratic differentials theory; interesting configurations
- Can be thought of as a higher dim'l generation of SL S^2's in Gibbons–Hawking ansatz (Lotay–Oliveira)
- In contrast to Hein–Sun's, the SL S^3's in Theorem are long, thin, and sharp.

Ideas of proof

$$L_\gamma = \bigcup_{y \in \gamma} L_y$$

$$\mathrm{Im}(e^{-i\theta}\Omega)\big|_{L_\gamma} = 0, \quad \tilde{\omega}_t\big|_{L_\gamma} = O(t^{\frac{1}{2}} y^{-1}) \text{ in } \{|y| > t^{\frac{9}{20}}\}.$$

L_γ is not a good model near $f^{-1}(S)$.

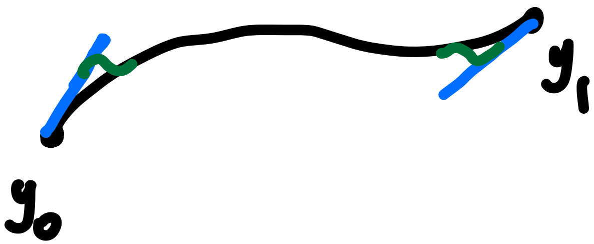
Prop In $(\mathbb{C}^3, \omega_{\mathbb{C}^3})$, $\exists$ SL thimble $L_{\mathbb{R}^3}$:

$\mathbb{C}^3 = \{ y = z_1^2 + z_2^2 + z_3^2 \} \subset \mathbb{C}^4$, $\iota(y, z) = (\bar{y}, \bar{z})$.

$\iota^* \omega_{\mathbb{C}^3} = -\omega_{\mathbb{C}^3}$

$L_{\mathbb{R}^3} = \mathrm{Fix}(\iota) \simeq \mathbb{R}^3$, fibers of $L_{\mathbb{R}^3} \to \mathbb{R}_{\geq 0}$ are SL S^2's in Eguchi-Hanson spaces

Ideas of proof (cont'd)



Replace the ends of γ by tangents

= replace the ends of L_γ by copies of $L_{\mathbb{R}^3}$

$\rightsquigarrow$ Get the approximate solutions $L_{\gamma,t}$

<u>Note</u> Both the construction of L_γ and $L_{\gamma,t}$ require holo. curves technique. How to generalize to higher dimensions?

Ideas of proof (cont'd)

General strategy : Write $L = L_{\gamma,t}$

For $\eta \in \Lambda^1(L)$, let $V \in \Gamma(TX|_L)$, $\tilde{\omega}_t(V,\cdot) = \eta$

$f_\eta : L \to X$, $f_\eta(x) = \exp_x V_x$

$F : \Lambda^1(L) \to \Lambda^0(L) \oplus \Lambda^2(L)$

$F(\eta) = \left(* f_\eta^* \operatorname{Im}\Omega , \ f_\eta^* \tilde{\omega}_t \right)$

Goal : find η s.t. $F(\eta) = 0$

Check : $\int_{L_{\gamma,t}} \operatorname{Im}\Omega = O(t^{\frac{9}{40}}) \xrightarrow[t\to 0]{} 0$

So $\int_{L_{\gamma,t}} \operatorname{Im}\Omega = 0$. Also $[\tilde{\omega}_t|_{L_{\gamma,t}}] = 0$ because $\underbrace{H^2(L_{\gamma,t})}_{=0}$

Ideas of proof (cont'd)

$$DF_0(\eta) = (d+d^*)\eta + E(\eta)$$

$$\|E(\eta)\|_{C^{0,\alpha}} \lesssim \left(\|\operatorname{Im}\Omega|_L\|_{C^{1,\alpha}} + \|\tilde{\omega}_t|_L\|_{C^{1,\alpha}}\right)\|\eta\|_{C^{1,\alpha}}$$

Can write

$$F(\eta) = F(0) + (d+d^*)\eta + Q(\eta)$$

Let $P = (d+d^*)^{-1} : \operatorname{Im}(d+d^*) \to \Lambda^1(L)$

$F(\eta) = 0 \iff \eta$ is a fixed point of

$$N(\eta) = -PF(0) - PQ(\eta)$$

Ideas of proof (cont'd)

Inverting $d+d^*$ wrt suitable weighted norms using parametrix method: need to find the (approximate) Green operator on

- $L_{\mathbb{R}^3}$
- $\mathbb{R} \times S^2$ (fiberwise zero average)
- Y equipped with a 'warped product' metric

- $\mathbb{R} \times S^2$: $d+d^*$ version of method of Székelyhidi, Walpuski, Brendle

Method of proof (cont'd)

- $d + d^*$ on $L_{\mathbb{R}^3}$:

  $L_{\mathbb{R}^3}$ polynomially asym. to $d\rho^2 + \rho^{\frac{1}{2}} g_{S^2}$

- Lockhart - McOwen theory not sufficient: either have cokernel or live with exponentially growing solutions

- Improve from exponential to polynomial: Solve $\Delta u = f$ at ∞, then apply Hein's result (need $f = O(\rho^{-2-\varepsilon})$)

Inspired by Székelyhidi, Haskins-Hein-Nordström

Ideas of proof (cont'd)

- $d+d^*$ on γ :

  Solutions have 'large' C^0 norm compared to other parts

In sum : $\exists\ P$ s.t. $(d+d^*)P = Id$,

$P(f, \omega) = \eta_1 + \eta_2$, where

- η_1 lives in the 'right' weighted space
- η_2 depends on γ, C^0 norm large

Consequence : the deformation $f_\eta : L \to X$
has the same shape.

Ideas of proof (cont'd)

Dealing with the "base direction" η_2 :

- η_2 lifts to a 1-form on L thanks to the right hand side having zero average
- PQ is not a contraction mapping: Do QP instead. Find $\psi \in \Lambda^0(L) \oplus \Lambda^2(L)$ s.t. ψ is a fixed point of

$$\tilde{N}[\psi] = -F(0) - Q(P\psi).$$

Remarks

- The case of 'geodesic' loop is easier : get SL $S^1 \times S^2$ for $0 < t \ll 1$

- Evidence for Thomas - Yau in 3 - dims :

  L_1 , L_2 stable $\Leftrightarrow \exists\, \gamma_3 \Rightarrow \exists\, L_3$ in the homology class of $L_1 \# L_2$

- Flow version (Lotay - Oliveira) ?

- Donaldson - Scaduto (Esfahni - Li) ?