Moduli Stacks and Categorical Symmetry

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Part I : Introduction

Introduction

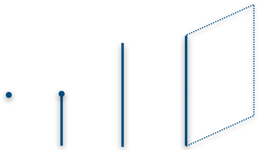
Supersymmetric quantum field theories often admit exact moduli spaces of vacua X parametrised by expectation values of *local* operators.

- Supersymmetry implies geometric structures e.g. Kähler, hyper-Kähler.
- Global symmetry groups F act as isometries of X.
- Quantum field theory naturally gives rise to constructions such as equivariant cohomology, K-theory, coherent sheaves.
- Many connections to geometric representation theory and enumerative geometry.

Introduction

Recent developments have lead to a vast generalisation of the notion of *symmetry* in quantum field theory [1].

- Ordinary symmetries F are groups and act on *local* operators.
- Generalised symmetries can form n-groups $\mathcal{F}$ acting on *extended* operators of dimension $0, 1, \cdots, n-1$.



The symmetries are categorical in nature and natural language is TQFT.

[1] Gaiotto-Kapustin-Seiberg-Willett '14

Introduction

In the context of supersymmetric field theories with eight supercharges, I will explain how generalised symmetries act on moduli *stacks* [2].

$$\pi : \mathcal{X} \ \to \ X$$

- The base X is the underlying moduli space.
- The fiber $\pi^{-1}(x)$ captures the existence of a TQFT at a point $x \in X$ on the moduli space.
- Ordinary symmetries F act on X.
- Generalised symmetries $\mathcal{F}$ act on $\mathcal{X}$.

[2] Hellerman-Sharpe '11, Sharpe '15

Part II : Moduli Spaces

Supersymmetry

I will consider supersymmetric quantum field theories eight supercharges.

- I will focus on three dimensions: 3d $\mathcal{N} = 4$.
- I will assume they flow to a superconformal fixed point.
- I expect at least some conclusions are applicable (with caveats and appropriate generalisations) to 4d $\mathcal{N} = 2$, 5d $\mathcal{N} = 1$.

Moduli Spaces

They admit moduli spaces that are *conical symplectic singularities* X_0. [3]

- The conical nature reflects the superconformal fixed point.
- There are sometimes parameters that allow symplectic resolutions

$$p : X \;\to\; X_0$$

- Examples are nilpotent orbit closures:

$$T^*\mathbb{CP}^1 \;\to\; \overline{\mathcal{N}_{\mathsf{min}}} \subset \mathfrak{sl}_2(\mathbb{C})$$

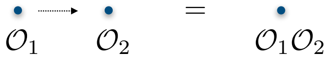
How are moduli spaces constructed *operationally* in quantum field theory?

[3] Beauville '99, Nakikawa '11 : see Antoine Bourget's talk.

Local operators I

We consider certain classes of BPS local operators.

- Translations are exact: $\partial_x \mathcal{O}(x) = Q(\cdots)$.
- Correlation functions $\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)$ are independent of positions $x_1, \cdots, x_2$.
- This defines a commutative *chiral ring* $\mathcal{A}_0$.
- We then define the affine variety $X_0 = \text{Spec}\,\mathcal{A}_0$.



Local operators II

There are additional properties of $\mathcal{A}_0$:

- Poincare supersymmetry implies a Poisson structure [4]

$$\{\,\cdot\,,\,\cdot\,\} : \mathcal{A}_0 \times \mathcal{A}_0 \to \mathcal{A}_0$$

  (The ring and Poisson structure come respectively from homology of $\mathrm{Conf}_2(\mathbb{R}^3) \sim S^2$ in degree 0 and 2.)

- Superconformal symmetry implies $\mathbb{Z}$-grading with
  1. All degrees $r \geq 0$, saturated only by 1.
  2. The Poisson structure has degree -2.

This ensures $X_0 = \mathrm{Spec}\,\mathcal{A}_0$ is a conical symplectic singularity.

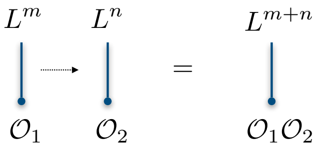
[4] Ben-Zvi, Beem, MB, Dimofte, Neitzke '18

Resolutions

How can we construct resolutions X?

The idea is to study local operators at the end of a line operator.

- Choose a BPS line operator L.
- Consider local operators $\mathcal{A}_n$ at the end of L^n, $n \geq 0$.
- Construct graded $\mathcal{A}_0$-algebra $\mathcal{A} := \oplus_{n\geq 0}\mathcal{A}_n$.
- Proj construction: $X = \operatorname{Proj} \mathcal{A}$



With some conditions on L, this is a symplectic resolution $X \to X_0$.

Gauge Theories

Consider a supersymmetric gauge theory labelled by

- A complex reductive group G (complexification of connected compact Lie group G_c).
- A complex symplectic representation $\rho : G \to \mathsf{Sp}(T^*V)$.

I will assume $\ker \rho$ is at most finite subgroup of $Z(G)$.

Higgs and Coulomb

There is in general an intricate moduli space of vacua. [5]

There are two components distinguished by which classes of BPS operators we consider:

1. Higgs branch.
2. Coulomb branch.

Today, I will focus on the Higgs branch.

Please feel free to ask about Coulomb branches at the end!

[5] See talks by Hiraku Nakajima and Antoine Bourget.

Local operators

The Higgs branch is constructed as follows:

- Hypermultiplet scalars are linear functions on T^*V.
- Local operators are invariant polynomial functions on T^*V subject to complex moment map constraint.
- In other words

$$\mathcal{A}_0 = \mathbb{C}[\mu^{-1}(0)]^G$$

  where $\mu : V \to \mathfrak{g}^*$ is the moment map.
- The Higgs branch is the affine GIT quotient

$$X_0 := \operatorname{Spec} \mathcal{A}_0 = \mu^{-1}(0)//G$$

Resolutions

To study resolutions, the relevant line operators are labelled by co-characters θ of the *topological symmetry*:

$$Z({}^{L}G) = \pi_1(G)^\vee$$

.

- Example: $G = GL(N)$, $\theta \in \mathbb{Z}$.
- The line operators are Wilson lines W_θ in representation $(\det V)^\theta$ where V is fundamental of $GL(N)$.
- Local operators ending Wilson lines are not gauge-invariant.
- $\mathcal{A}_n$ consists of semi-invariants $f(g \cdot x) = (\det g)^{n\theta} f(x)$ on $\mu^{-1}(0)$.

The resolution is the GIT quotient $X = \mathrm{Proj}\,\mathcal{A} = \mu^{-1}(0)^{\chi_\theta}//G$.

Example I

Consider supersymmetric QED: $G = \mathbb{C}^\times$, $V = \mathbb{C}^2$, weights $(1,1)$.

- Introduce linear coordinates (X_j, Y_j) on T^*V.
- Form 2×2 matrix of invariant functions $M_{ij} = X_i Y_j$.
- Complex moment map $\mu = \sum_j X_i Y_j = 0$ implies

$$\mathrm{Tr}(M) = 0 \qquad M^2 = 0$$

We recover minimal nilpotent orbit

$$X_0 = \overline{\mathcal{N}_{\mathrm{min}}} \subset \mathfrak{sl}(2, \mathbb{C})$$

Example I

Consider supersymmetric QED: $G = \mathbb{C}^\times$, $V = \mathbb{C}^2$, weights $(1, 1)$.

- Consider the Wilson line W of charge $+1$.
- Operators ending on W^n with $n \geq 0$ are

$$\mathcal{A}_n = \mathcal{A}_0 \times \{\text{homogeneous polynomials in } X_j \text{ of degree n}\}$$

We recover the resolution

$$X := \mathrm{Proj}\, \mathcal{A} = T^*\mathbb{CP}^1 \,.$$

Part III : Moduli Stacks

Motivation

This construction of X does not capture the presence of topological sectors e.g. discrete gauge theory.

- Promoted moduli spaces to moduli stacks $\mathcal{X}$, $\mathcal{X}_0$.
- The Higgs stack of a supersymmetric gauge theory:

$$\mathcal{X}_0 = [\mu^{-1}(0)/G] \qquad \mathcal{X} = [\mu^{-1}(0)^{\chi_\theta}/G]$$

Expected structure under my assumptions:

1. $\mathcal{X}_0$ is an algebraic stack whose generic point has finite automorphisms.
2. With appropriate θ, $\mathcal{X}$ is a Deligne-Mumford stack.

I will focus on the latter!

Heuristic Picture

The moduli stack can be understood heuristically as a fibration

$$\pi : \mathcal{X} \to X$$

- The base X is the underlying coarse moduli space.
- The fibres are classifying stacks $\pi^{-1}(x) = BA_x = [\mathrm{pt}/A_x]$
- The automorphism group $A_x \subset G$ is the discrete unbroken gauge symmetry at $x \in X$.

How to see this structure?

Lines Again

Discrete automorphism groups are detected by Wilson lines that *cannot* end on local operators.

- Consider the finite abelian group

$$\hat{A}_x = \{\text{Wilson lines}\}/\sim$$

  where $W_1 \sim W_2$ if there exists a non-trivial operator at their junction that does not vanish at x.
- This enumerates line operators that cannot end.
- Then the automorphism group / unbroken gauge symmetry is

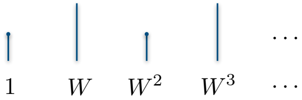
$$A_x = \text{Hom}(\hat{A}_x, U(1))\,.$$

Example I

Consider Supersymmetric QED: $G = U(1)$, $V = \mathbb{C}^2$, weights $(2, 2)$.

- The operators ending on Wilson lines are

$$\mathcal{A} = \begin{cases} \mathcal{A}_0 \times \{\text{homog. poly. in } X_1, X_2 \text{ degree } \frac{n}{2}\ \} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$



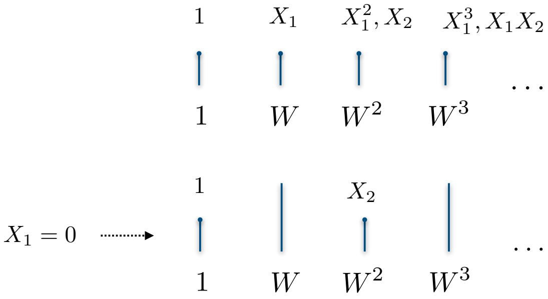
- The coarse moduli space is $X = T^*\mathbb{CP}^1$.
- Since $W^2 \sim 1$, there are automorphisms $A_x = \mathbb{Z}_2$ for all $x \in X$.
- $\mathcal{X}$ is the total space of a $\mathbb{Z}_2$-gerbe on $T^*\mathbb{CP}^1$.

This captures a discrete $\mathbb{Z}_2$ gauge theory everywhere on X.

Example II

Consider Supersymmetric QED: $G = U(1)$, $V = \mathbb{C}^2$, weights $(1, 2)$.

- Consider operators ending on Wilson lines:



- $\mathcal{X}$ is a a hyper-toric stack. [6]
- $\{X_1 = 0\}$ is a stacky locus with $\mathbb{Z}_2$ automorphism.

[6] Jiang-Tseng '08

Part IV : Symmetries

2-group Global Symmetry

The global symmetries acting on configurations of local operators and line operators may form a *2-group* $\mathcal{F}$.

In the present setup, this boils down to the following data:

- A compact connected Lie group F_c.
- A discrete abelian group A.
- A class in $H^3(BF_c, A)$

The proposal is that this captures automorphisms of the stack $\mathcal{X}$.

0-form Symmetry I

The compact connected Lie group F_c is the *0-form symmetry*.

- Genuine local operators transform in faithful representation of F_c.
- Local operators at the end of line operators may transform in representations of a central extension $\widetilde{F}_c$.
- The corresponding complex reductive F acts on X.

0-form Symmetry II

In a supersymmetric gauge theory $\rho : G \to \mathsf{Sp}(T^*V)$:

- The global symmetry acting on Higgs branch operators is

$$F = N_{\mathsf{Sp}(T^*V)}(G)/G$$

- This acts on the symplectic GIT quotient $X = \mu^{-1}(0)^{\chi_\theta}//G$.
- Example: in supersymmetric QED with $G = \mathbb{C}^\times$, $V = \mathbb{C}^2$ weights $(1,1)$ we have

$$X = T^*\mathbb{CP}^1 \qquad F = PGL(2)$$

1-form Symmetry I

The discrete abelian group A is the *1-form symmetry*.

This arises when there is a common subgroup $A \subset A_x$ of automorphisms at all points $x \in X$.

- Consider the finite abelian group

$$\hat{A} = \{\text{Wilson lines}\}/\sim$$

  where $W_1 \sim W_2$ if there are *any* non-trivial junction operators.
- They correspond to line operators that cannot end.
- Then $A = \text{Hom}(\hat{A}, U(1))$.
- BA acts on the fibers of $\pi : \mathcal{X} \to X$.

1-form Symmetry II

In a supersymmetric gauge theory $\rho : G \to \mathsf{Sp}(T^*V)$,

$$A = \ker \rho \subset Z(G)$$

In the examples considered earlier:

- Supersymmetric QED with weights $(1,1)$: $A = \emptyset$
- Supersymmetric QED with weights $(2,2)$: $A = \mathbb{Z}_2$
- Supersymmetric QED with weights $(1,2)$: $A = \emptyset$.

More generally in supersymmetric QED with charges $Q_1, \ldots, Q_N$,

$$A = \mathbb{Z}_{\mathsf{gcd}(Q_1,\ldots,Q_N)}$$

Postnikov Class I

The interaction between F, A is controlled by the Postnikov class

$$\Theta \in H^3(BF_c, A)$$

I will not explain how to construct it it general [7] but

- Suppose $\hat{A} \cong \mathbb{Z}_k$ is generated by a line operator $L^k \sim 1$.
- If L^k ends on local operators transforming in a central extension of F_c it is possible that $\Theta \neq 0$.

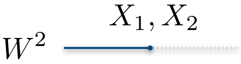
Here is an example...

[7] See Bhardwaj '21, Lee-Ohmori-Tachikawa '21

Example

Consider Supersymmetric QED: $G = U(1)$, $V = \mathbb{C}^2$, weights $(2,2)$.

- Recall that $\mathcal{X}$ is a $\mathbb{Z}_2$ gerbe over $T^*\mathbb{CP}^1$.
- $F_c = SO(3)$ acts on $T^*\mathbb{CP}^1$.
- $BA = B\mathbb{Z}_2$ acts in the $B\mathbb{Z}_2$ fibers.
- The Wilson line $W^2 \sim 1$ ends on local operators X_1, X_2 transforming in fundamental of $SU(2)$; an extension of $SO(3)$.



- Postnikov class $\Theta \in H^3(BSO(3), \mathbb{Z}_2) \cong \mathbb{Z}_2$ is non-trivial.

Part V : Final Thoughts

Mirror Symmetry

In a 3d $\mathcal{N} = 4$ theory there are two potential 2-group symmetries:

- $\mathcal{F}_A$ acting on the Coulomb branch stack $\mathcal{X}_A$.
- $\mathcal{F}_B$ acting on the Higgs branch stack $\mathcal{X}_B$.
- They have mixed 't Hooft anomalies!

Matching these structures provides new tests of mirror symmetry.

- An example is supersymmetric QED with charges $(2, 2)$:

$$\mathcal{X}_A = T^*\mathbb{CP}^1 / \mathbb{Z}_2$$

$$\mathcal{X}_B = \mathbb{Z}_2\text{-gerbe over } T^*\mathbb{CP}^1.$$

- Ask me about the mirror afterwards!