Transverse J-holomorphic curves in nearly Kähler $\mathbb{CP}^3$

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- The round $S^6 \subset \mathrm{Im}(\mathbb{O})$ has an orthogonal G_2-invariant almost complex structure J from the octonions
- It is not Kähler, *what is the underlying geometric structure?*
- Has special properties: Einstein, the cone has a G_2-structure
- Gray '70: An almost Hermitian manifold (M^{2n}, g, J) is called **nearly Kähler** if

$$(\nabla_\xi J)(\xi) = 0$$

The nearly Kähler structure on $\mathbb{CP}^3$

$\mathbb{CP}^3$ also has a nearly Kähler structure:

- The **twistor fibration**

$$\mathbb{CP}^3 \to \mathbb{HP}^1 \cong S^4, \quad [Z_0, Z_1, Z_2, Z_3] \mapsto [Z_0 + jZ_1, Z_2 + jZ_3]$$

  has a natural connection $T\mathbb{CP}^3 = \mathcal{H} \oplus \mathcal{V}$ coming from $\mathbb{CP}^3 \subset \Lambda^2_-(S^4)$. It is orthogonal w.r.t. the Fubini-Study metric g_{FS} on $\mathbb{CP}^3$
- $\mathcal{H}$ is a holomorphic contact distribution on $\mathbb{CP}^3$
- Non-integrable almost complex structure J reverses standard complex structure on $\mathcal{V}$
- Define a metric g_λ by squashing g_{FS} on $\mathcal{V}$ by $\lambda > 0$

There is λ such that $(\mathbb{CP}^3, g_\lambda, J)$ is nearly Kähler

- $\mathrm{Sp}(2)$ acts via automorphisms on $\mathbb{CP}^3$

Properties of nearly Kähler manifolds

Nearly Kähler manifolds in dimension < 6 are automatically Kähler. A strict nearly Kähler manifold M in dimension six

- admits an $\mathrm{SU}(3)$-structure $\omega \in \Omega^2(M), \psi \in \Omega^3(M, \mathbb{C})$ satisfying

$$\mathrm{d}\omega = 3c\mathrm{Re}(\psi)$$
$$\mathrm{dIm}(\psi) = -2c\omega \wedge \omega$$

  assume $c = 1$

- yields a torsion-free G_2-structures

$$\varphi = r^2 \mathrm{d}r \wedge \omega + r^3 \mathrm{Re}\psi$$

  on the cone $C(M) = M \times \mathbb{R}^{>0}$

- is Einstein with positive scalar curvature
- sine-cone is a nearly parallel G_2 structure
- $\mathrm{SU}(3)$ connection $\bar{\nabla}$ with totally skew symmetric torsion

Acharya-Bryant-Salamon ('20): Description of the $\mathrm{SU}(3)$ structure of a circle quotient of $C(\mathbb{CP}^3)$

Important Results for Nearly Kähler manifolds

- Grunewald ('94): Nearly Kähler manifolds are characterised by the existence of a real Killing spinor

$$\nabla_\xi \phi = \alpha \xi \cdot \phi \quad \phi \in \Gamma(X, \mathcal{S}), \xi \in TM, \alpha \in \mathbb{R}$$

- Nagy ('02): Nearly Kähler manifolds are locally a Riemannian product of homogenous nearly Kähler spaces, twistor spaces over quaternionic Kähler manifolds and 6-dimensional nearly Kähler manifolds
- Butruille ('10), there are exactly four homogeneous nearly Kähler structures (compact, simply-connected): On
  - $S^6 = G_2/\mathrm{SU}(3)$,
  - $\mathbb{CP}^3 = \mathrm{Sp}(2)/\mathrm{U}(1) \times \mathrm{Sp}(1)$,
  - $S^3 \times S^3 = \mathrm{SU}(2)^3/\Delta\mathrm{SU}(2)$
  - $\mathbb{F}_{1,2}(\mathbb{C}^3) = \mathrm{SU}(3)/\mathbb{T}^2$
- Foscolo-Haskins('17): Construction of inhomogeneous structures on $S^3 \times S^3$ and S^6

Lagrangian submanifold: $L \subset M$ with $\omega|_L \equiv 0$.
Our focus is on J-holomorphic curves $\varphi\colon (X, I) \to M$ with $\mathrm{d}\varphi \circ I = J \circ \mathrm{d}\varphi$:

- Have isolated singularities: $\mathrm{d}\varphi \in \Omega^{1,0}(X, \varphi^* TM)$ is holomorphic
- Not calibrated but minimal
- Are locally described (up to reparametrisation) by four functions of one variable: A real-analytic curve can locally uniquely be thickened to a J-holomorphic curve

Difficulties

- There are no J-holomorphic immersions $N^4 \to M^6$ or submersions $M^6 \to N^4$
- J is not 'generic' and not integrable

- Bryant '82: There is a class *J*-holomorphic curves, called **null-torsion curves**, in S^6 coming from certain holomorphic curves $X^2 \to Q \subset \mathbb{CP}^6$
- Bryant '82: Parametrisation of *J*-holomorphic curves in $\mathbb{CP}^3$, called **superminimal curves**, tangent to $\mathcal{H}$

$$\Theta(f,g) = [1, f - \frac{1}{2}g\left(\frac{\mathrm{df}}{\mathrm{dg}}\right), g, \frac{1}{2}\left(\frac{\mathrm{d}f}{\mathrm{d}g}\right)]$$

  for $f, g \colon X \to \mathbb{CP}^1$ meromorphic
- Xu '10: There is another copy of superminimal curves in $\mathbb{CP}^3$: **null-torsion curves**

J holomorphic curves in $\mathbb{CP}^3$ and other geometries

Eells-Salamon Correspondence

There is a one-to-one correspondence between (branched) minimal surfaces in S^4 and non-vertical J-holomorphic curves in $\mathbb{CP}^3$.

$\rightsquigarrow$ Minimal tori in S^4: Ferus-Pedit-Pinkall-Sterling '90, '92
Twistor perspective reduces second order to first order equations but more complicated ambient space.
Relation to G_2 and $\mathrm{Spin}(7)$ geometry:

- Associatives in the cone and sine-cone of $\mathbb{CP}^3$
- Karigiannis-Min-Oo '05: Associatives in $\Lambda^2_-(S^4)$ and Cayley submanifolds in $\mathcal{S}_-(S^4)$ as total spaces of vector bundles over minimal $X \subset S^4$
- Kawai '15, Ball-Madnick '20: Ruled associative submanifolds of nearly parallel S^7_{sq} and Berger space $\mathrm{SO}(5)/\mathrm{SO}(3)$

Reducing the frame bundle

- The splitting $T\mathbb{CP}^3 = \mathcal{H} \oplus \mathcal{V}$ is parallel wrt $\bar{\nabla}$,

$$\mathrm{Hol}(\bar{\nabla}) \subset S(\mathrm{U}(2) \times \mathrm{U}(1)) \subset \mathrm{U}(2)$$

- Up to double covers, $\mathrm{Sp}(2) \to \mathbb{CP}^3$ is the reduced frame budle with structure group $\mathrm{U}(1) \times \mathrm{Sp}(1)$

$$\Omega_{MC} = \begin{pmatrix} i\rho_1 + j\overline{\omega_3} & -\frac{\overline{\omega_1}}{\sqrt{2}} + j\frac{\omega_2}{\sqrt{2}} \\ \frac{\omega_1}{\sqrt{2}} + j\frac{\omega_2}{\sqrt{2}} & i\rho_2 + j\tau \end{pmatrix}$$

- $\omega_1, \omega_2, \omega_3 \in \Omega^1(\mathrm{Sp}(2), \mathbb{C})$ local unitary $(1,0)$-forms on $\mathbb{CP}^3$
- Over J-holomorphic curve $\varphi \colon X \to \mathbb{CP}^3$,

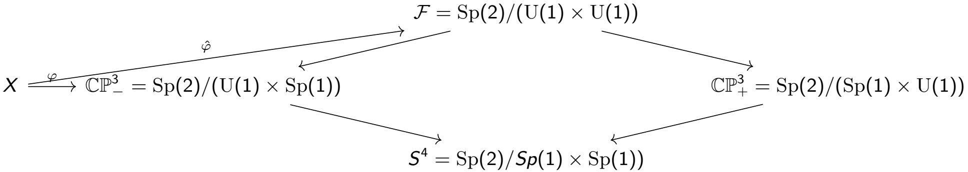
$$\omega_2 \equiv 0$$

reduces $\varphi^*\mathrm{Sp}(2)$ to an $\mathrm{U}(1) \times \mathrm{U}(1)$ and τ becomes a basic form

Angle functions

Equivalently, there is a J-holomorphic lift $\hat{\varphi}\colon X \to \mathcal{F} = \mathrm{Sp}(2)/\mathbb{T}^2$ and

$$T\mathcal{F} = \mathsf{H} \oplus \mathsf{V}_+ \oplus \mathsf{V}_-$$



Definition

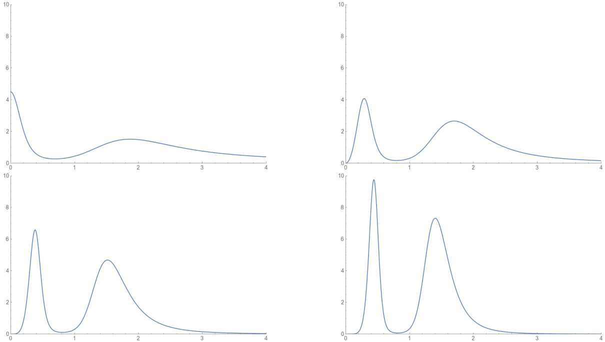
For $\varphi\colon X \to \mathbb{CP}^3$ with lift $\hat{\varphi}\colon X \to \mathcal{F}$ define

$$\alpha_\pm(x) = \frac{\|\xi\|_{\mathsf{V}_\pm}}{\|\xi\|_{\mathsf{H}}}, \qquad \text{for } \xi \in T_xX \subset \hat{\varphi}^*(T\mathcal{F}).$$

φ is superminimal iff $\alpha_- \equiv 0$ and null-torsion iff $\alpha_+ \equiv 0$

Some superminimal spheres

Immersion of superminimal spheres with $f(z) = z^k$ and $g(z) = z$, plot of α_+^2 for degree $k = 3, 4, 5, 6$



Zeros of α_+ correspond to totally geodesic points:

$$r_\tau = 6(g-1) + 2\deg - 2r_{\mathcal{H}}$$

for $g = 0$ and $r_{\mathcal{H}} = 0$

Transverse J-holomorphic curve

Call a J-holomorphic **transverse** if

$$\alpha_\pm \in (0,\infty) \Leftrightarrow \hat{\varphi} \text{ is nowhere tangent to } \mathsf{V}_\pm \text{ or } \mathsf{H}$$

- Allows reduction to discrete structure group:

$$\omega_3 = \alpha_- \omega_1, \quad \tau = \alpha_+ \omega_1$$

- Let $\Theta = -\alpha_-^2 \mathrm{Id}_{\mathcal{H}} \oplus \mathrm{Id}_{\mathcal{V}}$ and $\nu_1 = \Theta(TX) \subset \nu$

$$T\mathbb{CP}^3 = TX \oplus \nu_1 \oplus \nu_2$$

X is compact $\Rightarrow$ immersed torus with $\frac{1}{2}\mathrm{vol}_{\mathcal{H}} = \mathrm{vol}_{\mathsf{V}_-} = \mathrm{vol}_{\mathsf{V}_+}$.

- Non-transverse points governed by holomorphic differentials, local behaviour:

$$\alpha_\pm(z) = |z|^k u$$

with positive smooth $u \colon X \supset U \to \mathbb{C}$

Toda Lattice Equations

Theorem (A. '21)

The induced metric $g_{\mathcal{H}}$ *on* X *is conformally flat with factor* $\gamma = (\alpha_-\alpha_+)^{-1/2}$ *with Gauß curvature* $2(1-\alpha_-^2-\alpha_+^2)$ *and* $\alpha_\pm$ *satisfy*

$$\Delta_0 \log(\alpha_-^2) = -4(3\alpha_-^2+\alpha_+^2-2)\gamma$$
$$\Delta_0 \log(\alpha_+^2) = -4(3\alpha_+^2+\alpha_-^2-2)\gamma.$$

The second fundamental form $\mathrm{II} \in \Omega^1(X, TX^\vee \otimes \nu)$ *is*

$$\mathrm{II} = -\frac{2}{\alpha_-^2+1}\mathrm{d}^{1,0}\alpha_- \otimes f_2 \otimes f^1 + \frac{\alpha_+\omega_1}{\sqrt{\alpha_-^2+1}} \otimes f_3 \otimes f^1$$

Example: Unique flat curve $\alpha_- = \alpha_+ = \frac{1}{\sqrt{2}}$, lift of Clifford torus

Properties of the Angle Functions

- Solutions depend on four functions of one variable
- Negative result: Every J-holomorphic curve with holomorphic second fundamental form is superminimal or has zero-torsion
- Bolton-Pedit-Woodward '95: Solutions are integral surfaces of Hamiltonian distribution on finite-dim vector space
- $\alpha_\pm$ locally determine the J-holomorphic curve (up to constants)
- Minimal surface lies in $S^3 \subset S^4$ iff $\alpha_- \equiv \alpha_+$
  $\rightsquigarrow$ Sinh-Gordon equation $\Delta_0(u) = -\lambda \sinh(u)$
- **Lawson-torus**:

$$\{(z, w) \in S^3 \subset \mathbb{C}^2 \mid \mathrm{Im}(z^m \bar{w}^k) = 0\},$$
$$\alpha_-(x) = \alpha_+(x) = C_{k,m}(m^2 \cos(x)^2 + k^2 \sin(x)^2)$$

Imposing $\mathrm{U}(1)$-symmetry

- Fix $\mathrm{U}(1)_{k,m}$ subgroup of $\mathbb{T}^2 \subset \mathrm{Sp}(2)$ by choosing $\xi = \mathrm{diag}(ik, im) \in \mathfrak{sp}(2)$ with fundamental vector field K^ξ and $k \geq m$
- $\mathrm{U}(1)$-invariant J-holomorphic curves are integral surfaces of the distribution spanned by K^ξ and JK^ξ
$\rightsquigarrow$ Integrate JK^ξ
- For general $k, m \in \mathbb{Z}$ the action commutes with $\mathbb{T}^2 \subset \mathrm{Sp}(2)$
- Toric multi-moment-map Russo-Swann '19

$$\nu = \omega(K_1^\xi, K_2^\xi) = 12|Z|^{-4}\mathrm{Im}(Z_0 Z_1 \overline{Z_2 Z_3})$$

preserved for JK^ξ.
Are there more preserved quantities?

Parametrising the orbit space

The functions

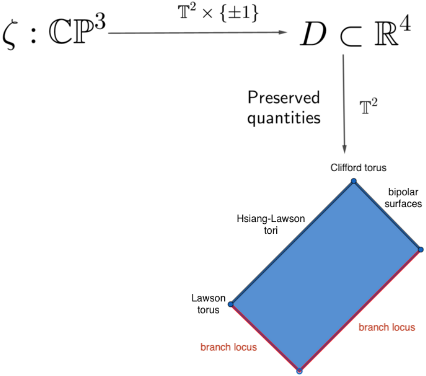
$$\zeta = (v_-, v_+, r_-, r_+) \colon \mathbb{CP}^3 \to D \subset \mathbb{R}^4$$
$$v_\pm = \|K^\xi\|^2_{\mathrm{V}_\pm}, \quad r_\pm = \frac{1}{2} JK^\xi \log(v_\pm)$$

parametrise $\mathbb{CP}^3/\mathbb{T}^2$, up to a singular set and $\mathbb{Z}_2$-action of complex conjugation. The branch locus in $\mathbb{CP}^3/\mathbb{T}^2$ is

$$\mathbb{RP}^3/\mathbb{Z}_2 = \nu^{-1}(0)/\mathbb{T}^2.$$

On D, the flow equation of JK^ξ is Hamiltonian and has a Lax representation.
$\rightsquigarrow$ two preserved quantities in involution

Liouville-Arnold: The fibres of $D \to \mathcal{R} \subset \mathbb{R}^2$ are two-tori and the flow equation admits action-angle coordinates



ξ corresponds to $Q = kZ_0Z_1 + mZ_2Z_3$ under $\mathfrak{sp}(2)_{\mathbb{C}} \cong S^2(\mathbb{C}^4)$ and $v_- = 4|Q|^2$.
The quadric $\{Q = 0\}$ is traced out by $\mathrm{U}(1)$-invariant superminimal curves.
The rectangle degenerates to the top left line if $k = m$ and top to right line if $m = 0$.

Further Directions

- Treat degenerate case $k = m$ with $\mathrm{U}(2)$-moment maps

$$M \to \mathbb{R}^3 \oplus \mathbb{R}^3 \oplus \mathbb{R}^3 \oplus \mathbb{R}$$

- Can $\mathrm{U}(1)$-invariant picture be generalised to τ-primitive maps in general flag manifolds?
- Is there a class of (special) Lagrangians in $\mathbb{CP}^3$ ruled over transverse J-holomorphic curves?