Symmetric Bow Varieties

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Connections between String Theory and Special Holonomy

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§ Review of Coulomb branches of 3d N=4 SUSY gauge theories

(G_c : a compact Lie group
($\mathbb{M}$: a quaternionic representation of G_c with an invariant inner product

↓ physics

3d N=4 SUSY gauge theory

↙ ↘

Higgs branch

$\mathcal{M}_H = \mathbb{M} /\!/\!/ G_c$

hyperKähler quotient

Coulomb branch

$\mathcal{M}_C \approx \mathfrak{t} \times T^\vee / W$

classically, but receives a quantum correction

→ no mathematical definition so far

Here $\mathbb{T}$ = maximal torus of complexification G of G_c
$\mathfrak{t}$ = Lie $\mathbb{T}$, $\mathbb{T}^\vee$ = dual torus

Current Status

- Suppose $\mathbb{M} = \mathbb{N} \oplus \mathbb{N}^*$
  - └ cpx representation of G_c

  Braverman-Finkelberg-N (2018) defined $\mathcal{M}_C$ as a Poisson variety

  More precisely,
  $$\mathcal{M}_C = \operatorname{Spec} H_*^{G[[z]]}(\mathcal{R})$$
  ($\mathcal{R}$: a certain ∞-dimensional variety; $H_*^{G[[z]]}(\mathcal{R})$: a commutative algebra with Poisson bracket)

- Braverman recently announced a generalization when
  $$\pi_4(G_c) \longrightarrow \pi_4(Sp(\mathbb{M}))$$
  ($\pi_4(G_c) \to \pi_4(Sp(\mathbb{M}))$ is $= 0$; $\pi_4(Sp(\mathbb{M})) = \{\pm 1\}$)

  This condition is anomaly cancellation, and is expected to be necessary

Remark

The definition satisfies all properties which physicists claim (as far as I know).
e.g. monopole formula by Cremonesi-Hanany-Zaffaroni

Torus action and deformation

- Suppose $\mathbb{G}_c \curvearrowright \mathbb{M}$. flavor symmetry

  $\mathbb{G}_c \triangleleft \tilde{\mathbb{G}}_c \curvearrowright$

  Then $\tilde{\mathbb{G}}_c / \mathbb{G}_c \curvearrowright \mathcal{M}_H = \mathbb{M} /\!\!/\!\!/ \mathbb{G}_c$ action

- Suppose $\mathbb{G}_c \xrightarrow{\chi} U(1)$ character

  FI parameter

  Then hyperKähler moment map for $\mathcal{M}_H$ can be shifted by $d\chi \in \mathfrak{g}_c^*$ deformation

☆ Flavor symmetry and FI parameter give opposite structures on $\mathcal{M}_C$.

- flavor $H_*^{\tilde{G}[\![z]\!]}(\mathcal{R})$ deformation of $H_*^{G[\![z]\!]}(\mathcal{R})$ parametrised by $H^*_{\tilde{G}/G}(pt)$

- FI $\pi_0(\mathcal{R}) = \pi_0(\Omega \mathbb{G}_c) = \pi_1(\mathbb{G}_c)$

  based loops

$\therefore$ $H_*^{G[\![z]\!]}(\mathcal{R})$ is graded by $\pi_1(\mathbb{G}_c)$

$\therefore$ $\mathcal{M}_C = \mathrm{Spec}\, H_*^{G[\![z]\!]}(\mathcal{R}) \curvearrowleft \pi_1(\mathbb{G}_c)^\wedge \xleftarrow{\pi_1(\chi)^\wedge} \pi_1(U(1))^\wedge \cong U(1)$

integrable system

$$H^*_G(pt) \longrightarrow H_*^{G[[z]]}(R)$$

$$\rightsquigarrow \operatorname{Spec} H^*_G(pt) \xleftarrow[\pi]{} M_C$$

$$\operatorname{Spec} H^*_G(pt) = \mathfrak{t}/W$$

generic fiber $\cong T^\vee$

$$\therefore M_C \approx \mathfrak{t} \times T^\vee / W$$

A recipe to identify M_C with a candidate M (as a variety)

① construct $M \xrightarrow{\pi} \mathfrak{t}/W$

② birational map $M \approx \mathfrak{t} \times T^\vee / W$

③ Show that M is normal and π is flat.

④ Show that $M \approx M_C$ extends up to codimension 2

We can use this recipe to identify M_C in many examples.

difficult parts ⓪ Find a candidate M and ③ above

§ quiver gauge theory

$Q = (Q_0, Q_1)$: quiver $\quad Q_0$: vertex, Q_1 : edges

V, W : Q_0 - graded vector spaces

$$G = \prod_{i \in Q_0} U(V_i),$$

$$\mathbb{N} = \bigoplus_{h \in Q_1} \mathrm{Hom}(V_{o(h)}, V_{i(h)}) \oplus \bigoplus_{i \in Q_0} \mathrm{Hom}(W_i, V_i),$$

$$\mathbb{M} = \mathbb{N} \oplus \mathbb{N}^*$$

<u>Th</u> [BFN 2019]

$(G_c, \mathbb{M})$: a framed quiver of type ADE

G_c = ADE group of adjoint type

Then $\mathcal{M}_C \cong$ moduli space of singular G_c - monopoles on $\mathbb{R}^3$

$(\dim W_i)$: singularity type

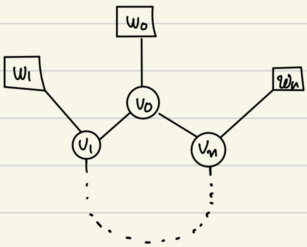
$(\dim V_i)$: monopole charge

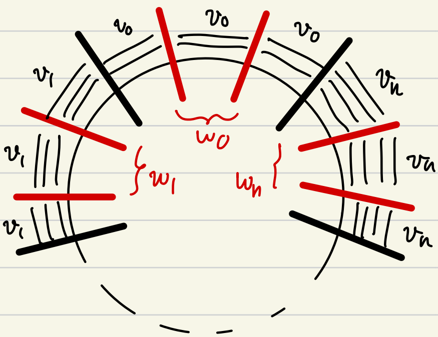
<u>Remark</u>

We used an algebro-geometric model of RHS (generalised slice)

§ Cherkis bow varieties

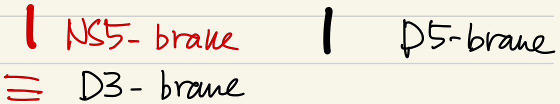
Hanany–Witten, de Boer–Hori–Ooguri–Oz–Yin
Cherkis, N–Takayama



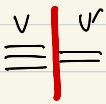


(balanced)

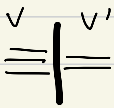
$\mathbb{G}_{\mathbb{C}} = \prod U(v_i)$
$\mathbb{M}$ = linear maps



Assign hyperKähler manifolds with $U(v) \times U(v')$-action



$\rightsquigarrow \mathbb{C}^v \rightleftarrows \mathbb{C}^{v'}$



$\rightsquigarrow$ moduli of solutions for Nahm's equation on $\underset{-1}{\bullet}\!\!-\!\!-\underset{0}{\times}\!\!-\!\!-\underset{1}{\bullet}$ with pole $\frac{\rho_\alpha}{s}$ $(\alpha = 1, 2, 3)$

$$\cong \begin{cases} T^* GL(v) \times \mathbb{C}^v \times (\mathbb{C}^v)^* & \text{if } v = v' \\ GL(v) \times (\text{slice to } \underbrace{\square}_{v-v'}) & \text{if } v > v' \end{cases}$$

Take products and hyperKähler quotients

Th. (N-Takayama 2017)
Suppose $(G_c, \mathbb{M})$ arises from a framed quiver of affine type A.
Then $\mathcal{M}_C \cong$ balanced bow variety.

<u>Remark</u>
Cherkis introduced bow varieties as ADHM type descriptions of instantons on multi Taub-NUT spaces

§ Quiver gauge theories of affine type D and symmetric bow varieties

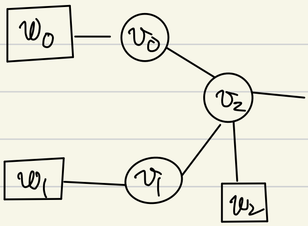
(originated by my student de Campos Affonso)

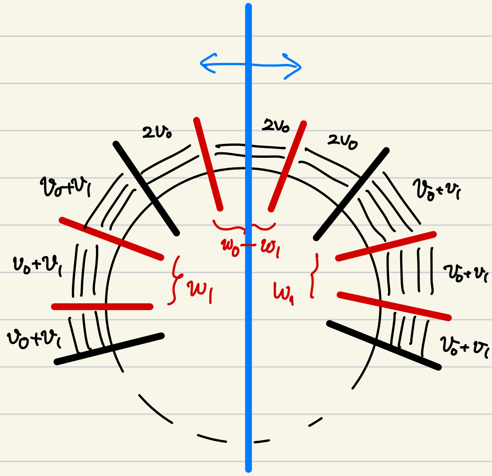
$\mathcal{M}_C$ should be partial compactification of a moduli space of SO(even)-instanton on a multi Taub-NUT space.

$\therefore$ $\mathcal{M}_C$ = fixed point locus of a bow variety with respect to an involution

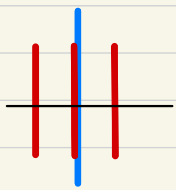
instanton $\mapsto$ dual instanton.

involution

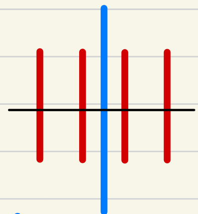




$w_0 - w_1$: odd



$w_0 - w_1$: even



$$\mathbb{C}^{2v_0} \underset{D}{\overset{C}{\rightleftarrows}} \mathbb{C}^{2v_0}$$

dual by symplectic form

$C^* = C$

$D^* = D$

$$\mathbb{C}^{2v_0} \rightleftarrows \mathbb{C}^{2v_0}$$

symplectic

$USp(v_0)$

☆ nontrivial part is ③ normality and flatness

<u>Remark</u> The involution appeared in the description of monopoles on $\mathbb{R}^3$ Hurtubise – Murray 1989.

technical caveat

The fixed point set in the usual bow variety
↖ affine algebraic
does not behave well.
→ not irreducible

Consider deformed or resolved bow varieties and take fixed points there