# Kondo line defect and affine oper/Gaudin correspondence

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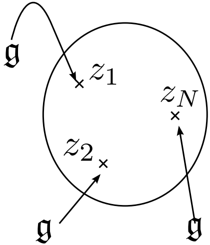
# Oper/Gaudin correspondence

# Gaudin model

finite Gaudin model is defined as follows[Gaudin '87]




- simple Lie algebra $\mathfrak{g}$, with basis $t^a$
- $N$ sites $z_1, z_2, \ldots, z_N \in \mathbb{C}$
- Hilbert space $V_{\underline{\lambda}} = V_{\lambda_1} \otimes V_{\lambda_2} \cdots \otimes V_{\lambda_N}$ where $\underline{\lambda}$ is a collection of integral dominant weights

The (quadratic) Hamiltonians are commutative

$$H_i = \sum_{j \neq i} \frac{t_i^a t_j^a}{z_i - z_j}, \quad i = 1, \ldots, N \tag{1}$$

Surprisingly, there are higher Hamiltonians. In fact, there is a large commutative subalgebra $\mathcal{Z} \subset U(\mathfrak{g})^{\otimes N}$, called *Gaudin algebra*
A natural question is to figure out the Gaudin algebra and its joint eigenvalues and eigenvectors.

# Oper

- Cartan decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$
- Chevalley generators $f_i, \alpha_i, e_i$, $i = 1, \ldots, r$

$$\mathrm{Op}_{\mathfrak{g}}\left(\mathbb{CP}^1\right) = \left(\partial_z + \sum_{i=1}^{r} f_i + v(z)\right) \Big/ \mathcal{N}_+(R), \qquad v(z) \in \mathfrak{b}(R) \quad (2)$$

For example, $\mathfrak{g} = \mathfrak{sl}_M$,

$$\partial_z - \begin{pmatrix} 0 & v_2(z) & v_3(z) & \cdots & v_n(z) \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \quad (3)$$

or equivalently

$$\partial_z^n - v_2(z)\partial_z^{n-2} - \cdots - v_n(z) \quad (4)$$

# Oper/Gaudin correspondence

Theorem [Frenkel '04]

$$\mathcal{Z}_{\{z_i\}}(\mathfrak{g}) \cong \mathrm{Fun}\left(\mathrm{Op}_{\mathfrak{g}^\vee}\left(\mathbb{CP}^1\right)_{\{z_i\},\infty}\right) \tag{5}$$

where the oper has regular singularities of trivial monodromy at $z_i, \infty$
For $\mathfrak{g} = \mathfrak{sl}_2$,

$$\partial_z^2 - \sum_{i=1}^{N} \frac{c_i}{\left(z - z_i\right)^2} - \sum_{i=1}^{N} \frac{\mu_i}{z - z_i}, \qquad \sum_{i=1}^{N} \mu_i = 0 \tag{6}$$

where $c_i$ and $\mu_i$ are eigenvalues of the Casimir $C_i \equiv \frac{1}{2} t^{a(i)} t_a^{(i)}$ and Gaudin Hamiltonians $H_i$.

# Outline

**Motivations:**

- What happens in affine Lie algebra?
  - Higher Gaudin Hamiltonians?
  - affine oper/affine Gaudin correspondence?
  - joint eigenvalues and eigenvectors? how to read eigenvalues from affine oper?
- Physical interpretation
- affine Geometric Langlands?

**Plan:**

- focus on single site $\widetilde{\mathfrak{sl}}_2$
- more direct statement using Kondo line defect, basically the same as ODE/IM correspondence
- potential proof from 4d Chern Simons?
- interesting Stokes phenomenon: exact-WKB analysis, wall-crossing, ...

4d CS

integrable model

Kondo line

affine oper/ODE

affine Gaudin

affine oper/Kondo correspondence

# Affine $\widetilde{\mathfrak{sl}}_2$ oper = $\mathfrak{sl}_2$ $\lambda$-oper

In the principal gradation

$$\widetilde{\mathfrak{sl}}_2 := \bigoplus_{n\in\mathbb{Z}} \left(\mathbb{C}H\otimes\lambda^{2n}\oplus(\mathbb{C}E\oplus\mathbb{C}F)\otimes\lambda^{2n+1}\right)\oplus\mathbb{C}\delta\oplus\mathbb{C}\lambda\partial_\lambda$$

$$\partial_x + \begin{pmatrix} \sum\lambda^{2n}a_n(x) & \lambda^{-1}+\sum\lambda^{2n+1}b_n(x) \\ \lambda^{-1}+\sum\lambda^{2n+1}c_n(x) & -\sum\lambda^{2n}a_n(x) \end{pmatrix} - \omega(x)\lambda\partial_\lambda \Big/ \widetilde{\mathcal{N}}_+(R)$$

With $\omega(x)\equiv\frac{1}{2}\frac{\partial_x P(x)}{P(x)}$, there is a nice gauge ($\mathfrak{sl}_2$ $\lambda$-oper)

$$\partial_x + \begin{pmatrix} 0 & P(x)\lambda^{-1}+t(x)\lambda \\ \lambda^{-1} & 0 \end{pmatrix} \tag{7}$$

which is equivalent to

$$\partial_x^2 - \left(\frac{1}{\lambda^2}P(x)+t(x)\right) \tag{8}$$

# Kondo line defect in single site $\widetilde{sl}(2)_k$

$$\mathrm{Tr}_{\mathcal{R}_n}\mathcal{P}\exp\left(ig\int_L d\sigma\ t_a J^a(\sigma)\right) \rightsquigarrow \hat{T}_n[L,\lambda]$$

where $t_a$ are the $n$-dim rep $\mathcal{R}_n$ of $su(2)$. $\lambda^{-1}$ is the scale

- UV (small $g$ or large $\lambda$): only small $g$ perturbative expansion is known.
- IR (large $g$ or small $\lambda$): cannot be seen from perturbation theory

This is precisely the Kondo line defect in chiral $SU(2)_k$ WZW CFT. Physically it describes a single magnetic impurity coupled to the current by[Kondo, Wilson]

$$H = H_{bulk} + gS^a J^a \delta(x=0) \tag{9}$$

with $SU(2)$ spin $S^a$ and $SU(2)$ current $J^a$.

$$
\begin{aligned}
\hat{T}_j =&2j+1,\\
&+g^2 4\pi^2\lambda^2 x_j J_0^a J_0^a\\
&+g^3(-16)i\pi^2\lambda^3 x_j\left\{\sum_{n>0}\frac{i}{2n}f_{abc}J_{-n}^a J_0^b J_n^c+\sum_{n>0}\frac{2}{n}J_{-n}^a J_n^a-\log R\, J_0^a J_0^a-\frac{k}{2}\right\}\\
&+g^4 8\pi^2\lambda^4 x_j\left\{\sum_{n,m,n+m\neq 0}\Big[\frac{2}{3m(m+n)}:J_{-m-n}^a J_0^b J_m^a J_n^b:-\frac{1}{3nm}:J_{-m-n}^a J_0^a J_m^b J_n^b:\Big]\right.\\
&\quad+\sum_{n,m>0}\frac{1}{nm}\Big[:J_{-n}^a J_{-m}^a J_m^b J_n^b:-:J_{-n}^a J_{-m}^b J_m^a J_n^b:\Big]\\
&\quad+\sum_{n>0}\frac{1}{n^2}\Big[2J_{-n}^a J_0^b J_0^b J_n^a-J_{-n}^a J_0^a J_0^b J_n^b-J_{-n}^a J_0^b J_0^a J_n^b\Big]\\
&\quad+\sum_{n,m>0}\frac{3i}{nm}f_{abc}:J_{-n}^a J_{-m+n}^b J_m^c:+\sum_{n>m>0}\frac{i}{nm}f_{abc}[J_{-n}^a J_{-m}^b J_{m+n}^c+J_{-m-n}^a J_m^b J_n^c]\\
&\quad+\sum_{n>0}\frac{6i}{n}\Big[\log R-\frac{1}{3}(H_{\lfloor\frac{n}{2}\rfloor}+H_{\lfloor\frac{n-1}{2}\rfloor})\Big]f_{abc}J_{-n}^a J_0^b J_n^c+\sum_{n>0}\frac{2i}{n^2}f_{abc}J_{-n}^a J_0^b J_n^c\\
&\quad+\ldots\ldots
\end{aligned}
$$

# Kondo defect is **Integrable**

We perform the explicit checks to the order $g^4$:
Commutativity:

$$\left[\hat{T}_n[\lambda], \hat{T}_{n'}[\lambda']\right] = 0 \qquad (10)$$

$\Rightarrow$ Coefficients of $g$-expansion or $\lambda$-expansion are Gaudin Hamiltonians
Hirota relations:

$$\hat{T}_n\left[e^{i\frac{\pi}{2}}\lambda\right]\hat{T}_n\left[e^{-i\frac{\pi}{2}}\lambda\right] = 1 + \hat{T}_{n-1}[\lambda]\hat{T}_{n+1}[\lambda] \qquad (11)$$

[Kondo, Lesage-Saleur-Kivelson, Bachas-Gaberdiel, Runkel, Andrei,Wiegmann,Destri,Tsvelick-Weigmann, Affleck-Ludwig, ...]

*Kondo defects carry the integrability structure of the bulk CFT*

# affine oper/Kondo correspondence

| | |
|---|---|
| quantum | Expectation values of the Kondo line defect |
| | can be computed by |
| classical | the generalized monodromy data of an auxiliary ODE |

**Claim:** for a single site $\widetilde{\mathfrak{sl}}(2)_k$, $P(x) = e^{2x} x^k$

$$\psi''(x) = \frac{1}{\lambda^2} e^{2x} x^k \psi$$

where we put $t(x) = 0$ for now.

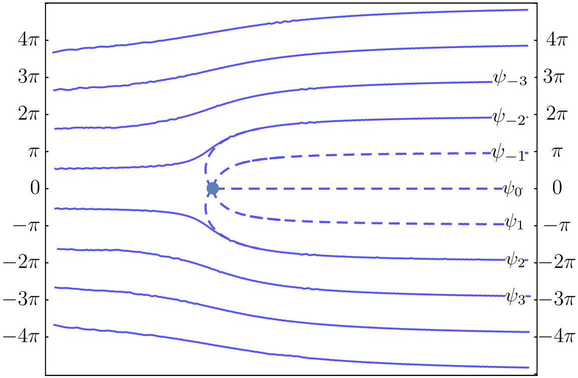
# Stokes phenomenon: essential singularity

The oper is a flat connection on $\mathbb{P}^1$

$$\partial_x + \begin{pmatrix} 0 & P(x)\lambda^{-1} \\ \lambda^{-1} & 0 \end{pmatrix}, \qquad P(x)dx^2 = e^{2x}x^k dx^2 \tag{12}$$

Define *spectral network* by the union of the trajectories

$$\sqrt{P(x)}dx \cdot \partial_t \in \mathbb{R}^\times \tag{13}$$




Generalized monodromy data are encoded by Wronskians

# New examples of ODE/IM correspondence

We claim the ODE for chiral $\mathrm{SU}(2)_k$ WZW is

$$\psi''(x) = \left[\frac{1}{\lambda^2} e^{2x} x^k + t(x)\right]\psi$$

- Vacuum: $t(x) = 0$.

$$\langle 0|\hat{T}_n[\lambda]|0\rangle \triangleq i\left(\psi_{-\frac{n}{2}}(x;\lambda), \psi_{\frac{n}{2}}(x,\lambda)\right)$$

- other states with quantum number

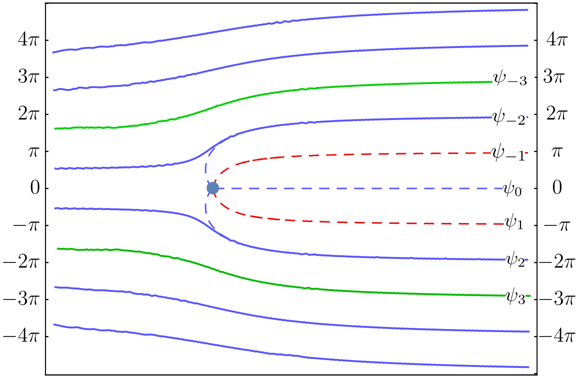
  $l =$ spin, $\quad \#w'_a = L_0, \quad l + \#w'_a - \#w_i = J_0^0$

  we choose $t(x) = a(x)^2 + \partial_x a(x)$ and

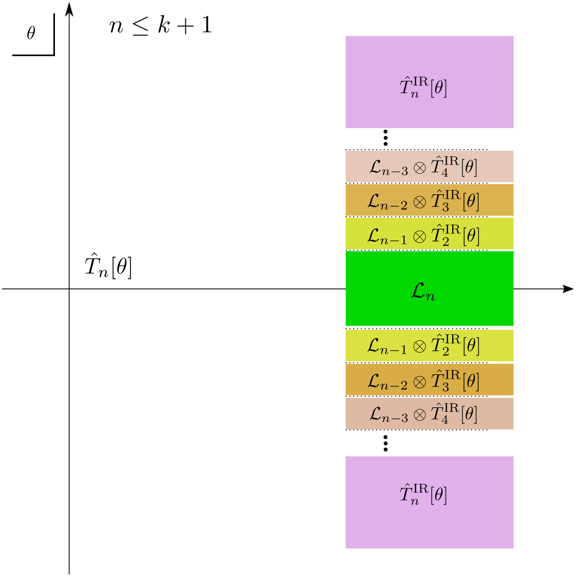
$$a(x) = -\frac{l}{x} + \sum_i \frac{1}{x - w_i} - \sum_a \frac{1}{x - w'_a}$$

  $w_i$ and $w'_a$ are determined so that we have *trivial monodromy* at all the regular singularities $0, w_i, w'_a$ for all $\lambda$.

$$i(\psi_{-\frac{n}{2}}, \psi_{\frac{n}{2}}) = \begin{cases} d_n^{(k)} + \ldots, & \text{connected at the zero}, n \le k+1 \\ n - k + \ldots, & \text{connected at } -\infty, n > k+1 \end{cases}$$



# Wall-crossing from exact WKB



# Construction in 4D Chern Simons

# 4D Chern Simons

4D cousin of the three dimensional Chern Simons [Costello, Costello-Witten-Yamazaki, . . . ]

$$S = \int_{\mathbb{R}^2 \times \mathbb{C}} dz \wedge \mathrm{CS}[A] \tag{14}$$

topological in $\mathbb{R}^2$, holomorphic in $\mathbb{C}$

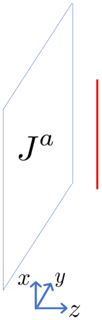
# Construction of Kondo problems in 4D CS

- Let's put a chiral WZW on $\mathbb{R}^2 \times \{z_0\}$ and couples to the bulk via $J^a_w A^a_{\bar{w}}$
- gauge anomaly[Costello]:

$$dz \to \omega(z)dz \equiv \left(1 + \frac{k}{2}\frac{1}{z - z_0}\right) dz \qquad (15)$$

- RG flow implemented by shifting $\theta$ [Costello]

$$d\theta = \omega(z)dz \qquad (16)$$



# Construction of Kondo problems in 4D CS

CLAIM: reduce to 2d, we get Kondo line in chiral WZW

- commutativity of the line
- Hirota relation: fusion of the line
- at leading order, the Wilson line becomes [Costello-Witten-Yamazaki]

$$\int d\sigma \frac{1}{z - z_0} t^a J^a \tag{17}$$

# Construction of Kondo problems in 4D CS

CLAIM: reduce to 2d, we get Kondo line in chiral WZW

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## Conjecture

The holomorphic (twisted) one form of the 4d CS is related to the quadratic differential of the ODE($\psi'' = \left[e^{2\theta} P(x) + t(x)\right] \psi$) by

$$\omega(z) dz = \frac{1}{2} \frac{\partial P(z)}{P(z)} dz \tag{18}$$

# Generalization 1: multichannel Kondo

chiral $\prod_i \mathrm{SU}(2)_{k_i}$ WZW

Kondo line

$$\mathrm{Tr}_{\mathcal{R}} \mathcal{P} \exp \left( i \int_0^{2\pi} d\sigma \; g_i t^a J_i^a(\sigma) \right) \rightsquigarrow \hat{T}_{\mathcal{R}}(\{g_i\}) \tag{19}$$

4d CS multiple surface defect at $\mathbb{R}^2 \times \{z_1, z_2, \dots\}$

$$w(z) = 1 + \sum_i \frac{k_i}{2} \frac{1}{z - z_i} \tag{20}$$

ODE

$$P(x) = e^{2x} \prod_i (x - x_i)^{k_i} \tag{21}$$

# Generalization 2: anisotropic Kondo

chiral $\prod_i \mathrm{SU}(2)_{k_i}$ WZW

Kondo line $SU(2) \to U(1)$

$$\mathrm{Tr}_{\mathcal{R}} \mathcal{P} \exp \left[ i \int_0^{2\pi} d\sigma \ \lambda t^0 \sum_i J_i^0 + \sum_i g_i \left( t^+ J_i^-(\sigma) + t^- J_i^+(\sigma) \right) \right]$$
$$\rightsquigarrow \hat{T}_{\mathcal{R}}(\lambda, \{g_i\}) \tag{22}$$

4d CS multiple surface defect at $\mathbb{R}^2 \times \{z_1, z_2, \dots\}$ in the trigonometric setting

$$w(z) = \frac{1}{z} \left( 1 + \sum_i \frac{k_i}{2} \frac{\hbar}{z/z_i - 1} \right) \tag{23}$$

ODE

$$P(x) = e^{2x} \prod_i (e^{-\epsilon x} - e^{-\epsilon x_i})^{k_i} \tag{24}$$

# Generalization 3: coset

coset

$$\frac{\prod_i su(2)_{k_i}}{su(2)_{\sum_i k_i}} \tag{25}$$

Kondo line $\mathcal{L} + \Phi_{1,3}$

4d CS multiple surface defect at $\mathbb{R}^2 \times \{z_1, z_2, \dots\}$

$$w(z) = \sum_i \frac{k_i}{2} \frac{1}{z - z_i} \tag{26}$$

ODE $P(x) = \prod_i (x - x_i)^{k_i}$

Cases for minimal models are known before
[Dorey-Dunning-Gliozzi-Tateo]
KdV Kondo/oper correspondence is precisely ODE/IM correspondence

# Conclusions and Future directions

**Conclusions**

- the role of Kondo defect in affine oper/Gaudin correspondence, ODE/IM correspondence
  - Kondo defect is the generating function of Gaudin Hamiltonians
  - Eigenvalues are given by Wronskians
- strong hints that we can 'prove' affine oper/Gaudin correspondence from (string embedding of) 4d Chern Simons.
- new examples of ODE/IM correspondence for the chiral $\prod_i \mathrm{SU}(2)_{k_i}$ WZW
- wall-crossing phenomenon in the IR

**Future directions**

- *local* integral of motions
- higher rank Lie algebra
- Hitchin system/Kondo line correspondence
- string theory construction of affine oper/Gaudin?
  - meaning of $t(x)$ in 4d CS

# Thank You!

# Checks and Remarks

PHYSICAL ORIGIN? UNKNOWN!
*How we wrote down this ODE will be presented in the third section.*

Practically

1. UV: calculation of $\langle\phi|\hat{T}_n|\phi\rangle$
2. IR: phase diagram for complex scale $\lambda$
3. find all the states in the spectrum
4. Hirota equation $\Leftrightarrow$ Plucker relation of the Wronskians
5. reproduce beta function
6. ....

# complex $\theta$, complex scale?

While it is very important to consider complexified spectral parameter $\theta$ in the study of integrability, in the current context it is not immediately natural to consider a complex scale.
I will provide two physical interpretation:

Generically, for a chiral defect, **scaling transformation** (shift of $\theta$) and **rotation** (shift of $\phi$) can be combined into a single parameter $\theta + i\phi$

$$\left[T - \bar{T}\right]_{x^1=0^+} - \left[T - \bar{T}\right]_{x^1=0^-} = 2i\partial_{x^0} t^{00} \tag{27}$$

$$\left[T + \bar{T}\right]_{x^1=0^+} - \left[T + \bar{T}\right]_{x^1=0^-} = 2\partial_{x^0} \tilde{t}^{00} \tag{28}$$

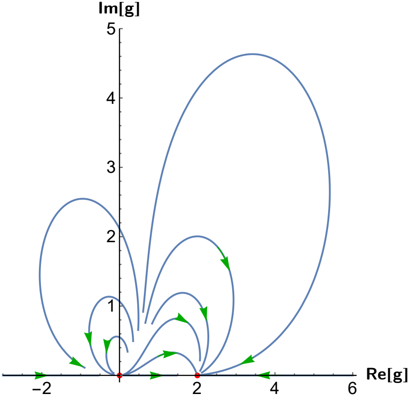
therefore

$$\int \left[t^{00}\partial_0 v^0 + \tilde{t}^{00}\partial_0 v^1\right] dx^0 = \int \left[\partial_0 v^0 + i\partial_0 v^1\right] t^{00} dx^0 \tag{29}$$

therefore, imaginary scale transformation = rotation.

# complex $\theta$, complex scale?

Second interpretation is that complex couplings models non-Hermitian process that involves **particle loss** and **dissipation**.
[Nakagawa-Kawakami-Ueda]
phase transition is simply due to the lack of the ability of screening.



# beta function from ODE

One of the nice parametrization in the UV is

$$\partial_x^2 \psi(x) = e^{2\theta} e^{2x} (1 + gx)^k \psi(x) \tag{30}$$

With the shift $x \mapsto x - \frac{1}{g}$, we get

$$\partial_x^2 \psi(x) = e^{2\theta} e^{-\frac{2}{g}} g^k e^{2x} x^k \psi(x) \tag{31}$$

Therefore if we parametrize the RG flow by $g_{\text{eff}}(\theta)$,

$$e^{-\frac{2}{g_{\text{eff}}(\theta)}} g_{\text{eff}}(\theta)^k \equiv e^{2\theta} e^{-\frac{2}{g}} g^k \tag{32}$$

with the beta function given by

$$\partial_\theta g_{\text{eff}}(\theta) = \frac{g_{\text{eff}}(\theta)^2}{1 + \frac{k}{2} g_{\text{eff}}(\theta)} \tag{33}$$

# from 4d CS to 2d Kondo

There exists a gauge where gluon 3-vertex vanishes [Costello-Yamazaki]

$$\int d\bar{w}dw\langle A_{a\bar{w}}(w,\bar{w},z,\bar{z})A_{b\bar{w}}(w',\bar{w}',z',\bar{z}')$$

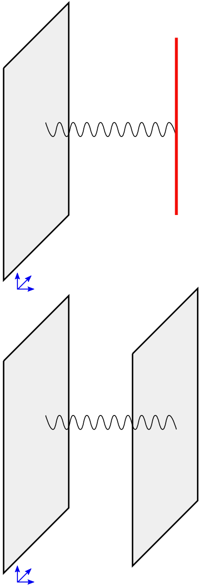
$$= r_{ab}(z-z') \propto \frac{\delta_{ab}}{z-z'}$$

We then get the leading classical action

$$\int r_{ab}(z-z_0)\sigma^a J^b(t)dt \qquad (34)$$

Similarly, in [Costello-Yamazaki]

$$\int_{\mathbb{R}^2} r_{ab}(z-z')J^a \bar{J}^b \qquad (35)$$



# Gaudin model and affine Gaudin model?

Given Lie algebra $\mathfrak{g}$, highest weight $\lambda$, irrep $V_\lambda$. A $N$-site Gaudin model is defined by

- a collection of integral dominant weights $\underline{\lambda} = \{\lambda_1, \ldots, \lambda_N\}$. Then the Hilbert space is given by $V_{\underline{\lambda}} = V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_N}$.
- a set $\underline{z} = \{z_1, \ldots, z_N\}$

The algebra of observables is $U(\mathfrak{g})^{\otimes N}$. There is a large commutative subalgebra called Gaudin subalgebra, which in particular contains the quadratic Gaudin Hamiltonian

$$H_i = \sum_{k \neq i} \frac{t_i^a t_k^a}{z_i - z_k} \tag{36}$$

In the case of $\mathfrak{gl}_2$, $H_i$ are all we need. $\textsc{Diagonalize}$?

- Bethe ansatz ?
- ${}^L G$ oper !

# Q operators

In the case of vacuum state,

$$\psi_0(x;\theta) \sim -Q(\theta)\Big[x + \frac{1}{g}\Big] - \widetilde{Q}(\theta) \tag{37}$$

then

$$\begin{aligned} T_n(\theta) &= i\left(\psi\Big(x;\theta - \frac{i\pi n}{2}\Big), \psi\Big(x,\theta + \frac{i\pi n}{2}\Big)\right) \\ &= \left[Q_\ell\left(\theta + \frac{i\pi n}{2}\right)\widetilde{Q}_\ell\left(\theta - \frac{i\pi n}{2}\right) - Q_\ell\left(\theta - \frac{i\pi n}{2}\right)\widetilde{Q}_\ell\left(\theta + \frac{i\pi n}{2}\right)\right] \end{aligned}$$

# coset scaling limit

Rescale the coordinate $x \mapsto \beta x$

$$\partial_x^2 \psi(x) = \left( \frac{\beta^{2+\sum_i k_i}}{\lambda^2} e^{2\beta x} \prod_i (x - \frac{z_i}{\beta})^{k_i} + \alpha^2 t(\alpha x) \right) \psi(x). \qquad (38)$$

- Take the limit $\beta \to 0$ while keeping $\frac{z_i}{\beta}$ and $\frac{\beta^{2+\sum_i k_i}}{\lambda^2}$ fixed.
- the line defects become effectively transparent to the overall WZW currents
- they can be identified with defect lines in a coset model.

OPEN QUESTION: its relation with BLZ oper? a nontrivial duality?

# Examples of minimal model: critical Ising model

$$\mathcal{L}_\epsilon \leftarrow \mathcal{L}_\sigma \rightarrow \mathcal{L}_I \tag{39}$$

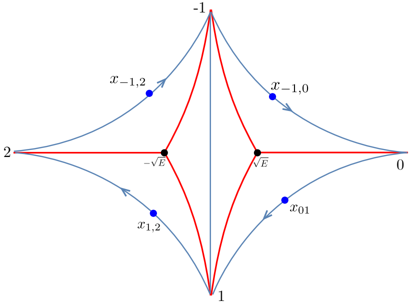
Fusion relation

$$\mathcal{L}_\sigma \times \mathcal{L}_\sigma = \mathcal{L}_I + \mathcal{L}_\epsilon \tag{40}$$

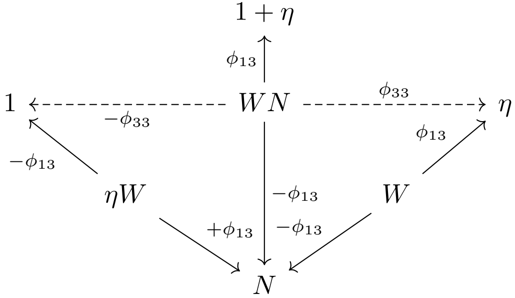
Define the Kondo line $L_\sigma[\theta]$ as deforming $\mathcal{L}_\sigma$ by $\Phi_{1,3}$. after the deformation

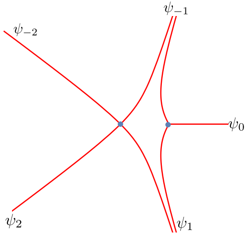
$$L_\sigma[\theta - i\frac{\pi}{2}]L_\sigma[\theta + i\frac{\pi}{2}] = \mathcal{L}_I + e^{-2\pi e^\theta}\mathcal{L}_\epsilon \tag{41}$$

$$\partial_x^2\psi(x) = e^{2\theta}(x^2 - 2)\psi(x)$$
$$T_\sigma[\theta] = i(\psi_{-1}, \psi_1)$$



# Examples of minimal model: tricritical Ising model




We can define the Kondo line $L_{\eta W}[\theta]$ as deforming from $\mathcal{L}_{\eta W}$ by $\Phi_{1,3}$.

$$\partial_x^2 \psi(x) = e^{2\theta} x^2 (x-1) \psi(x) \quad (42)$$

$$T_{\eta W}[\theta] = i(\psi_{-1}, \psi_1) \quad (43)$$

# Oper, affine oper

$$\partial_x + \begin{pmatrix} a_+(x) & P(x)\lambda^{-1} \\ \lambda^{-1} & -a_+(x) \end{pmatrix} \longrightarrow \partial_x^2 - \frac{P(x)}{\lambda^2} - \left[a_+^2 + \partial_x a_+\right]$$

Miura $\mathfrak{sl}_2$ $\lambda$-oper $\longrightarrow$ $\mathfrak{sl}_2$ $\lambda$-oper

Miura $\widetilde{\mathfrak{sl}}_2$ oper $\leftarrow\dashrightarrow$ Miura $\mathfrak{sl}_2$ $\lambda$-oper

Miura $\widetilde{\mathfrak{sl}}_2$ oper $\leftarrow\dashrightarrow$ dual Miura $\mathfrak{sl}_2$ $\lambda$-oper

$$\partial_x + \begin{pmatrix} -a_-(x) & \lambda^{-1} \\ P(x)\lambda^{-1} & a_-(x) \end{pmatrix} \longrightarrow \partial_x^2 - \frac{P(x)}{\lambda^2} - \left[a_-^2 + \partial_x a_-\right]$$

dual Miura $\mathfrak{sl}_2$ $\lambda$-oper $\longrightarrow$ dual $\mathfrak{sl}_2$ $\lambda$-oper

Miura $\widetilde{\mathfrak{sl}}_2$ oper $\longrightarrow$ $\widetilde{\mathfrak{sl}}_2$ oper

where $a_+(x) + a_-(x) = -\frac{1}{2}\frac{\partial_x P(x)}{P(x)} \equiv -\varphi(x)$

$$a_+(x) = -\alpha_+ - \sum_a \frac{l_a}{x - z_a} + \sum_i \frac{1}{x - w_i} - \sum_i \frac{1}{x - w_i'}, \quad (44)$$

$$a_-(x) = -\alpha_- - \sum_a \frac{\frac{k_a}{2} - l_a}{x - z_a} + \sum_i \frac{1}{x - w_i'} - \sum_i \frac{1}{x - w_i} \quad (45)$$

The Gaudin Bethe equation is simply

$$\mathrm{Reg}_{w_i} a_+(x) = 0, \quad \mathrm{Reg}_{w_i'} a_-(x) = 0 \quad (46)$$