Non-compact Spin(7)-manifolds

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- $\mathrm{G}_2 \subset \mathrm{Spin}(7)$: If we write $\mathbb{R}^8 = \mathbb{R}_t \times \mathbb{R}^7$ and $\varphi_0 \in \Lambda^3(\mathbb{R}^7)^*$ is the standard G_2-structure on $\mathbb{R}^7$, then

$$\psi_0 = dt \wedge \varphi_0 + *_7\varphi_0.$$

- A 4-form $\psi \in \Omega(M)$ on an 8-manifold M is called *admissible* if at each point $p \in M$ there is a linear isomorphism $T_pM \cong \mathbb{R}^8$ which identifies $\psi|_p = \psi_0$

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- If (M, ψ) satisfies $\nabla\psi = 0$ then ψ is called *torsion-free* and (M, ψ) a Spin(7)-manifold
- Restrictions on the holonomy group put algebraic constraints on the curvature tensor. $\mathrm{Hol}(g) \subset \mathrm{Spin}(7)$ implies $\mathrm{Ric}(g) = 0$.

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- $d\psi = 0$ locally non-linear PDE system of 56 equations for 43 unknown functions depending on 8 variables.
- Idea: Use symmetries to simplify.
- There are some restrictions: There are no non-flat homogeneous Spin(7)-manifolds and no compact Spin(7)-holonomy manifolds with continuous symmetries.

Cohomogeneity one manifolds

- Let G be a compact Lie group. G acts on M with *cohomogeneity one* if M/G is 1-dimensional.

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- In the context of Spin(7)-manifolds only one topological type of cohomogeneity one manifolds is interesting:
- $M/G \cong [0, \infty)$
- Denote the quotient map by $q : M \to M/G$. Then

$$M = \underbrace{(G/K)}_{q^{-1}(0)} \cup \underbrace{(0, \infty) \times (G/H)}_{q^{-1}(0,\infty)}.$$

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- G/K is called *singular orbit* and G/H is called *principal orbit*.

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- This is the local model for the general case:
  total space: vector space $\to$ vector bundle
  principal orbits: spheres $\to$ sphere bundles
  singular orbit: zero $\to$ zero section

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- Suppose M is a 8-dimensional cohomogeneity one manifold as above and ψ a G-invariant Spin(7)-structure.
- On each principal orbit $\{t\} \times G/H$ there exists a G-invariant G_2-structure φ_t such that

$$\psi = dt \wedge \varphi_t + *_t \varphi_t,$$
$$g = dt^2 + h_t.$$

Here the Hodge star depends on φ_t. The condition $d\psi = 0$ for ψ to be torsion-free then is equivalent to the system

$$d_{G/H} * \varphi_t = 0,$$
$$\frac{\partial}{\partial t} * \varphi_t = d_{G/H} \varphi_t.$$

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- Eschenburg–Wang (2000) in the context of coh1 Einstein manifolds considered a singular initial value problem on the singular orbit. In general one obtains families of local solution in a neighbourhood of the singular orbit which depend on several free parameters.

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- $\Psi_\mu, \mu \in (0, \infty)$, has principal orbit $N(1, -1) \cong \mathrm{SU}(3)/\mathrm{U}(1)$ and singular orbit $S^5 = \mathrm{SU}(3)/\mathrm{SU}(2)$. Total space is unique non-trivial rank 3 vector bundle over S^5.

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- $\Upsilon_\tau, \tau \in (-\infty, \infty)$, has principal orbit $N(1, 0) \cong \mathrm{SU}(3)/\mathrm{U}(1)$ and singular orbit $\mathbb{C}P^2 \cong \mathrm{SU}(3)/(\mathrm{SU}(2) \times \mathrm{U}(1))$. Total space is universal quotient bundle of $\mathbb{C}P^2$.

- For every pair (k, l) of integers which are not both zero, $\mathrm{U}(1)$ can be embedded in the maximal torus of diagonal matrices in $\mathrm{SU}(3)$ as

$$e^{i\theta} \mapsto \begin{pmatrix} e^{ik\theta} & 0 & 0 \\ 0 & e^{il\theta} & 0 \\ 0 & 0 & e^{-i(k+l)\theta} \end{pmatrix}. \tag{1.1}$$

We also denote this subgroup of $\mathrm{SU}(3)$ by $\mathrm{U}(1)_{k,l}$. The Aloff–Wallach space $N(k, l)$ is the homogeneous space $\mathrm{SU}(3)/\mathrm{U}(1)_{k,l}$.

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- An important property of complete non-compact manifolds is the asymptotic geometry.

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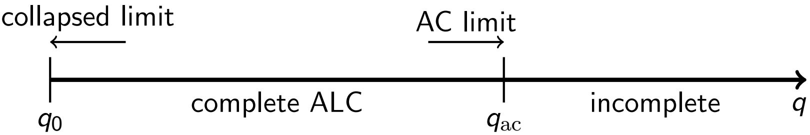
- $(C(\Sigma), \psi)$ is a Spin(7)-cone iff (Σ, φ) is a *nearly parallel* G_2-manifold.
- Every Aloff–Wallach space $N(k, l)$ carries a homogeneous, *nearly parallel* G_2-structure.

- The other asymptotic type that we will meet are *asymptotically locally conical* Spin(7)-manifolds. Close to infinity they locally look like the product of a 7-dimensional cone and a circle of fixed length.
- These are analogues of *asymptotically locally flat* (ALF) hyperkähler manifolds in real dimension 4 such as the Taub-NUT and Atiyah–Hitchin metrics.

Behaviour of the families Ψ_μ and Υ_τ

Theorem (L-, '20)

The two families Ψ_μ and Υ_τ behave as in the following figure:



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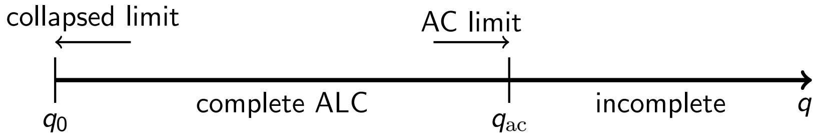

As $\mu \to 0$ the curvatures stay bounded and Ψ_μ converges to the Bryant–Salamon G_2-holonomy metric on $\Lambda^2_- \mathbb{C}P^2$.
As $\tau \to -\infty$ the curvatures of Υ_τ blow up on the zero section.

- The AC Spin(7) holonomy manifolds $\Psi_{\mu_{\mathrm{ac}}}$ and $\Upsilon_{\tau_{\mathrm{ac}}}$ resemble the well-known conifold transition of Calabi-Yau 3-folds: the *smoothing* T^*S^3 of the conifold $\{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 | \ z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0\}$ and its *small resolution* $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{C}P^1$ are topologically different spaces which carry cohomogeneity one AC Calabi-Yau metrics asymptotic to the same Calabi-Yau cone, the conifold.

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- For each τ the singular orbit $\mathbb{C}P^2$ is Cayley with respect to Υ_τ.

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- (a, b, c) describe a metric on $\mathrm{SU}(3)/\mathrm{U}(1)^2$.
- The Bryant–Salamon metric on $\Lambda^2_- \mathbb{C}P^2$ is asymptotic to the cone over the nearly Kähler metric $(a, b, c) = (1, 1, 1)$ on the flag manifold. Therefore, ALC asymptotics correspond to

$$(a(t), b(t), c(t), f(t)) \to (t, t, t, \ell) \quad \text{as } t \to \infty,$$

where $\ell > 0$ is a constant.

- Denote by (a_c, b_c, c_c, f_c) the coefficients of the nearly G_2-metric on $N(1,-1)$. Then AC asymptotics correspond to

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- $d\psi = 0$ then gives the ODE system

$$\frac{\dot{a}}{a} = \frac{b^2 + c^2 - a^2}{abc},$$
$$\frac{\dot{b}}{b} = \frac{c^2 + a^2 - b^2}{abc} - \frac{f}{b^2},$$
$$\frac{\dot{c}}{c} = \frac{a^2 + b^2 - c^2}{abc} + \frac{f}{c^2},$$
$$\frac{\dot{f}}{f} = \frac{f}{b^2} - \frac{f}{c^2}.$$

Projective coordinates

- Idea: The right-hand side of the ODE system is a scale-invariant rational expression. Switch to projective coordinates

$$A = \frac{a}{c}, \quad B = \frac{b}{c}, \quad F = \frac{f}{c}.$$

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$$A = \frac{a}{c}, \quad B = \frac{b}{c}, \quad F = \frac{f}{c}.$$

- With respect to the parameter s given by $dt = \frac{ab}{c} ds$ the ODE system for (A, B, F) takes the form

$$\begin{aligned}
\dot{A} &= A\left(2 - 2A^2 - ABF\right), \\
\dot{B} &= B\left(2 - 2B^2 - ABF - \frac{AF}{B}\right), \\
\dot{F} &= F\left(1 - A^2 - B^2 - 2ABF + \frac{AF}{B}\right).
\end{aligned}$$

Good news:

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- The asymptotic geometries now correspond to critical points.
- The ODE system for (A, B, F) is less singular.

Bad news:

- Still no control on F.

Another coordinate transform

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Another coordinate transform

- Key: Well-behaved behaviour of A, B dominates bad-behaviour of F.
- A good set of coordinates to approach the problem is

$$X = A^2, \quad Y = B^2, \quad Z = ABF.$$

Still denoting differentiation with respect to the variable s by a dot, the ODE system takes the form

$$\dot{X} = 2X(2 - 2X - Z),$$
$$\dot{Y} = 4Y - 4Y^2 - 2YZ - 2Z,$$
$$\dot{Z} = Z(5 - 3X - 3Y - 4Z).$$

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- The family Ψ_μ of trajectories emanates from fixed point $(0,1,0)$ and enters $\mathcal{W}$.
- The family Υ_τ emanates from the fixed point $(1,0,0)$ and enters $\mathcal{W}$.

Fixed points

The fixed points in the closure of $\mathcal{W}$ are:

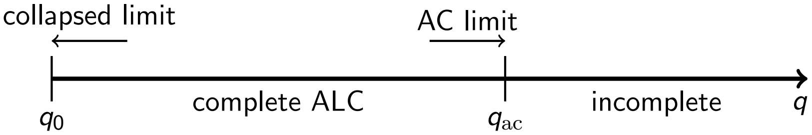
- $(0,1,0)$: S^5, index is $(2,1)$
- $(1,0,0)$: $\mathbb{C}P^2$, index is $(2,1)$
- $(1,1,0)$: ALC end, index is $(0,3)$
- The Spin(7)-cone

$$(X_c, Y_c, Z_c) := \left(\frac{15 - 3\sqrt{5}}{10}, \frac{3 - \sqrt{5}}{2}, \frac{3\sqrt{5} - 5}{5} \right) \approx (0.83, 0.38, 0.34)$$

  index is $(1,2)$.
- $(0,0,0)$: index is $(3,0)$

Proof strategy

- Recall: We have the two families $\Psi_\mu, \mu \in (0, \infty)$ and $\Upsilon_\tau, \tau \in (-\infty, \infty)$ and want to show



- Problem: AC space is elusive.
- Define

$$\mathfrak{X}_{\mathrm{alc}} := \{\mu \,|\, \Psi_\mu \text{ is complete and ALC}\},$$
$$\mathfrak{X}_{\mathrm{ac}} := \{\mu \,|\, \Psi_\mu \text{ is complete and AC}\},$$
$$\mathfrak{X}_{\mathrm{inc}} := \{\mu \,|\, \Psi_\mu \text{ is incomplete}\}.$$

Proof strategy: continuity principle

Our strategy is to prove

- $\mathfrak{X}_{\mathrm{alc}}$ and $\mathfrak{X}_{\mathrm{inc}}$ are both open and non-empty,
- $(0, \infty) = \mathfrak{X}_{\mathrm{alc}} \cup \mathfrak{X}_{\mathrm{ac}} \cup \mathfrak{X}_{\mathrm{inc}}$.

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This then implies that $\mathfrak{X}_{\mathrm{a}}$ is non-empty! Moreover, a comparison argument then shows

$$\mathfrak{X}_{\mathrm{alc}} = (0, \mu_1), \quad \mathfrak{X}_{\mathrm{ac}} = [\mu_1, \mu_2], \quad \mathfrak{X}_{\mathrm{inc}} = (\mu_2, \infty).$$

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- another key ingredient:
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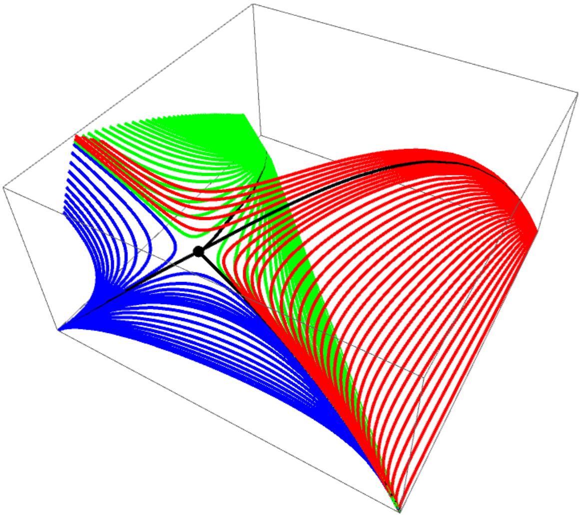
- There are only two fixed points in $\overline{\mathcal{W}}$ the trajectories Ψ_μ can converge: $\Rightarrow (0,\infty) = \mathfrak{X}_{\mathrm{alc}} \cup \mathfrak{X}_{\mathrm{ac}} \cup \mathfrak{X}_{\mathrm{inc}}$



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- For $z \in (0, z_0)$ there exists 1-parameter family of Spin(7) holonomy metrics Ω^z_κ, $\kappa \in (0,1)$, which behave as Ψ_μ and Υ_τ. Collapsed limit is singular version of Bryant–Salamon AC metric. While $a, b \to 0$ at singularity, $c \to \infty$.

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- Recent work by Chi on existence of ALC metrics

Deformations of AC Spin(7)-manifolds

Let (M, ψ) AC Spin(7)-manifold with

$$|\psi - \psi_C| = \mathcal{O}(r^{\nu}).$$

ν is called decay rate.

- A tensor which decays faster than -4 is square integrable on (M, ψ).
- Denote by $\mathcal{M}_{\nu}$ space of torsion-free AC Spin(7)-structures on M which decay at least like $r^{-\nu}$ to a fixed Spin(7)-cone, up to diffeomorphisms decaying to the identity.

Theorem (L-, '21)

In the non-L^2 regime ($-4 < \nu < 0$), for generic rates $\mathcal{M}_\nu$ is an orbifold. Dimension can be computed in terms of topology and solutions to differential equations on link.

- Derivative of

$$\mathrm{GL}(7,\mathbb{R}) \to \Lambda^3(\mathbb{R}^7)^*, A \mapsto A^*\varphi_0$$

gives map

$$\mathfrak{gl}(7,\mathbb{R}) = \Lambda^2_7 \oplus \Lambda^2_{14} \oplus \mathbb{R}\mathrm{Id} \oplus S^2_0(\mathbb{R}^7) \to \Lambda^3.$$

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- Infinitesimal deformations in the range $\nu \in (-4, 0)$ come from solutions of

$$d\zeta = -(\nu + 4) * \zeta, \quad \zeta \in \Gamma(\Lambda^3_{27}).$$

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- The nearly parallel G_2-structures on the Aloff–Wallach spaces $N(k,l)$ are not standard. $\mathrm{SU}(3)$-invariance $\to$ polynomial equation.