Higgs Bundles for G_2-manifolds and Brane/Particle Probes

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Special Holonomy: Progress and Open Problems 2021

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Overview

Introduction and Motivation

- M-theory on a compact G_2 manifolds engineers a 4d theory with minimal supersymmetry. [Joyce, 1996], [Kovalev, 2003], [Corti, Haskins, Nordström, Pacini, 2015], [Joyce, Karigiannis, 2017], [Acharya, 1998], [Halverson, Morrison, 2015], [Braun, Schäfer-Nameki, 2017], [Braun, Del Zotto, 2017], [Xu, 2020]
- The gauge theory sector can be isolated by considering non-compact (local) G_2 manifolds. [Bryant, Salamon, 1989], [Acharya 2000], [Acharya, Witten, 2001], [Witten, 2001], [Atiyah, Witten, 2003], [Pantev, Wijnholt, 2009], [Braun, Cizel, H, Schäfer-Nameki, 2018], [Barbosa, Cvetič, Heckman, Lawrie, Torres, Zoccarato, 2019], [Cvetič, Heckman, Rochais, Torres, Zoccarato 2020], [H, 2020], [Karigiannis, Lotay, 2020]

- F-theory methods relying on Higgs bundles and their spectral covers can be applied to study the physics of local G_2 manifolds. [Beasley, Heckman, Vafa, 2009], [Hayashi, Kawano, Tatar, Watari, 2009], [Marsano, Saulina, Schäfer-Nameki, 2010], [Blumenhagen, Grimm, Jurke, Weigand, 2010], [Donagi, Wijnholt, 2011], [Donagi, Wijnholt, 2014], [Cvetič, Heckman, Rochais, Torres, Zoccarato 2020]
- Supersymmetric sigma models probing the geometries give insight into non-perturbative classical effects. [Alvarez-Gaume, Witten, 1981], [Witten, 1982], [Pantev, Wijnholt, 2009], [Atiyah, Witten, 2003], [Pantev, Wijnholt, 2009], [Braun, Cizel, H, Schäfer-Nameki, 2018], [H, 2020], [Cvetič, Heckman, Torres, Zoccarato, 2021]

ALE-Fibered, Local G_2 Manifolds

Geometric data

Local G_2 Manifold: $\widetilde{\mathbb{C}^2/\Gamma_{ADE}} \hookrightarrow X_7 \to M_3$

Fibral 2-Spheres: $\sigma_I \in H_2(\widetilde{\mathbb{C}^2/\Gamma_{ADE}}, \mathbb{R})$

Hyperkähler Triple: $(\omega_1, \omega_2, \omega_3) \in H^2(\widetilde{\mathbb{C}^2/\Gamma_{ADE}}, \mathbb{R})$

The Higgs field collects the Kähler periods

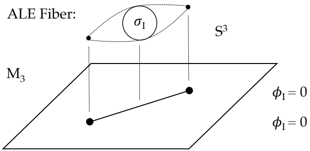
$$\text{Higgs field:} \qquad \phi_I = \left(\int_{\sigma_I} \omega_i \right) dx^i \in \Omega^1(M_3)$$

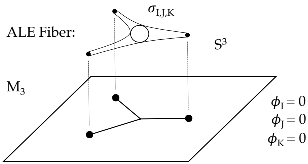
where $I = 1, \ldots, \operatorname{rank} \mathfrak{g}_{ADE}$.

Singularities and Supersymmetric 3-cycles

$$\text{Singularity Enhancement at } x \in M_3 : \quad \phi_I(x) = 0 \quad (\text{codim. } 7)$$
$$\text{Morse-Bott Degenerate Set-up} : \quad \phi_I|_{S^1} = 0 \quad (\text{codim. } 6)$$

The vanishing cycles trace out 3-spheres:





Questions 1

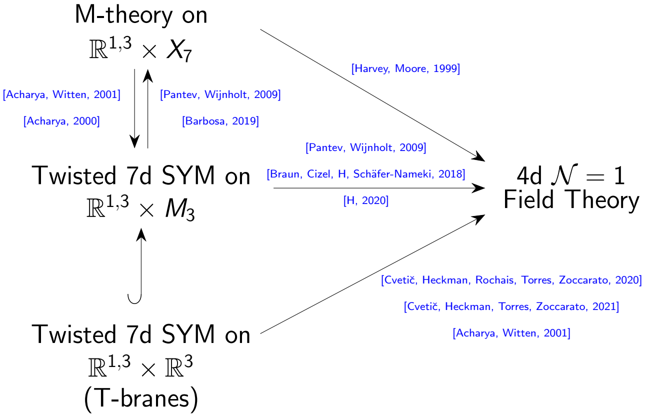
Local to Global.

- What is the physics of a local patch containing a single component of $\phi = 0$? Zero mode analysis in an ultra local patch on M_3. [Acharya, Witten, 2001], [Witten, 2001], [Barbosa, Cvetič, Heckman, Lawrie, Torres, Zoccarato, 2019]
- How does the physics of ultra local patches glue globally across M_3? M2-Instanton analysis. [Harvey, Moore, 1999], [Pantev, Wijnholt, 2009], [Braun, Cizel, H, Schäfer-Nameki, 2018], [H, 2020]
- How does the local analysis apply to compact G_2 manifolds? Analysis of the local model associated to TCS G_2 manifolds. [Braun, Cizel, H, Schäfer-Nameki, 2018]

Questions 2

- What do the supersymmetric 3-spheres descend to in the Higgs bundle? [Acharya, Witten, 2001], [Pantev, Wijnholt, 2009]
- What is the global structure of the network of supersymmetric 3-spheres? [Fukaya, 1999], [Pantev, Wijnholt, 2009], [Braun, Cizel, H, Schäfer-Nameki, 2018], [H, 2020]

Work Done


M-theory on
$\mathbb{R}^{1,3} \times X_7$
[Acharya, Witten, 2001]
[Acharya, 2000]
[Pantev, Wijnholt, 2009]
[Barbosa, 2019]
[Harvey, Moore, 1999]
Twisted 7d SYM on
$\mathbb{R}^{1,3} \times M_3$
[Pantev, Wijnholt, 2009]
[Braun, Cizel, H, Schäfer-Nameki, 2018]
[H, 2020]
4d $\mathcal{N} = 1$
Field Theory
Twisted 7d SYM on
$\mathbb{R}^{1,3} \times \mathbb{R}^3$
(T-branes)
[Cvetič, Heckman, Rochais, Torres, Zoccarato, 2020]
[Cvetič, Heckman, Torres, Zoccarato, 2021]
[Acharya, Witten, 2001]

Effective 7d Physics

M-theory on the local G_2 manifold X_7 with ADE singularities gives

Partially twisted 7d SYM on $\mathbb{R}^{1,3} \times M_3$
with gauge group G_{ADE}

Topological twist

$$SU(2)_{M_3} \times SU(2)_R \quad \rightarrow \quad SU(2)_{\text{twist}} = \text{diag}\left(SU(2)_{M_3}, SU(2)_R\right)$$

Complex bosonic 1-form on M_3: $\varphi = \phi + iA \in \Omega^1(M_3, \mathfrak{g}_{\text{ADE}})$

Supersymmetric backgrounds are solutions of a Hitchin system:

$$i(F_A)_{ij} + [\phi_i, \phi_j] = 0\,, \qquad (d_A\phi)_{ij} = 0\,, \qquad *d_A * \phi = 0$$

For a given background zero modes along M_3 are determined by

$$H = \frac{1}{2}\left\{Q, Q^\dagger\right\}\,, \qquad Q = d + \varphi$$

and counted by the cohomologies

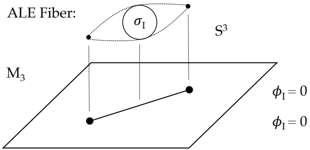
$$H^*_Q(M_3, \mathfrak{g}_{\mathrm{ADE}}).$$

The operator Q is a complex flat connection.

Alternatively, consider approximate zero modes

$$\chi_a \in \Omega^*(M_3, \mathfrak{g}_{\text{ADE}}) \quad \leftrightarrow \quad \text{Codimension 7 Singularity}$$

Non-perturbative masses corrections are generated by M2 brane instantons. The 7d SYM determines these mass corrections to M_{ab} and zero modes are recovered from Ker M_{ab}.



$$M_{ab} = \int_{M_3} \langle \chi_b, Q\chi_a \rangle$$

Morse-Bott/Novikov Theory and colored SQMs

Motivation: M2 brane probing the local G_2 manifold descends to a particle (W-boson) probing M_3 when reducing along ALE fibers.

We find a colored supersymmetric quantum mechanics (SQM) probing the Higgs bundle.

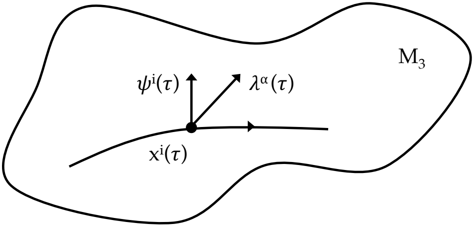
Relevant Data: Physical Hilbertspace of the SQM are Lie algebra valued forms and supercharge Q is

$$\mathcal{H}_{\text{phys.}} = \Lambda(M_3, \mathfrak{g}_{\text{ADE}})\,, \qquad Q = d + \varphi.$$

The colored SQM is an extension of Witten's SQM [Witten, 1982] by an adjoint bundle on the target space.

The dynamical fields, mapping from $\mathbb{R}_\tau$, are

$$\begin{aligned} \text{Bosonic coordinates on } M_3 : &\quad x^i, \quad i = 1,2,3 \\ \text{Fermions in } x^*(TM_3) : &\quad \psi^i, \quad i = 1,2,3 \\ \text{Color Fermions in } x^*(adG_{\text{ADE}}) : &\quad \lambda^\alpha, \quad \alpha = 1,\dots,\dim\mathfrak{g}_{\text{ADE}} \end{aligned}$$

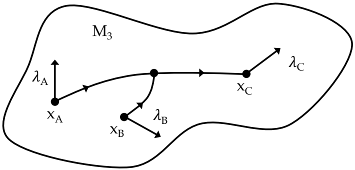


Perturbative ground states of $H = \frac{1}{2}\left\{Q, Q^\dagger\right\}: \quad (x, \lambda)$

Colored instantons are piecewise solutions to the flow equations

$$\dot{x}^i - \phi^i_\lambda = \dot{x}^i - ic^\alpha_{\beta\gamma}\phi^i_\alpha\bar{\lambda}^\beta\lambda^\alpha = 0\,, \qquad D_\tau\lambda^\alpha = 0$$

Colored instantons are in correspondence to flow trees on M_3 and three-cycles in X_7. The latter are conjectured to be associatives.



The colored SQM simplifies depending on the Higgs field background. Consider Higgs fields solving

$$[\phi_i, \phi_j] = 0\,, \qquad (d\phi)_{ij} = (*j)_{ij}\,, \qquad *d*\phi = \rho\,.$$

The 1-form j and 0-form ρ are supported in codimension 2. We also set $d_A = d$ and the adjoint bundle is trivial.

Such backgrounds allow for geometric interpretation and admit a spectral cover description. The eigenvalue 1-forms Λ_I of the Higgs field sweep out

$$\mathcal{C} = \{(x, \Lambda_I(x))\,|\, x \in M_3)\} \subset T^*M_3$$

We distinguish three types of spectral cover.

- Fully reducible and exact: Eigenvalues $\Lambda_I = df_I$ are globally defined and exact on $\mathcal{M}_3$. Spectral cover $\mathcal{C}$ is fully reducible, $Q = d + df$. Morse-Bott theory on $\mathcal{M}_3$. [Pantev, Wijnholt, 2009], [Braun, Cizel, H, Schäfer-Nameki, 2018], [H, 2020]
- Fully reducible and closed: Eigenvalues Λ_I are globally defined on $\mathcal{M}_3$ and closed $d\Lambda = 0$. Spectral cover $\mathcal{C}$ is fully reducible, $Q = d + \phi$. Novikov theory on $\mathcal{M}_3$. [Pantev, Wijnholt, 2009], [H, 2020]
- Irreducible and closed: Eigenvalues Λ_I are locally defined on $\mathcal{M}_3$ and mixed by monodromies. Spectral cover $\mathcal{C}$ not fully reducible, $Q = d + \phi$. Novikov theory on covering space of $\mathcal{M}_3$. [H, 2020]

Here $\mathcal{M}_3 = M_3 \setminus \text{sing}(\phi)$.

Fully Reducible and Exact Backgrounds

Writing $\phi = df_I \mathfrak{t}^I$ the first class are solutions to Poission's equation

$$\Delta f_I = \rho_I \,,$$

Where the f_I (and their integer sums) are generically Morse.

The supercharge $Q = d + df_I \mathfrak{t}^I$ and Hamiltonian are trivial at the Lie algebra level. The restrictions

$$Q^{(\alpha)} : \quad \Omega^*(M_3, \mathfrak{g}_{\text{ADE}})\big|_{E^\alpha} \to \Omega^{*+1}(M_3, \mathfrak{g}_{\text{ADE}})\big|_{E^\alpha}$$

are well defined for all Lie algebra generators E^α. We can associate to each E^α a Morse theory.

Example: $SU(2) \to U(1)$

Start with A_1 singularity along M_3 and gauge group $G = SU(2)$. The resolution of the singularity is informed by the Higgs field background $\phi \in \Omega^1(M_3, \mathfrak{su}(2))$

$$\phi = df\,\mathfrak{t}\,, \qquad \Delta f = \rho\,, \qquad \mathfrak{t} = \text{diag}(1,-1)\,.$$

with Morse function f. The A_1 singuarity locus is resolved everywhere except $df = 0$. The gauge groups breaks

$$SU(2) \to U(1)$$

and the adjoint representation decomposes

$$\text{ad}\,\mathfrak{su}(2) \to \text{ad}\,\mathfrak{u}(1) \oplus \mathbf{1}_+ \oplus \mathbf{1}_-$$

The representations $\mathbf{1}_+ \oplus \mathbf{1}_-$ are spanned by the generators

$$E^\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad E^{-\alpha} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

The supercharge $Q = d + df\,\mathfrak{t}$ restricts to subspaces spanned by $E^\alpha, E^{-\alpha}$ as

$$Q^{(\alpha)} = d + 2df\wedge\,, \qquad Q^{(-\alpha)} = d - 2df\wedge\,,$$

respectively.

Denote the critical points as $\mathrm{Crit}(2f) = \{p_i\,;\, i = 1, \ldots, n\}$ and their Morse indices as $\mu_i = 1, 2$. In coordinates $x(p_i) = 0$ we expand

$$f(x) = \pm c_1 x_1^2 \pm c_2 x_2^2 \pm c_3 x_3^2 + \ldots\,, \qquad c_k > 0\,,$$

and a single approximate zero mode localizes

$$\chi_{\alpha,i} = \exp(-c_1 x_1^2 - c_2 x_2^2 - c_3 x_3^2) dx^{\mu_i} \otimes E^\alpha + \ldots$$

where dx^{μ_i} is a μ_i-form. Concentrating on this sector one finds gradient flow line lines connection $\chi_{\alpha,i}$ and $\chi_{\beta,j}$ and this builds a Morse complex.

Colored SQM and Witten's SQM

Every root gives a copy of Witten's SQM

$$\text{Root } \alpha \quad \to \quad \text{Witten's SQM with Morse function } f_\alpha = \alpha^I f_I$$

The Morse-Witten complex associated to a Higgs field ϕ is the collection of the Morse-Witten complexes of all these SQMs.

Denote the number of critical points of f_α by n_α and the number of roots of $\mathfrak{g}_{\text{ADE}}$ by n_r. The set of all perturbative zero modes are

$$\chi_{\alpha,i} \qquad i = 1, \ldots, n_\alpha\,, \quad \alpha = 1, \ldots, n_r\,.$$

The Morse-Witten complex of the colored SQM is given by

$$0 \to C_{\mu=1} \xrightarrow{Q} C_{\mu=2} \to 0\,.$$

where the chains C_μ collect all degree $\mu = 1, 2$ forms

$$C_\mu = \bigoplus_{i,\alpha} \chi_{\alpha,i,\mu}$$

The complex is graded by color α.

The physical spectrum is characterized by

$$H^1_Q(M_3, \mathfrak{g}_{\text{ADE}}) \cong \text{Ker}\, Q\,, \qquad H^2_Q(M_3, \mathfrak{g}_{\text{ADE}}) \cong \text{CoKer}\, Q$$

The operator Q in the Morse-Witten complex has the matrix representation

$$M_{\alpha\beta,ij} = \int_{M_3} \langle \chi_{\alpha,i}, Q\chi_{\beta,j} \rangle = \delta_{\alpha+\beta,0} \sum_{\Gamma_{ij}} (\pm)_{\Gamma_{ij}} \exp\left\{- \left[f_\alpha(p_i) + f_\beta(p_j) \right] \right\}$$

which obeys the selection rules $\alpha + \beta = 0$.

Example: $SU(n+2) \to SU(n) \times U(1)_a \times U(1)_b$

In physically interesting situations the correspondence between roots and SQMs often degenerates.

The Higgs field $\phi = df_a \mathfrak{t}^a + df_b \mathfrak{t}^b$ breakes the gauge symmetry

$$SU(n+2) \to SU(n) \times U(1)_a \times U(1)_b$$

and the adjoint representation decomposes

$$\begin{aligned} \text{ad}\, SU(n+2) \quad \to \quad & \text{ad}\, SU(n) \oplus \sum_{q=(q_1,q_2)} (\mathbf{n}_{q_1,q_2} \oplus \bar{\mathbf{n}}_{-q_1,-q_2}) \\ & \oplus \text{ad}\, U(1)^2 \oplus \mathbf{1}_{0,1} \oplus \mathbf{1}_{0,-1} \end{aligned}$$

The fundamental representations $\mathbf{n}_{q_1,q_2}$ are spanned by n Lie algebra generators carrying the same $U(1)_a \times U(1)_b$ weight. Their associated copy of Witten's SQM are identical

$$\text{Irreps. } \mathbf{R}_q \quad \leftrightarrow \quad \text{Witten's SQM with } Q = d + q^I df_I \,.$$

With this the number of chiral and conjugate-chiral fields are computed to

$$\begin{aligned} \text{Rank}\, H^1_Q(M_3, \mathbf{R}_q) &= \#\,\text{chiral mode in } \mathbf{R}_q \\ \text{Rank}\, H^2_Q(M_3, \mathbf{R}_q) &= \#\,\text{conjugate-chiral mode in } \bar{\mathbf{R}}_q \end{aligned}$$

If the source $\rho = q^I \rho_I$ has $k_\pm, l_\pm$ postively/negatively charged components, loops respectively one has [Pantev, Wijnholt, 2009]

$$\text{Rank}\, H^1_Q(M_3, \mathbf{R}_q) = l_+ + k_- - r - 1$$
$$\text{Rank}\, H^2_Q(M_3, \mathbf{R}_q) = l_- + k_+ - r - 1$$

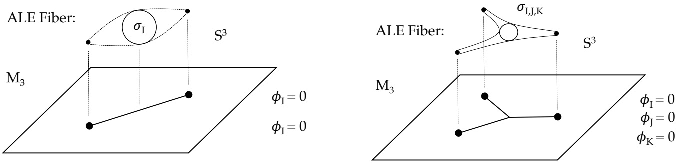
where r counts the number of negative loops which are independent in homology when embedded in $M_3 \setminus \text{supp}\, \rho_+$.

The chiral index for matter in $\mathbf{R}_q$ is

$$\chi(M_3, \mathbf{R}_q) = l_+ - l_- + k_- - k_+$$

and whenever $\chi(M_3, \mathbf{R}_q) \neq 0$ the spectrum is chiral.

This completes the analysis of supersymmetric 3-spheres connecting two codimension 7 singularities. What about 3-spheres connecting three (or more) codimension 7 singularities?



Yukawa Couplings and Flow Trees

The 7d SYM theory gives the Yukawa couplings between three perturbatively massless chiral multiplets to [Braun, Cizel, H, Schäfer-Nameki, 2018]

$$Y_{ijk,\alpha\beta\gamma} = \int_{M_3} \langle \chi_{\alpha,i}, \left[\chi_{\beta,j}, \chi_{\gamma,k}\right]\rangle$$

which obey the selection rule

$$\alpha + \beta + \gamma = 0$$

This is equivalent to topological consistency in the ALE fibration.

Via methods of supersymmetric localization in the colored SQM this overlap integral computes to

$$Y_{ijk,\alpha\beta\gamma} = \delta_{\alpha+\beta+\gamma,0} \sum_{\Gamma_{ijk}} (\pm)_{\Gamma_{ijk}} \exp\{-[f_\alpha(p_i) + f_\beta(p_j) + f_\gamma(p_k)]\}$$

We find a cup-product on the Morse-Witten complex of the colored SQM

$$\cup : \ C_{\mu=1} \times C_{\mu=1} \to C_{\mu=2}$$

mapping as

$$(\chi_{\beta,j}, \chi_{\gamma,k}) \ \mapsto \ \sum_{i,\alpha} Y_{ijk,\alpha\beta\gamma} \chi_{\alpha,i}$$

This cup product descends to cohomology $H^*_Q(M_3, \mathfrak{g}_{\text{ADE}})$.

Comments:

- Flow trees corresponding to three-spheres connecting n codimension 7 singularities exist. They correspond to irrelevant couplings in 4d and are not captured by the 7d SYM.
- The spectrum can alternatively be counted by analyzing and counting intersections between components of the spectral cover.
- Yet another way of computing the spectrum is given by excising the source loci and map the problem to de Rham cohomology on a manifold with boundary.
- The presented analysis persists when considering Morse-Bott degenerate cases with matter along circles $\phi|_{S^1} = 0$ with codimension 6 singularities.

Irreducible and Closed Backgrounds

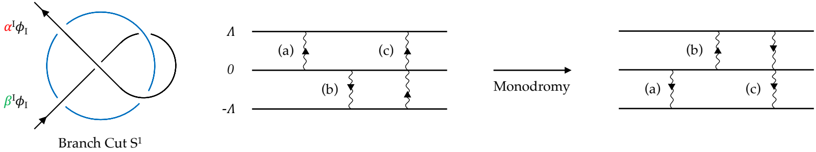
Consider Higgs fields with an irreducible spectral cover $\mathcal{C}$. This introduces a branch locus $\mathcal{B}$ along circles (codim. 2) embedded as knots K_i into M_3

$$\mathcal{B} = \cup_i K_i \subset M_3 \,.$$

Monodromy along paths linking $\mathcal{B}$

$$\text{Monodromy Action}: \quad \phi \rightarrow g\phi g^{-1}$$

$$\text{Color Mixing}: \quad E^\alpha \rightarrow gE^\alpha g^{-1}$$



The monodromy action gives orbits of Lie algebra generators E^α

$$[E^\alpha] = \{E^\alpha, gE^\alpha g^{-1}, g^2 E^\alpha g^{-2}, \dots\}$$

to which one associates an orbit of roots $[\alpha]$.

The ultra local analysis of approximate zero modes is unaltered. We again obtain a Morse-Witten complex

$$0 \to C_{\mu=1} \xrightarrow{Q} C_{\mu=2} \to 0\,.$$

which is now graded by color orbits $[\alpha]$.

The color orbits describe which resolution 2-spheres $\alpha^I \sigma_I \in H_2(\mathbb{C}/\Gamma_{\text{ADE}})$ are identified under monodromy.

$$\alpha \sim \beta \;\rightarrow\; \alpha^I \sigma_I = \beta^I \sigma_I \,.$$

The cycle $\alpha^I \sigma_I$ is homologous to $\beta^I \sigma_I$ by moving it some number of times around the branch locus $\mathcal{B}$.

Monodromies break the gauge symmetry

$$\text{Commutant of } \phi \quad \rightarrow \quad \text{Stabilizer of } \phi$$

From the monodromies construct a covering space [Cecotti, Córdova, Vafa, 2011]. Pick a Seifert surface F for the Branch locus $\mathcal{B} = \partial F$. Now glue

$$\mathcal{C} = (M_3 \setminus F)\# \ldots \#(M_3 \setminus F)$$

where the number of gluing components equals the order of the monodromy action. This space is topologically equivalent to the spectral cover.

The Higgs field $\alpha^I \phi_I$ glues across branch surfaces F to closed 1-forms

$$\phi_{[\alpha]} \in \Omega^1(\mathcal{C})$$

on the spectral cover.

Degenerate case: the irreducible representations $\mathbf{R}_q$ are grouped by the orbits $[q]$.

These combine to the representation $\mathbf{R}_{[q]}$ under the monodromy reduced gauge symmetry. Associate Higgs field $\phi_{[q]} \in \Omega^1(\mathcal{C})$.

The matter spectrum in the representation $\mathbf{R}_{[q]}$ labelled by $[q]$ is computed by

$$\begin{aligned} \text{Rank}\, H^1_{\text{Nov.}}(\mathcal{C}, \phi_{[q]}) &= \#\,\text{chiral mode in } \mathbf{R}_{[q]} \\ \text{Rank}\, H^2_{\text{Nov.}}(\mathcal{C}, \phi_{[q]}) &= \#\,\text{conjugate-chiral mode in } \mathbf{R}_{[q]} \end{aligned}$$

These numbers are computable in highly symmetric situations.

Summary and Conclusion

- We started from an ALE fibered G2 manifold and mapped it to a Higgs bundle.
- M2 branes probing the G2 manifold reduce to particles (W-bosons) probing the Higgs bundle.
- Particle probes associate a quantum mechanical model to the Higgs bundle. This model we dubbed colored SQM.
- We derived Morse-theoretic structures from the colored SQM which describe classical, non-perturbative effects. Quantum effects are not included.
- We characterized the gauge symmetry, spectrum and interactions of the final 4d $\mathcal{N} = 1$ gauge theory.

Outlook: Open Problems

Construction of Higgs field backgrounds solving

$$[\phi_i, \phi_j] = 0 \,, \qquad (d\phi)_{ij} = (*j)_{ij} \,, \qquad *d * \phi = \rho \,.$$

These have singularities modeled on $1/\sqrt{z}$ similar to [Donaldson, 2021] with singularities modled on $\sqrt{z}$.

Lift Higgs bundles to geometry. What are the constraints of

$$\text{Higgs field } \phi \quad \mapsto \quad \text{ALE-fibered G2-manifold } X_7\,.$$

See [Pantev, Wijnholt, 2009], [Barbosa, 2019].

Computation of Q-cohomologies. Find the map

$$\text{Source Data of } \rho, j \quad \mapsto \quad \text{Rank}\, H^*_Q(M_3)\,.$$

For reducible and exact Higgs field only topological data enters.

Study non-commuting Higgs field configurations

$$[\phi_i, \phi_j] \neq 0 \, .$$

See [Bielawski, Foscolo, 2020], [Cvetič, Heckman, Rochais, Torres, Zoccarato 2020].

End

Extra Slide: cSQM

Lagrangian

$$\mathcal{L} = \frac{1}{2}\dot{x}^i\dot{x}_i + i\bar{\psi}^i\nabla_\tau\psi_i + i\bar{\lambda}^\alpha D_\tau\lambda_\alpha + \frac{i}{2}\left(F_{ij}\right)_\lambda\bar{\psi}^i\psi^j - \frac{1}{2}R_{ijkl}\psi^i\bar{\psi}^j\psi^k\bar{\psi}^l$$
$$- \left(D_{(i}\phi_{j)}\right)_\lambda\bar{\psi}^i\psi^j - \frac{1}{2}\phi^i_\lambda\phi_{\lambda,i} - \frac{1}{2}[\phi_i,\phi_j]_\lambda\bar{\psi}^i\psi^j + \zeta\left(\bar{\lambda}^\alpha\lambda_\alpha - n\right) .$$

Variations

$$\begin{aligned}
\delta x^i &= \epsilon\bar{\psi}^i - \bar{\epsilon}\psi^i \,, \\
\delta\psi^i &= i\epsilon\dot{x}^i + \epsilon\phi^i_\lambda - \epsilon\Gamma^i_{jk}\bar{\psi}^j\psi^k \,, \\
\delta\lambda^\alpha &= -i\epsilon c^\alpha{}_{\beta\gamma}\bar{\psi}^i\varphi^\beta_i\lambda^\gamma - i\bar{\epsilon}c^\alpha{}_{\beta\gamma}\psi^i\bar{\varphi}^\beta_i\lambda^\gamma \,.
\end{aligned}$$