Higgs Bundles for $G_2$-manifolds and Brane/Particle Probes

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Special Holonomy: Progress and Open Problems 2021

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Overview

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Introduction and Motivation


F-theory methods relying on Higgs bundles and their spectral covers can be applied to study the physics of local $G_2$ manifolds. \cite{Beasley, Heckman, Vafa, 2009}, \cite{Hayashi, Kawano, Tatar, Watari, 2009}, \cite{Marsano, Saulina, Schäfer-Nameki, 2010}, \cite{Blumenhagen, Grimm, Jurke, Weigand, 2010}, \cite{Donagi, Wijnholt, 2011}, \cite{Donagi, Wijnholt, 2014}, \cite{Cvetič, Heckman, Rochais, Torres, Zoccarato 2020}

Supersymmetric sigma models probing the geometries give insight into non-perturbative classical effects. \cite{Alvarez-Gaume, Witten, 1981}, \cite{Witten, 1982}, \cite{Pantev, Wijnholt, 2009}, \cite{Atiyah, Witten, 2003}, \cite{Pantev, Wijnholt, 2009}, \cite{Braun, Cizel, H, Schäfer-Nameki, 2018}, \cite{H, 2020}, \cite{Cvetič, Heckman, Torres, Zoccarato, 2021}
ALE-Fibered, Local $G_2$ Manifolds

Geometric data

Local $G_2$ Manifold: \( \mathbb{C}^2 / \Gamma_{ADE} \rightarrow X_7 \rightarrow M_3 \)

Fibral 2-Spheres: \( \sigma_I \in H_2(\mathbb{C}^2 / \Gamma_{ADE}, \mathbb{R}) \)

Hyperkähler Triple: \( (\omega_1, \omega_2, \omega_3) \in H^2(\mathbb{C}^2 / \Gamma_{ADE}, \mathbb{R}) \)

The Higgs field collects the Kähler periods

Higgs field: \( \phi_I = \left( \int_{\sigma_I} \omega_i \right) dx^i \in \Omega^1(M_3) \)

where \( I = 1, \ldots, \text{rank } g_{ADE} \).
Singularities and Supersymmetric 3-cycles

Singularity Enhancement at $x \in \mathcal{M}_3$:
\[ \phi_I(x) = 0 \quad \text{(codim. 7)} \]

Morse-Bott Degenerate Set-up:
\[ \phi_I|_{S^1} = 0 \quad \text{(codim. 6)} \]

The vanishing cycles trace out 3-spheres:

ALE Fiber:
\[ \sigma_1 \]
\[ M_3 \]
\[ \phi_1 = 0 \]
\[ \phi_1 = 0 \]

\[ S^3 \]
Questions 1

Local to Global.

- What is the physics of a local patch containing a single component of $\phi = 0$? Zero mode analysis in an ultra local patch on $M_3$. [Acharya, Witten, 2001], [Witten, 2001], [Barbosa, Cvetič, Heckman, Lawrie, Torres, Zoccarato, 2019]

- How does the physics of ultra local patches glue globally across $M_3$? M2-Instanton analysis. [Harvey, Moore, 1999], [Pantev, Wijnholt, 2009], [Braun, Cizel, H, Schäfer-Nameki, 2018], [H, 2020]

- How does the local analysis apply to compact $G_2$ manifolds? Analysis of the local model associated to TCS $G_2$ manifolds. [Braun, Cizel, H, Schäfer-Nameki, 2018]
Questions 2

- What do the supersymmetric 3-spheres descend to in the Higgs bundle? [Acharya, Witten, 2001], [Pantev, Wijnholt, 2009]
- What is the global structure of the network of supersymmetric 3-spheres? [Fukaya, 1999], [Pantev, Wijnholt, 2009], [Braun, Cizel, H, Schäfer-Nameki, 2018], [H, 2020]
Work Done

**M-theory on** $\mathbb{R}^{1,3} \times X_7$

- [Acharya, Witten, 2001]
- [Acharya, 2000]
- [Harvey, Moore, 1999]

**Twisted 7d SYM on** $\mathbb{R}^{1,3} \times M_3$

- [Pantev, Wijnholt, 2009]
- [Barbosa, 2019]

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- [Pantev, Wijnholt, 2009]
- [Braun, Cizel, H, Schäfer-Nameki, 2018]
- [H, 2020]
- [Cvetič, Heckman, Rochais, Torres, Zoccarato, 2020]
- [Cvetič, Heckman, Torres, Zoccarato, 2021]
- [Acharya, Witten, 2001]

**4d $\mathcal{N} = 1$ Field Theory**
Effective 7d Physics

M-theory on the local $G_2$ manifold $X_7$ with ADE singularities gives

Partially twisted 7d SYM on $\mathbb{R}^{1,3} \times M_3$

with gauge group $G_{\text{ADE}}$

Topological twist

$SU(2)_{M_3} \times SU(2)_R \rightarrow SU(2)_{\text{twist}} = \text{diag} (SU(2)_{M_3}, SU(2)_R)$

Complex bosonic 1-form on $M_3$: $\varphi = \phi + iA \in \Omega^1(M_3, g_{\text{ADE}})$
Supersymmetric backgrounds are solutions of a Hitchin system:

\[ i(F_A)_{ij} + [\phi_i, \phi_j] = 0, \quad (d_A \phi)_{ij} = 0, \quad *d_A * \phi = 0 \]

For a given background zero modes along \( M_3 \) are determined by

\[ H = \frac{1}{2} \left\{ Q, Q^\dagger \right\}, \quad Q = d + \varphi \]

and counted by the cohomologies

\[ H^*_Q(M_3, g_{\text{ADE}}). \]

The operator \( Q \) is a complex flat connection.
Alternatively, consider approximate zero modes

\[ \chi_a \in \Omega^*(M_3, g_{ADE}) \leftrightarrow \text{Codimension 7 Singularity} \]

Non-perturbative masses corrections are generated by M2 brane instantons. The 7d SYM determines these mass corrections to \( M_{ab} \) and zero modes are recovered from \( \text{Ker} M_{ab} \).

\[ M_{ab} = \int_{M_3} \langle \chi_b, Q \chi_a \rangle \]
Morse-Bott/Novikov Theory and colored SQMs

Motivation: M2 brane probing the local $G_2$ manifold descends to a particle ($W$-boson) probing $M_3$ when reducing along ALE fibers.

We find a colored supersymmetric quantum mechanics (SQM) probing the Higgs bundle.

Relevant Data: Physical Hilbertspace of the SQM are Lie algebra valued forms and supercharge $Q$ is

$$\mathcal{H}_{\text{phys.}} = \Lambda(M_3, g_{\text{ADE}}), \quad Q = d + \varphi.$$  

The colored SQM is an extension of Witten’s SQM [Witten, 1982] by an adjoint bundle on the target space.
The dynamical fields, mapping from $\mathbb{R}_\tau$, are

- **Bosonic coordinates on $M_3$**: $x^i$, $i = 1, 2, 3$
- **Fermions in $x^*(TM_3)$**: $\psi^i$, $i = 1, 2, 3$
- **Color Fermions in $x^*(adG_{ADE})$**: $\lambda^\alpha$, $\alpha = 1, \ldots, \text{dim}g_{ADE}$
Perturbative ground states of $H = \frac{1}{2} \{Q, Q^\dagger\} : (x, \lambda)$

Colored instantons are piecewise solutions to the flow equations

$$\dot{x}^i - \phi^i_\lambda = \dot{x}^i - i c^\alpha_{\beta\gamma} \phi^i_\alpha \bar{\lambda}^\beta \lambda^\alpha = 0, \quad D_\tau \lambda^\alpha = 0$$

Colored instantons are in correspondence to flow trees on $M_3$ and three-cycles in $X_7$. The latter are conjectured to be associatives.
The colored SQM simplifies depending on the Higgs field background. Consider Higgs fields solving

\[ [\phi_i, \phi_j] = 0, \quad (d\phi)_{ij} = (\ast j)_{ij}, \quad \ast d \ast \phi = \rho. \]

The 1-form \( j \) and 0-form \( \rho \) are supported in codimension 2. We also set \( d_A = d \) and the adjoint bundle is trivial.

Such backgrounds allow for geometric interpretation and admit a spectral cover description. The eigenvalue 1-forms \( \Lambda_l \) of the Higgs field sweep out

\[ \mathcal{C} = \{ (x, \Lambda_l(x)) \mid x \in M_3 \} \subset T^* M_3 \]
We distinguish three types of spectral cover.

- **Fully reducible and exact**: Eigenvalues $\Lambda_I = df_I$ are globally defined and exact on $M_3$. Spectral cover $C$ is fully reducible, $Q = d + df$. Morse-Bott theory on $M_3$. [Pantev, Wijnholt, 2009], [Braun, Cizel, H, Schäfer-Nameki, 2018], [H, 2020]

- **Fully reducible and closed**: Eigenvalues $\Lambda_I$ are globally defined on $M_3$ and closed $d\Lambda = 0$. Spectral cover $C$ is fully reducible, $Q = d + \phi$. Novikov theory on $M_3$. [Pantev, Wijnholt, 2009], [H, 2020]

- **Irreducible and closed**: Eigenvalues $\Lambda_I$ are locally defined on $M_3$ and mixed by monodromies. Spectral cover $C$ not fully reducible, $Q = d + \phi$. Novikov theory on covering space of $M_3$. [H, 2020]

Here $M_3 = M_3 \setminus \text{sing}(\phi)$. 
Fully Reducible and Exact Backgrounds

Writing $\phi = df_I t^I$ the first class are solutions to Poission’s equation

$$\Delta f_I = \rho_I,$$

Where the $f_I$ (and their integer sums) are generically Morse.

The supercharge $Q = d + df_I t^I$ and Hamiltonian are trivial at the Lie algebra level. The restrictions

$$Q^{(\alpha)} : \Omega^*(M_3, g_{ADE})|_{E^\alpha} \rightarrow \Omega^{*+1}(M_3, g_{ADE})|_{E^\alpha}$$

are well defined for all Lie algebra generators $E^\alpha$. We can associate to each $E^\alpha$ a Morse theory.
Example: \( SU(2) \rightarrow U(1) \)

Start with \( A_1 \) singularity along \( M_3 \) and gauge group \( G = SU(2) \). The resolution of the singularity is informed by the Higgs field background \( \phi \in \Omega^1(M_3, \mathfrak{su}(2)) \)

\[
\phi = df t, \quad \Delta f = \rho, \quad t = \text{diag}(1, -1)
\]

with Morse function \( f \). The \( A_1 \) singularity locus is resolved everywhere except \( df = 0 \). The gauge groups break

\[
SU(2) \rightarrow U(1)
\]

and the adjoint representation decomposes

\[
\text{ad } \mathfrak{su}(2) \rightarrow \text{ad } \mathfrak{u}(1) \oplus 1_+ \oplus 1_-
\]
The representations $\mathbf{1}_+ \oplus \mathbf{1}_-$ are spanned by the generators

$$
E^\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E^{-\alpha} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}
$$

The supercharge $Q = d + df \, t$ restricts to subspaces spanned by $E^\alpha, E^{-\alpha}$ as

$$
Q^{(\alpha)} = d + 2df \wedge, \quad Q^{(-\alpha)} = d - 2df \wedge,
$$

respectively.
Denote the critical points as \( \text{Crit}(2f) = \{ p_i \mid i = 1, \ldots, n \} \) and their Morse indices as \( \mu_i = 1, 2 \). In coordinates \( x(p_i) = 0 \) we expand

\[
 f(x) = \pm c_1 x_1^2 \pm c_2 x_2^2 \pm c_3 x_3^2 + \ldots, \quad c_k > 0,
\]

and a single approximate zero mode localizes

\[
 \chi_{\alpha,i} = \exp(-c_1 x_1^2 - c_2 x_2^2 - c_3 x_3^2) dx^{\mu_i} \otimes E^\alpha + \ldots
\]

where \( dx^{\mu_i} \) is a \( \mu_i \)-form. Concentrating on this sector one finds gradient flow line lines connection \( \chi_{\alpha,i} \) and \( \chi_{\beta,j} \) and this builds a Morse complex.
Colored SQM and Witten’s SQM

Every root gives a copy of Witten’s SQM

Root \( \alpha \quad \rightarrow \quad \) Witten’s SQM with Morse function \( f_\alpha = \alpha^l f_i \)

The Morse-Witten complex associated to a Higgs field \( \phi \) is the collection of the Morse-Witten complexes of all these SQMs.

Denote the number of critical points of \( f_\alpha \) by \( n_\alpha \) and the number of roots of \( g_{\text{ADE}} \) by \( n_r \). The set of all perturbative zero modes are

\[ \chi_{\alpha,i} \quad i = 1, \ldots, n_\alpha, \quad \alpha = 1, \ldots, n_r. \]
The Morse-Witten complex of the colored SQM is given by

$$0 \rightarrow C_{\mu=1} \xrightarrow{Q} C_{\mu=2} \rightarrow 0.$$ 

where the chains $C_{\mu}$ collect all degree $\mu = 1, 2$ forms

$$C_{\mu} = \bigoplus_{i,\alpha} \chi_{\alpha,i,\mu}$$

The complex is graded by color $\alpha$. 
The physical spectrum is characterized by

$$H^1_Q(M_3, g_{ADE}) \cong \text{Ker } Q, \quad H^2_Q(M_3, g_{ADE}) \cong \text{CoKer } Q$$

The operator $Q$ in the Morse-Witten complex has the matrix representation

$$M_{\alpha\beta,ij} = \int_{M_3} \langle \chi_{\alpha,i}, Q\chi_{\beta,j} \rangle = \delta_{\alpha+\beta,0} \sum_{\Gamma_{ij}} (\pm) \Gamma_{ij} \exp \{- [f_\alpha(p_i) + f_\beta(p_j)]\}$$

which obeys the selection rules $\alpha + \beta = 0$. 

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Higgs Bundles for $G_2$-manifolds and Brane/Particle Probes
Example: $SU(n + 2) \rightarrow SU(n) \times U(1)_a \times U(1)_b$

In physically interesting situations the correspondence between roots and SQMs often degenerates.

The Higgs field $\phi = df_a t^a + df_b t^b$ breaks the gauge symmetry

$$SU(n + 2) \rightarrow SU(n) \times U(1)_a \times U(1)_b$$

and the adjoint representation decomposes

$$\text{ad } SU(n+2) \rightarrow \text{ad } SU(n) \oplus \sum_{q=(q_1,q_2)} (n_{q_1,q_2} \oplus \bar{n}_{-q_1,-q_2}) \oplus \text{ad } U(1)^2 \oplus 1_{0,1} \oplus 1_{0,-1}$$
The fundamental representations \( \mathfrak{n}_{q_1, q_2} \) are spanned by \( n \) Lie algebra generators carrying the same \( U(1)_a \times U(1)_b \) weight. Their associated copy of Witten’s SQM are identical

\[
\text{Irreps. } R_q \leftrightarrow \text{Witten’s SQM with } Q = d + q^l df_l.
\]

With this the number of chiral and conjugate-chiral fields are computed to

\[
\begin{align*}
\text{Rank } H^1_Q(M_3, R_q) &= \# \text{ chiral mode in } R_q \\
\text{Rank } H^2_Q(M_3, R_q) &= \# \text{ conjugate-chiral mode in } \bar{R}_q
\end{align*}
\]
If the source $\rho = q^I \rho_I$ has $k_\pm, l_\pm$ positively/negatively charged components, loops respectively one has [Pantev, Wijnholt, 2009]

$$\text{Rank } H^1_Q(M_3, R_q) = l_+ + k_- - r - 1$$
$$\text{Rank } H^2_Q(M_3, R_q) = l_- + k_+ - r - 1$$

where $r$ counts the number of negative loops which are independent in homology when embedded in $M_3 \setminus \text{supp } \rho_+$. The chiral index for matter in $R_q$ is

$$\chi(M_3, R_q) = l_+ - l_- + k_- - k_+$$

and whenever $\chi(M_3, R_q) \neq 0$ the spectrum is chiral.
This completes the analysis of supersymmetric 3-spheres connecting two codimension 7 singularities. What about 3-spheres connecting three (or more) codimension 7 singularities?
Yukawa Couplings and Flow Trees

The 7d SYM theory gives the Yukawa couplings between three perturbatively massless chiral multiplets to [Braun, Cizel, H, Schäfer-Nameki, 2018]

$$Y_{ijk,\alpha\beta\gamma} = \int_{M_3} \langle \chi_{\alpha,i}, [\chi_{\beta,j}, \chi_{\gamma,k}] \rangle$$

which obey the selection rule

$$\alpha + \beta + \gamma = 0$$

This is equivalent to topological consistency in the ALE fibration.
Via methods of supersymmetric localization in the colored SQM this overlap integral computes to

\[ Y_{ijk,\alpha\beta\gamma} = \delta_{\alpha+\beta+\gamma,0} \sum_{\Gamma_{ijk}} (\pm)\Gamma_{ijk} \exp \left\{ - \left[ f_\alpha(p_i) + f_\beta(p_j) + f_\gamma(p_k) \right] \right\} \]

We find a cup-product on the Morse-Witten complex of the colored SQM

\[ \bigcup : C_{\mu=1} \times C_{\mu=1} \rightarrow C_{\mu=2} \]

mapping as

\[ (\chi_{\beta,j}, \chi_{\gamma,k}) \mapsto \sum_{i,\alpha} Y_{ijk,\alpha\beta\gamma} \chi_{\alpha,i} \]

This cup product descends to cohomology \( H^*_Q(M_3, g_{ADE}) \).
Comments:

- Flow trees corresponding to three-spheres connecting $n$ codimension 7 singularities exist. They correspond to irrelevant couplings in 4d and are not captured by the 7d SYM.

- The spectrum can alternatively be counted by analyzing and counting intersections between components of the spectral cover.

- Yet another way of computing the spectrum is given by excising the source loci and map the problem to de Rham cohomology on a manifold with boundary.

- The presented analysis persists when considering Morse-Bott degenerate cases with matter along circles $\phi|_{S^1} = 0$ with codimension 6 singularities.
Consider Higgs fields with an irreducible spectral cover $\mathcal{C}$. This introduces a branch locus $\mathcal{B}$ along circles (codim. 2) embedded as knots $K_i$ into $M_3$

$$\mathcal{B} = \bigcup_i K_i \subset M_3.$$  

Monodromy along paths linking $\mathcal{B}$

**Monodromy Action**: $\phi \rightarrow g\phi g^{-1}$

**Color Mixing**: $E^\alpha \rightarrow gE^\alpha g^{-1}$

![Diagram](attachment:diagram.png)
The monodromy action gives orbits of Lie algebra generators $E^\alpha$

$$[E^\alpha] = \{ E^\alpha, gE^\alpha g^{-1}, g^2 E^\alpha g^{-2}, \ldots \}$$

to which one associates an orbit of roots $[\alpha]$.

The ultra local analysis of approximate zero modes is unaltered. We again obtain a Morse-Witten complex

$$0 \rightarrow C_{\mu=1} \xrightarrow{Q} C_{\mu=2} \rightarrow 0.$$

which is now graded by color orbits $[\alpha]$. 
The color orbits describe which resolution 2-spheres 
\( \alpha^I \sigma_I \in H_2(\mathbb{C}/\Gamma_{\text{ADE}}) \) are identified under monodromy.

\[ \alpha \sim \beta \quad \rightarrow \quad \alpha^I \sigma_I = \beta^I \sigma_I. \]

The cycle \( \alpha^I \sigma_I \) is homologous to \( \beta^I \sigma_I \) by moving it some number of times around the branch locus \( \mathcal{B} \).

Monodromies break the gauge symmetry

\[
\text{Commutant of } \phi \quad \rightarrow \quad \text{Stabilizer of } \phi
\]
From the monodromies construct a covering space [Cecotti, Córdova, Vafa, 2011]. Pick a Seifert surface $F$ for the Branch locus $\mathcal{B} = \partial F$. Now glue

$$C = (M_3 \setminus F) \# \ldots \# (M_3 \setminus F)$$

where the number of gluing components equals the order of the monodromy action. This space is topologically equivalent to the spectral cover.

The Higgs field $\alpha^I \phi_I$ glues across branch surfaces $F$ to closed 1-forms

$$\phi[\alpha] \in \Omega^1(C)$$

on the spectral cover.
Degenerate case: the irreducible representations $R_q$ are grouped by the orbits $[q]$.

These combine to the representation $R_{[q]}$ under the monodromy reduced gauge symmetry. Associate Higgs field $\phi_{[q]} \in \Omega^1(C)$.

The matter spectrum in the representation $R_{[q]}$ labelled by $[q]$ is computed by

$$\text{Rank } H^1_{\text{Nov.}}(C, \phi_{[q]}) = \# \text{ chiral mode in } R_{[q]}$$

$$\text{Rank } H^2_{\text{Nov.}}(C, \phi_{[q]}) = \# \text{ conjugate-chiral mode in } R_{[q]}$$

These numbers are computable in highly symmetric situations.
Summary and Conclusion

- We started from an ALE fibered G2 manifold and mapped it to a Higgs bundle.
- M2 branes probing the G2 manifold reduce to particles (W-bosons) probing the Higgs bundle.
- Particle probes associate a quantum mechanical model to the Higgs bundle. This model we dubbed colored SQM.
- We derived Morse-theoretic structures from the colored SQM which describe classical, non-perturbative effects. Quantum effects are not included.
- We characterized the gauge symmetry, spectrum and interactions of the final 4d $\mathcal{N} = 1$ gauge theory.
Outlook: Open Problems

Construction of Higgs field backgrounds solving

$$[\phi_i, \phi_j] = 0, \quad (d\phi)_{ij} = (*j)_{ij}, \quad *d * \phi = \rho.$$ 

These have singularities modeled on $1/\sqrt{z}$ similar to [Donaldson, 2021] with singularities modeled on $\sqrt{z}$. 
Lift Higgs bundles to geometry. What are the constraints of
Higgs field $\phi \mapsto$ ALE-fibered $G_2$-manifold $X_7$.

See [Pantev, Wijnholt, 2009], [Barbosa, 2019].
Computation of $Q$-cohomologies. Find the map

$$\text{Source Data of } \rho, j \mapsto \text{Rank } H^{*}_Q(M_3).$$

For reducible and exact Higgs field only topological data enters.
Study non-commuting Higgs field configurations

\[ [\phi_i, \phi_j] \neq 0. \]

See [Bielawski, Foscolo, 2020], [Cvetič, Heckman, Rochais, Torres, Zoccarato 2020].
End
Extra Slide: cSQM

Lagrangian

\[
\mathcal{L} = \frac{1}{2} \dot{x}^i \dot{x}_i + i \bar{\psi}^i \nabla_\tau \psi_i + i \bar{\psi}^\alpha D_\tau \lambda_\alpha + \frac{i}{2} (F_{ij})_\lambda \bar{\psi}^i \psi^j - \frac{1}{2} R_{ijkl} \psi^i \bar{\psi}^j \psi^k \bar{\psi}^l \\
- (D_{(i} \phi_{j)})_\lambda \bar{\psi}^i \psi^j - \frac{1}{2} \phi^i_\lambda \phi_{\lambda,i} - \frac{1}{2} [\phi_i, \phi_j]_\lambda \bar{\psi}^i \psi^j + \zeta (\bar{\lambda}_\alpha \lambda_\alpha - n).
\]

Variations

\[
\delta x^i = \epsilon \bar{\psi}^i - \bar{\epsilon} \psi^i, \\
\delta \psi^i = i \epsilon \dot{x}^i + \epsilon \phi^i_\lambda - \epsilon \Gamma^i_{jk} \bar{\psi}^j \psi^k, \\
\delta \lambda^\alpha = -i \epsilon c^\alpha_{\beta \gamma} \bar{\psi}^i \varphi^\beta_i \lambda^\gamma - i \bar{\epsilon} c^\alpha_{\beta \gamma} \psi^i \bar{\varphi}^\beta_i \lambda^\gamma.
\]