New Spin(7)-instantons on compact manifolds

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Simons Collaboration on Special Holonomy in Geometry, Analysis, and Physics, Duke University

Abstract: Spin(7)-instantons are interesting principal bundle connections on 8-dimensional manifolds with a Spin(7)-structure. Not many examples of such instantons are known. In the talk I will explain a new construction method for Spin(7)-instantons generating more than 20,000 examples. The construction takes place on Joyce's first examples of compact Spin(7)-manifolds. In the talk, I will briefly review the manifold construction, which glues together an orbifold, an ALE space (Eguchi-Hanson space), and a product of two ALE spaces, which is a QALE space. I will then explain the instanton construction. It makes use of weighted Hölder norms that are known from other gluing constructions, but the presence of a QALE piece makes the analysis more interesting in our case. Time permitting, I will explain how we obtained a large number of examples. This is joint work with Mateo Galdeano, Yuuji Tanaka, and Luya Wang, started at a summer school in 2019. (arXiv:2310.03451)

Motivation

Theorem ([Joyce, 1996])

There exist compact (M^8, g) with $\mathrm{Hol}(g) = \mathrm{Spin}(7)$.

- 65536 examples with 181 different sets of Betti numbers
- Question: how many of them are homotopic as torsion-free Spin(7)-manifolds?
- Idea [Donaldson and Thomas, 1998]: construct invariants using gauge theory to distinguish them

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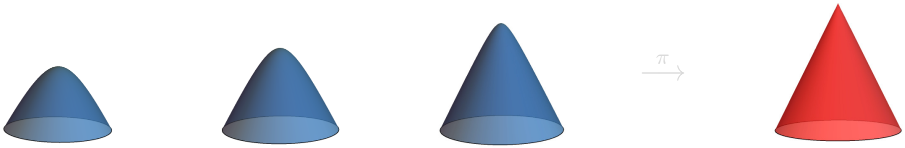
Eguchi-Hanson space and Spin(7)

- $\omega_1, \omega_2, \omega_3 \in \Omega^2(\mathbb{C}^2/\{\pm 1\})$ Hyperkähler triple
- Blowup $\Rightarrow$ complex manifold X_{EH} with $\pi : X_{EH} \to \mathbb{C}^2/\{\pm 1\}$
  Eguchi-Hanson $\Rightarrow$ ex. $\widetilde{\omega}_1^{(t)}, \widetilde{\omega}_2^{(t)}, \widetilde{\omega}_3^{(t)} \in \Omega^2(X_{EH})$ Hyperkähler triple



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- $p_{1/2} : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}^2$ projection onto 1st/2nd component, $\mu_i = p_1^* \omega_i$, $\sigma_i = p_2^* \omega_i$

$$\Omega_{\text{flat}} = \frac{1}{2}\mu_1^2 + \frac{1}{2}\sigma_1^2 - \sum_{i=1}^{3} \mu_i \wedge \sigma_i \in \Omega^4(\mathbb{R}^8), \qquad \text{Spin}(7) := \text{Stab}_{GL(8,\mathbb{R})}(\Omega_{\text{flat}})$$

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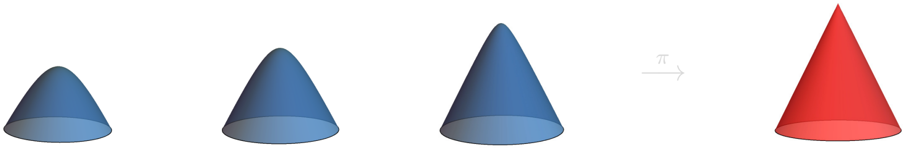

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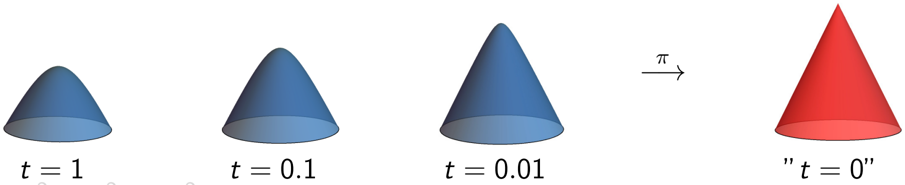

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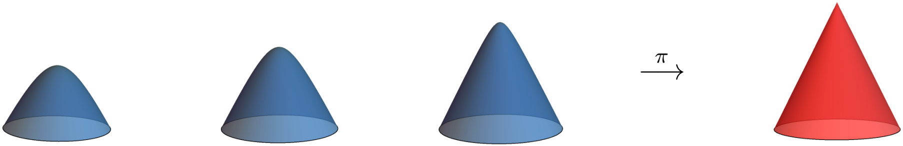

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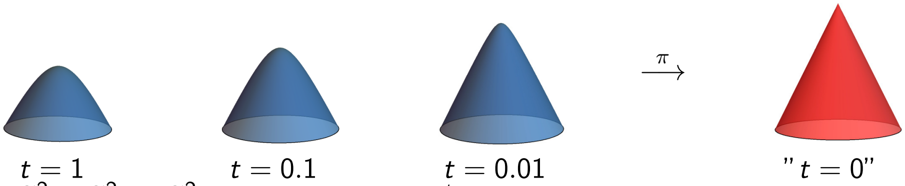

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Example of a Spin(7)-manifold

$$\alpha, \beta, \gamma : T^8 \to T^8$$

$$\alpha : (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \mapsto (-x_1, -x_2, -x_3, -x_4, x_5, x_6, x_7, x_8)$$

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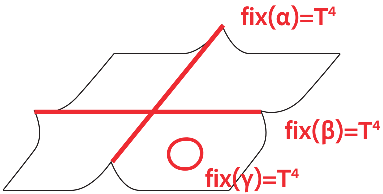
$$\gamma : (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \mapsto \left(\frac{1}{2} - x_1, -x_2, x_3, x_4, -x_5, -x_6, x_7, x_8\right)$$



$$T^8/\langle \alpha, \beta, \gamma \rangle$$

$$U_1 = (B^4/\{\pm 1\}) \times (B^4/\{\pm 1\})$$

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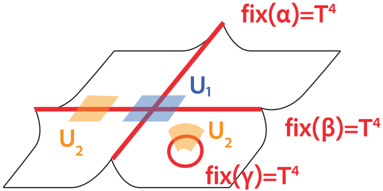

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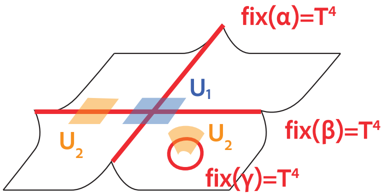

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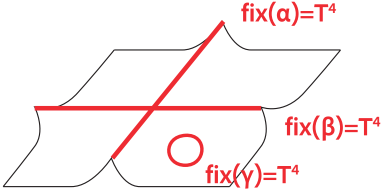
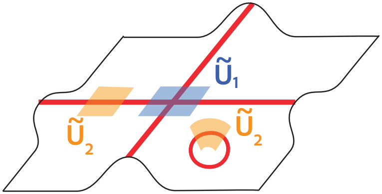


Ω_{flat}

$\rightsquigarrow$



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$$M^8 := T^8/\langle \alpha, \beta, \gamma \rangle \# \widetilde{U}_1 \cup \widetilde{U}_2$$

Theorem ([Joyce, 1996])

Ex. torsion-free Spin(7)*-structure* $\widetilde{\Omega}_t \in \Omega^4(M)$ *such that* $\left|\left|\Omega_{glued,t} - \widetilde{\Omega}_t\right|\right|_{C^{0,\alpha}} \leq ct^{3/10}$.

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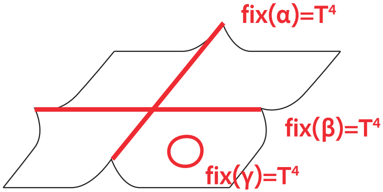

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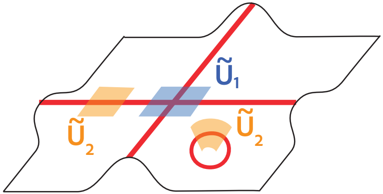

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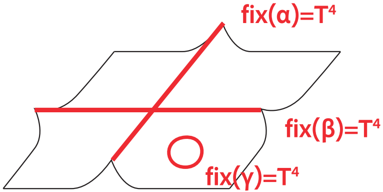

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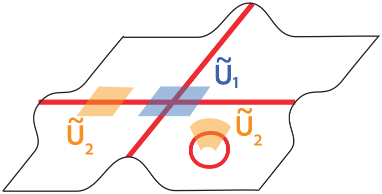

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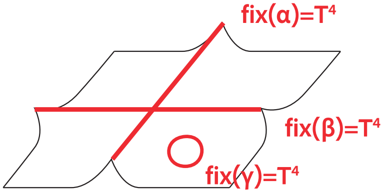

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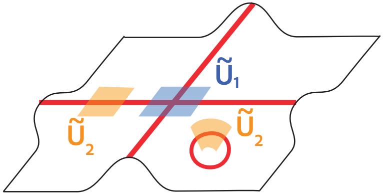

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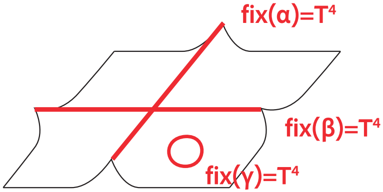

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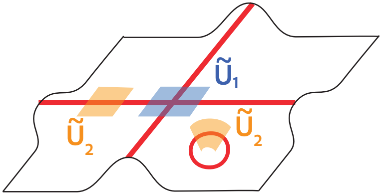

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- Dim 8: $\begin{array}{c} P \\ \downarrow \\ (M^8, \Omega) \end{array}$, connection A Spin(7)-instanton if $*(F_A \wedge \Omega) = -F_A$
- E.g.: A ASD on $(X^4, \omega_i) \Rightarrow p_1^* A$ is Spin(7)-instanton on $X \times \mathbb{R}^4$ w.r.t. Ω_{product}
- Plan: Glue $\underbrace{\text{Spin(7)-instanton w.r.t. } \Omega_{\text{flat}}}_{\text{(I)}}$ and $\underbrace{\text{w.r.t. } \Omega_{\text{product},t}}_{\text{(II)}}$

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- Dim 8: $\begin{array}{c} P \\ \downarrow \\ (M^8, \Omega) \end{array}$, connection A Spin(7)-instanton if $*(F_A \wedge \Omega) = -F_A$
- E.g.: A ASD on $(X^4, \omega_i) \Rightarrow p_1^* A$ is Spin(7)-instanton on $X \times \mathbb{R}^4$ w.r.t. Ω_{product}
- Plan: Glue $\underbrace{\text{Spin(7)-instanton w.r.t. } \Omega_{\text{flat}}}_{\text{(I)}}$ and $\underbrace{\text{w.r.t. } \Omega_{\text{product},t}}_{\text{(II)}}$

Instantons

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- For (I) get $20,000$ examples with $G = SO(n)$ for $n = 2, \ldots, 8$:

$$\left\{\begin{array}{l}\text{flat connections}\\ \text{on } T^8/\langle\alpha,\beta,\gamma\rangle\end{array}\right\} \longleftrightarrow \left\{\text{homo } \rho : \pi_1(T^8/\langle\alpha,\beta,\gamma\rangle \setminus \text{singularities}) \to G\right\}$$

$$\theta \mapsto \rho_\theta = \text{monodromy of } \theta$$

- Require:
  1. $\rho(\alpha) = \rho(\beta) = \text{Id}$
     (To remove this, need unobstructed Spin(7)-instantons on $X_{EH} \times X_{EH}$ with $\rho(\alpha)$ and $\rho(\beta)$ as its monodromy at infinity)
  2. θ rigid and unobstructed
     (To remove *rigid*: need estimate for right-inverse of lin. operator L_t on M^8. $L_{(\text{I})}$, $L_{(\text{II})}$ linearisations of (I), (II). Can show L_t injective on $\left(\text{Ker}(L_{(\text{I})}) \oplus \text{Ker}(L_{(\text{II})})\right)^\perp$. To check unobstructed, must know $\text{ind}(L_t)$)
- For (II): ASD-inst $\begin{array}{c}(A,E)\\ \downarrow\\ \left(X_{EH}, \tilde{\omega}_i^{(t)}\right)\end{array}$ and flat $\begin{array}{c}(A_0,E_0)\\ \downarrow\\ \mathbb{C}^2/\{\pm 1\}\end{array}$, $|A - A_0|_{\tilde{\omega}_i^t} = \mathcal{O}(t^{-1}(r+1)^{-3})$

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Theorem (GPTW '24)

Ex. Spin(7)*-instanton* $\widetilde{A}_t$ *s.t.* $\left|\left|A_t - \widetilde{A}_t\right|\right|_{C^{1,\alpha}_{\beta}} \leq ct^{3/10}$.

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Proof. For $\beta < 0$ let $||f||_{C^0_\beta} := \left|\left|f \cdot (t + d(\cdot, \text{sing}))^{-\beta}\right|\right|_{C^0}$ and $C^{k,\alpha}_\beta$ analog

- $e_t := *(F_{A_t} \wedge \Omega) + F_{A_t}$ has $||e_t||_{C^{0,\alpha}_{\beta-1}} \leq ct^{3/10}$
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  Cf. Calabi-Yaus with max volume growth [Chiu, 2021, Donaldson and Sun, 2014]
  - On $\mathbb{C}^2/\{\pm 1\} \times \mathbb{C}^2/\{\pm 1\}$ no decaying harmonic 1-forms
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    $\Rightarrow$ by Arzelà–Ascoli $a^* \in \Omega^1(\mathbb{C}^2/\{\pm 1\} \times \mathbb{C}^2/\{\pm 1\})$ harmonic (contradiction)

- $\rightsquigarrow \left|\left|L^{-1}_{A_t}\right|\right|_{C^{0,\alpha}_{\beta-1}\to C^{1,\alpha}_\beta} \leq c$, implicit function thm $A_t + a$ Spin(7)-instanton □

Existence proof

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Thank you for the attention!

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