New estimates for G2-structures on resolutions of orbifolds

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Abstract: Joyce and Karigiannis extended the generalised Kummer construction and constructed torsion-free G2-structures on resolutions of G2-orbifolds. In the talk I will explain a different analytic setup to study the same problem, using weighted Hölder norms, which gives improved control over the torsion-free G2-structure and a slightly simpler proof compared to the original proof. This has applications in G2-instantons, and potential applications in associative submanifolds and resolutions of singularities at different length scales. This is the content of arXiv:2011.00482.

The exceptional holonomy group G_2

- $\phi = dx_{123} - dx_{145} - dx_{167} - dx_{246} + dx_{257} - dx_{347} - dx_{356}$ has $\mathrm{Stab}_{GL(7)}(\phi) = G_2$
- On M^7, $\varphi \in \Omega^3(M)$ s.t. $\varphi_x \simeq \phi$ for all $x \in M$ is called G_2*-structure*
- $G_2 \subset SO(7)$, so φ induces a metric g_φ on M
  Theorem (Fernández-Gray '82): $\mathrm{Hol}(M, g_\varphi) \subset G_2$ if and only if $d\varphi = 0$ and $d^*\varphi = 0$.
- Example: identify $\mathbb{R}^7 \simeq \mathbb{R}^3 \oplus \mathbb{H}$ with coordinates $((x_1, x_2, x_3), (y_1, y_2, y_3, y_4))$, then

$$\varphi = dx_{123} - \sum_{i=1}^{3} dx_i \wedge \omega^i, \text{ where}$$

$$\omega^1 = dy_{12} + dy_{34}, \qquad \omega^2 = dy_{13} + dy_{42}, \qquad \omega^3 = dy_{14} + dy_{23},$$

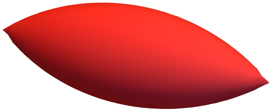
$$\psi = *\varphi = \frac{1}{2}\omega^1 \wedge \omega^1 - \sum_{\substack{(1,2,3),(2,3,1),\\(3,1,2)}} dx_{ij} \wedge \omega^k$$

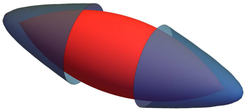
- Linear model extends to product manifolds: $T^3 \times X^4$, X Calabi-Yau manifold

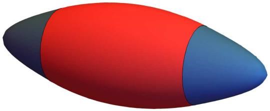
Compact manifolds with holonomy G_2

Three constructions:

1. Joyce '96: Generalised Kummer Construction
2. Kovalev '03, Corti-Haskins-Nordström-Pacini '12: Twisted Connected Sum Construction
3. Joyce-Karigiannis '17: Resolutions of G_2-orbifolds





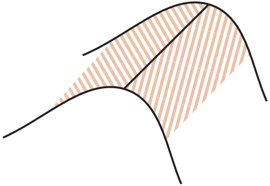


Resolutions of G_2-orbifolds: the smooth manifold

G_2-manifold (M, φ), $\iota : M \to M$ such that $\iota^2 = \mathrm{Id}$, $\iota^* \varphi = \varphi$
$L = \mathrm{fix}(\iota)$ associative, assume there is $\lambda \in \Omega^1(L)$ closed+co-closed, nowhere 0

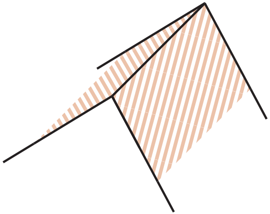
$$N_t = (M/\langle \iota \rangle \setminus L) \cup (\exp_t \circ \rho)^{-1}(U)/\sim$$

for $t \in (0,1)$



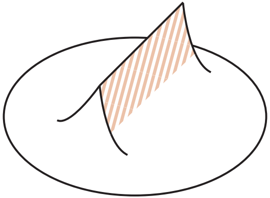
$P = \mathrm{Fr}(\nu) \times_{\mathrm{U}(2)} X_{\mathrm{EH}}$, with fibre Eguchi-Hanson space

$\downarrow \rho$



$\nu/\{\pm 1\}$ normal bundle over L
complex structure $I(\cdot) = \frac{\lambda}{|\lambda|} \times (\cdot)$
$\nu/\{\pm 1\} = \mathrm{Fr}(\nu) \times_{\mathrm{U}(2)} (\mathbb{C}^2/\{\pm 1\})$

$\xleftarrow{\exp_t = \exp \circ (\cdot t)}$



$M/\langle \iota \rangle$
U neighbourhood of L

Resolutions of G_2-orbifolds: the G_2-structure

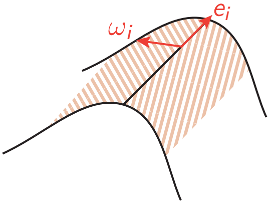
$\widehat{\omega}^i \in \Omega^2(\nu/\{\pm 1\})$, $\omega_i \in \Omega^2(P)$ s.t.

for $x \in L$: $\omega_i|_{P_x}$ blowup of $\widehat{\omega}^i|_{(\nu/\{\pm 1\})_x}$

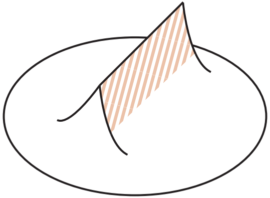
$$r = d_\varphi(L, \cdot) : N_t \to \mathbb{R}$$

$$\varphi^N_t = \begin{cases} \widetilde{\varphi}^P_t & \text{, where } r \leq t^{4/5} \\ \varphi & \text{, where } r \geq 2t^{4/5} \end{cases}$$

and interpolate. ψ^P_t, $\widetilde{\psi}^P_t$, ψ^N_t analog



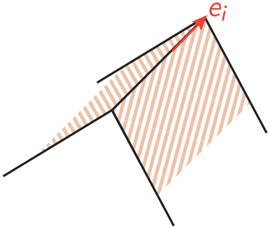
$$\varphi^P_t = e_{123} - t^2 \sum_{i=1}^3 e_i \wedge \omega_i$$
$$\widetilde{\varphi}^P_t = \varphi^P_t + \xi \text{ closed}$$
$$\downarrow \rho$$



$\varphi/\langle\iota\rangle$

$(e_1 = \frac{\lambda}{|\lambda|}, e_2, e_3)$ orthonormal frame on L

$\xleftarrow{\exp_t = \exp \circ (\cdot t)}$



$x \in L : \varphi|_x = e_{123} - \sum_{i=1}^3 e_i \wedge \widehat{\omega}_i$

get $\widehat{\omega}^i \in \Omega^2(\nu/\{\pm 1\})$ parallel on fibres

Perturbing to a torsion-free G_2-structure

Theorem (Joyce '96)

$(M^7, \varphi, g_\varphi)$ *with* $\mathrm{d}\varphi = 0$. inj $\geq ct$, $|\,\mathrm{Riem}\,| \leq ct^{-2}$. $\psi \in \Omega^4(M)$ *closed such that*

$$||{*\varphi} - \psi||_{L^2} \leq ct^{7/2+\epsilon} \text{ for some } c, \epsilon > 0$$

(plus C^0 and L^{14}_1 estimate). Then: for small t, ex. $\eta \in \Omega^2(M)$ such that $\widetilde{\varphi} = \varphi + \mathrm{d}\eta$ is a torsion-free G_2-structure satisfying $||\widetilde{\varphi} - \varphi||_{C^0} \leq ct^{\epsilon}$.

- Problem: $\left|\left|{*\varphi_t^N} - \psi_t^N\right|\right|_{L^2} \leq ct^3$
- Solution in Joyce-Karigiannis '17: "solve $*_{\varphi+\mathrm{d}\eta}(\varphi + \mathrm{d}\eta) - (\psi + \mathrm{d}\xi) = 0$ to first order" and get $\varphi_t^{N,corr}, \psi_t^{N,corr}$
- Then $\left|\left|{*\varphi_t^{N,corr}} - \psi_t^{N,corr}\right|\right|_{L^2} \leq ct^{32/9} = ct^{3.555...}$
  $\rightsquigarrow$ can apply Theorem to get torsion-free G_2-structure

Alternative proof

- Alternative: norms adapted to manifold (Walpuski '13). r dist. from L and
$$||f||_{L^\infty_{\beta;t}} = \left|\left|(t+r)^{-\beta}\cdot f\right|\right|_{L^\infty} \text{ and } ||f||_{C^{k,\alpha}_{\beta;t}} \text{ analog}$$
- e.g. for $\beta < 0$:
$$||\cdot||_{L^\infty} \sim ||\cdot||_{L^\infty_{\beta;t}} \qquad \text{on } \{x \in N_t : r(x) > 1\}$$
$$||\cdot||_{C^{k,\alpha}_{\beta;t}, g_t^N} \overset{\text{rescale}}{\sim} ||\cdot||_{C^{k,\alpha}_{\beta;1}, g_{\mathbb{R}^3}\oplus g_X} \qquad \text{near } L$$

Theorem (Platt '21, true version)

$(N_t, \varphi = \varphi_t^N, g_\varphi)$. $\psi \in \Omega^4(N_t)$ closed such that
$$||\mathrm{d}(*\varphi - \psi)||_{C^{0,\alpha}_{\beta-2;t}} \leq ct^{1+\epsilon-\beta+\alpha} \text{ for some } c, \epsilon > 0, \alpha \in (0,1), \beta \in (-2,0)$$
(plus $C^{0,\alpha}$ estimate). Then: for small t, ex. $\eta \in \Omega^2(M)$ such that $\widetilde{\varphi} = \varphi + \mathrm{d}\eta$ is a torsion-free G_2-structure satisfying $||\widetilde{\varphi} - \varphi||_{C^{1,\alpha/2}_{\beta-1;t}} \leq ct^{1+\epsilon}$.

- Check: $\left|\left|\mathrm{d}(*\varphi_t^N - \psi_t^N)\right|\right|_{L^\infty_{-2;t}} \leq ct^{8/5} \rightsquigarrow$ Theorem gives torsion-free G_2-structure

Special case: T^7/Γ

- Generalised Kummer construction: $\Gamma \simeq \mathbb{Z}_2^3$ acts through G_2-involutions on T^7
  $\rightsquigarrow$ G_2-orbifold T^7/Γ
- In this case:

Old estimate:	$\|\|{*\varphi} - \psi\|\|_{L^2} \leq ct^4$	$\Rightarrow$	$\|\|\widetilde{\varphi} - \varphi\|\|_{L^2} \leq ct^{1/2}$
New estimate:	$\|\|\mathrm{d}({*\varphi} - \psi)\|\|_{L^\infty_{-2;t}} \leq ct^4$	$\Rightarrow$	$\|\|\widetilde{\varphi} - \varphi\|\|_{L^\infty_{\beta-1;t}} \leq ct^4$

- Application:
  Construct G_2-instanton w.r.t. φ from G_2-instanton on T^7/Γ and Fueter section.
  Can be perturbed to G_2-instanton w.r.t. $\widetilde{\varphi}$ if $||\widetilde{\varphi} - \varphi||_{C^0}$ is small.

Proof of the correction theorem: overview

Theorem 1 (Joyce '96): Let $\vartheta = \varphi - *\psi$. If $\eta \in \Omega^2(N)$ such that $\Delta\eta = \mathrm{d}^*\vartheta + \frac{3}{7}\,\mathrm{d}^*(\langle \mathrm{d}\eta, \varphi\rangle\vartheta) + h.o.t.(\eta)$, then $\varphi + \mathrm{d}\eta$ is torsion-free.

Lemma 2: There exists a unique (up to scaling) harmonic form $\mu \in \Omega^2(X_{\mathrm{EH}})$ which decays at ∞.

- Let $\alpha_1, \dots, \alpha_k$ be basis of $\mathcal{H}^2(M/\langle\iota\rangle)$
- $\dim \operatorname{Ker} \Delta_{N_t} = b^2(N_t) = b^2(M/\langle\iota\rangle) + b^0(L)b^2(X)$ by Künneth formula. This motivates definition of approximate kernel of same dimension:

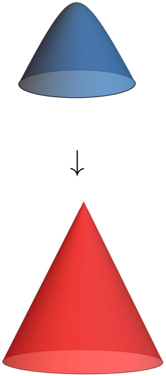
$$\mathcal{K}_{\mathrm{ap}} = \langle \text{cut-offs of } \alpha_i \rangle_{i=1,\dots,k} \oplus \langle \text{cut-offs of } \mu \text{ on connected cpts of } P \rangle$$

Lemma 3: For $\beta \in (-2, 0)$, $a \perp \mathcal{K}_{\mathrm{ap}}$: $\|a\|_{C^{2,\alpha}_{\beta;t}} \leq c\,\|\Delta a\|_{C^{0,\alpha}_{\beta-2;t}}$.

- Let $\eta_0 = 0$, η_{j+1} such that $\Delta\eta_{j+1} = \mathrm{d}^*\vartheta + \frac{3}{7}\,\mathrm{d}^*(\langle \mathrm{d}\eta_j, \varphi\rangle\vartheta) + h.o.t.(\eta_j)$ Sequence (η_j) is bounded by Lemma 3 $\rightsquigarrow \eta = \lim \eta_j$ solves equation from Theorem 1.

Proof of the correction theorem: Lemma 2

Lemma 2: There exists a unique (up to scaling) harmonic form $\mu \in \Omega^2(X_{\mathrm{EH}})$ which decays at ∞.



- Lockhart '87: on asymp conical 4-folds have in deg 2

$$L^2 \text{ deRham cohomology } = \text{ singular cohomology}$$

- X_{EH} is asymptotic to the cone $C(S^3/\{\pm 1\}) = \mathbb{R}^+ \times (S^3/\{\pm 1\})$ and $X_{\mathrm{EH}} \simeq T^*S^2$, so $H^2(X_{\mathrm{EH}}) = \mathbb{R}$
- Let r be a coordinate on $\mathbb{R}^+$. Forms in L^2 decay faster than r^{-2}, otherwise their integral would be ∞. Are there *more* harmonic forms that decay, but not fast enough to be in L^2?
- Lockhart-McOwen '85: harmonic *homogeneous* forms on cone $\rightsquigarrow$ ind Δ
  (1) Count harmonic homogeneous forms on cone $\rightsquigarrow$ representations of SO(4);
  (2) Guess explicit basis of CoKer Δ on forms that decay at least like r^{-2}
  (3) Find that dim Ker Δ is the same for decays $[-2, 0)$
- $\rightsquigarrow$ claim

Proof of the correction theorem: Lemma 3

Lemma 3: For $\beta \in (-2, 0)$, $a \perp \mathcal{K}_{\mathrm{ap}}$: $\|a\|_{\cancel{C^{2,\alpha}_{\beta;t}}\; L^\infty_{\beta;t}} \leq c\,\|\Delta a\|_{C^{0,\alpha}_{\beta-2;t}}$.

Assume not, then there exist sequences $a_i \perp \mathcal{K}_{\mathrm{ap}}$, $x_i \in N_{t_i}$, $t_i \to 0$ such that

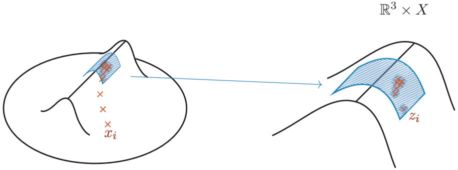
$$|a_i(x_i)| \cdot (t_i + r(x_i))^{-\beta} = 1, \qquad \|\Delta a_i\|_{C^{0,\alpha}_{\beta-2;t_i}} \to 0.$$

Get contradiction if $\lim x_i = p \in \rho^{-1}(L)$: nbhd of p, say $\rho^{-1}(B^3(\sqrt{t}) \times B^4(\sqrt{t}))$,

$$\text{w.r.t. } g_{\mathbb{R}^3} \oplus g_X \text{ looks like } \rho^{-1}(B^3(1/\sqrt{t}) \times B^4(1/\sqrt{t})) \subset \mathbb{R}^3 \times X_{\mathrm{EH}}$$

$1/\sqrt{t} \to \infty$, so get a limit $a^* \in \Omega^2(\mathbb{R}^3 \times X_{\mathrm{EH}})$ which

- is harmonic,
- is perpendicular to μ because $a_i \perp \mathcal{K}_{\mathrm{ap}}$,
- decays at ∞ because $\|a_i\|_{L^\infty_{\beta;t_i}} = 1$,



$\Rightarrow a^* = 0$, as μ the only harmonic form with decay.
Contradiction to $a^*(p) \neq 0$.

Similar: contradiction if x_i has accumulation point elsewhere $\rightsquigarrow$ claim

Summary

1. Analysis on the model space X_{EH}
2. $\Rightarrow$ estimate for the Laplacian on N_t
3. $\Rightarrow$ solution for PDE $\Delta\eta = \mathrm{d}^*\vartheta + \frac{3}{7}\mathrm{d}^*(\langle \mathrm{d}\eta, \varphi\rangle\vartheta) + h.o.t.$
4. $\Rightarrow$ torsion-free G_2-structure $\varphi_t^N + \mathrm{d}\eta$ with good estimate for η

Thank you for the attention!

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