Gluing Eguchi-Hanson Metrics and a Question of Page
(by Simon Brendle and NK)

NIKOLAOS KAPOULEAS
Simons workshop

May 27, 2021
Our motivation was to develop gluing constructions for Einstein four-manifolds and related geometric objects like solitons, ancient solutions, etc.
Our motivation was to develop gluing constructions for Einstein four-manifolds and related geometric objects like solitons, ancient solutions, etc.


In 1978 Gibbons-Pope and Page in an effort to understand the geometry of the Ricci-flat metrics on the K3 surface proposed the following gluing construction for those metrics:

Starting with a flat torus $\mathbb{T}^4 = \mathbb{R}^4/\mathbb{Z}^4$, quotient out further with a reflection with respect to a point. This produces a flat orbifold with 16 singular points modeled after $\mathbb{R}^4/\mathbb{Z}^2$. 
Our motivation was to develop gluing constructions for Einstein four-manifolds and related geometric objects like solitons, ancient solutions, etc.


In 1978 Gibbons-Pope and Page in an effort to understand the geometry of the Ricci-flat metrics on the K3 surface proposed the following gluing construction for those metrics:
Our motivation was to develop gluing constructions for Einstein four-manifolds and related geometric objects like solitons, ancient solutions, etc.


In 1978 Gibbons-Pope and Page in an effort to understand the geometry of the Ricci-flat metrics on the K3 surface proposed the following gluing construction for those metrics:

Starting with a flat torus $\mathbb{T}^4 = \mathbb{R}^4 / \mathbb{Z}^4$, quotient out further with a reflection with respect to a point. This produces a flat orbifold with 16 singular points modeled after $\mathbb{R}^4 / \mathbb{Z}_2$. 
Resolve the singular points by attaching in their place Eguchi-Hansom (EH) manifolds scaled down by small factors.

\[ \text{Eguchi-Hansom (EH) manifolds are Ricci-flat hyper-Kähler four-manifolds which are topologically } T^2 \text{ and they carry the metric } g_{eh}, \epsilon = r^2 \left( \epsilon^4 + r^4 \right)^{1/2} \left( dr \otimes dr + r^2 \alpha_1 \otimes \alpha_1 \right) + \left( \epsilon^4 + r^4 \right)^{1/2} \left( \alpha_2 \otimes \alpha_2 + \alpha_3 \otimes \alpha_3 \right) \sim \epsilon^2 g_{eh}, 1, \right. \]

They are Asymptotically Locally Euclidean (ALE) with infinity asymptotically $\mathbb{R}^4 / \mathbb{Z}^2$, hyper-Kähler, and self-dual.
Resolve the singular points by attaching in their place Eguchi-Hansom (EH) manifolds scaled down by small factors.

EH manifolds are Ricci-flat hyper-Kähler four-manifolds which are topologically $T\mathbb{S}^2$ and they carry the metric

$$g_{eh,\varepsilon} = \frac{r^2}{(\varepsilon^4 + r^4)^{\frac{1}{2}}} \left( dr \otimes dr + r^2 \alpha_1 \otimes \alpha_1 \right)$$

$$+ (\varepsilon^4 + r^4)^{\frac{1}{2}} (\alpha_2 \otimes \alpha_2 + \alpha_3 \otimes \alpha_3) \sim \varepsilon^2 g_{eh,1},$$

where

$$\alpha_1 = \frac{1}{r^2} (x_1 \, dx_2 - x_2 \, dx_1 + x_3 \, dx_4 - x_4 \, dx_3),$$

$$\alpha_2 = \frac{1}{r^2} (x_1 \, dx_3 - x_3 \, dx_1 + x_4 \, dx_2 - x_2 \, dx_4),$$

$$\alpha_3 = \frac{1}{r^2} (x_1 \, dx_4 - x_4 \, dx_1 + x_2 \, dx_3 - x_3 \, dx_2).$$
Resolve the singular points by attaching in their place Eguchi-Hansom (EH) manifolds scaled down by small factors.

EH manifolds are Ricci-flat hyper-Kähler four-manifolds which are topologically $T\mathbb{S}^2$ and they carry the metric

$$g_{eh, \varepsilon} = \frac{r^2}{(\varepsilon^4 + r^4)^{1/2}} \left( dr \otimes dr + r^2 \alpha_1 \otimes \alpha_1 \right)$$

$$+ (\varepsilon^4 + r^4)^{1/2} \left( \alpha_2 \otimes \alpha_2 + \alpha_3 \otimes \alpha_3 \right) \sim \varepsilon^2 g_{eh,1},$$

where

$$\alpha_1 = \frac{1}{r^2} \left( x_1 \, dx_2 - x_2 \, dx_1 + x_3 \, dx_4 - x_4 \, dx_3 \right),$$

$$\alpha_2 = \frac{1}{r^2} \left( x_1 \, dx_3 - x_3 \, dx_1 + x_4 \, dx_2 - x_2 \, dx_4 \right),$$

$$\alpha_3 = \frac{1}{r^2} \left( x_1 \, dx_4 - x_4 \, dx_1 + x_2 \, dx_3 - x_3 \, dx_2 \right).$$

They are Asymptotically Locally Euclidean (ALE) with infinity asymptotically $\mathbb{R}^4/\mathbb{Z}^2$, hyper-Kähler, and self-dual.
The construction has clearly $16 \times 3 + 10 - 1$ parameters.
The construction has clearly $16 \times 3 + 10 - 1$ parameters.
This construction was carried out rigorously by Topiwala, LeBrun-Singer, and Donaldson.
The construction has clearly $16 \times 3 + 10 - 1$ parameters.

This construction was carried out rigorously by Topiwala, LeBrun-Singer, and Donaldson.

Since there is a unique Ricci-flat metric for given Kähler class there are no obstructions of the type we will discuss later.
Page in 1981 asked the question whether modifying the construction by reversing the orientation of some but not all the EH manifolds being attached could produce new Ricci-flat non-Kähler metrics (known such metrics are extremely few).
Page in 1981 asked the question whether modifying the construction by reversing the orientation of some but not all the EH manifolds being attached could produce new Ricci-flat non-Kähler metrics (known such metrics are extremely few).

The signature of the new manifolds is different and they cannot be Kähler.
Page in 1981 asked the question whether modifying the construction by reversing the orientation of some but not all the EH manifolds being attached could produce new Ricci-flat non-Kähler metrics (known such metrics are extremely few).

The signature of the new manifolds is different and they cannot be Kähler.

It turns out that for such a construction there are obstructions which have to be understood.
Page in 1981 asked the question whether modifying the construction by reversing the orientation of some but not all the EH manifolds being attached could produce new Ricci-flat non-Kähler metrics (known such metrics are extremely few).

The signature of the new manifolds is different and they cannot be Kähler.

It turns out that for such a construction there are obstructions which have to be understood.

In this direction the understanding of the kernel of the Lichnerowicz Laplacian by Biquard and Page is important. Recall that the Lichnerowicz Laplacian is defined by

\[ \Delta_L h_{ik} = \Delta h_{ik} + 2R_{ijkl} h^{jl} - \text{Ric}^l_i h_{kl} - \text{Ric}^l_k h_{il}. \]
By Biquard’s work already the interaction of the EH manifolds with the flat orbifold is not obstructing the construction as expected.
By Biquard’s work already the interaction of the EH manifolds with the flat orbifold is not obstructing the construction as expected.

The EH manifolds however interact with each other through the flat orbifold.
By Biquard’s work already the interaction of the EH manifolds with the flat orbifold is not obstructing the construction as expected.

The EH manifolds however interact with each other through the flat orbifold.

These interactions can be understood in a way similar to the understanding of the interactions between catenoidal bridges in doubling constructions for minimal surfaces.
By Biquard’s work already the interaction of the EH manifolds with the flat orbifold is not obstructing the construction as expected.

The EH manifolds however interact with each other through the flat orbifold.

These interactions can be understood in a way similar to the understanding of the interactions between catenoidal bridges in doubling constructions for minimal surfaces.

The approach (called *linearized doubling* in the minimal doubling case) constructs as an intermediate step singular solutions of the Jacobi equation on the given object (in our case the orbifold and in the minimal doubling case the surface being doubled).
By Biquard’s work already the interaction of the EH manifolds with the flat orbifold is not obstructing the construction as expected.

The EH manifolds however interact with each other through the flat orbifold.

These interactions can be understood in a way similar to the understanding of the interactions between catenoidal bridges in doubling constructions for minimal surfaces.

The approach (called *linearized doubling* in the minimal doubling case) constructs as an intermediate step singular solutions of the Jacobi equation on the given object (in our case the orbifold and in the minimal doubling case the surface being doubled).

The obstructions (that is balancing/unbalancing questions) can be (approximately) calculated in terms of these singular solutions, which are easier to understand, because they satisfy a *linear* instead of a *nonlinear* equation.
We imposed maximal symmetry on the construction to simplify the situation while keeping the main features.
We imposed maximal symmetry on the construction to simplify the situation while keeping the main features.

Actually before we proceed it is worth pointing out that the main issue has to do with the size of the EH necks: if all sizes work out as in the Kähler case, then there is no obstruction, if not, then the construction is not a perturbation construction (at best).
We imposed maximal symmetry on the construction to simplify the situation while keeping the main features.

Actually before we proceed it is worth pointing out that the main issue has to do with the size of the EH necks: if all sizes work out as in the Kähler case, then there is no obstruction, if not, then the construction is not a perturbation construction (at best).

To discuss the construction in detail we let $\hat{g}_{eh,\varepsilon}$ denote the pull-back of $g_{eh,\varepsilon}$ under the map $(x_1, x_2, x_3, x_4) \mapsto (-x_1, x_2, x_3, x_4)$. Clearly, $\hat{g}_{eh,\varepsilon}$ is a Ricci flat metric.
We imposed maximal symmetry on the construction to simplify the situation while keeping the main features.

Actually before we proceed it is worth pointing out that the main issue has to do with the size of the EH necks: if all sizes work out as in the Kähler case, then there is no obstruction, if not, then the construction is not a perturbation construction (at best).

To discuss the construction in detail we let $\hat{g}_{eh,\varepsilon}$ denote the pull-back of $g_{eh,\varepsilon}$ under the map $(x_1, x_2, x_3, x_4) \mapsto (-x_1, x_2, x_3, x_4)$.

Clearly, $\hat{g}_{eh,\varepsilon}$ is a Ricci flat metric.
Near infinity, we have the asymptotic expansions

\[ g_{eh, \varepsilon} = g_{eucl} + \frac{1}{2} \varepsilon^4 T + O(\varepsilon^8 r^{-8}) \]

and

\[ \hat{g}_{eh, \varepsilon} = g_{eucl} + \frac{1}{2} \varepsilon^4 \hat{T} + O(\varepsilon^8 r^{-8}), \]

where
Near infinity, we have the asymptotic expansions
\[ g_{\text{eh},\varepsilon} = g_{\text{eucl}} + \frac{1}{2} \varepsilon^4 T + O(\varepsilon^8 r^{-8}) \]
and
\[ \hat{g}_{\text{eh},\varepsilon} = g_{\text{eucl}} + \frac{1}{2} \varepsilon^4 \hat{T} + O(\varepsilon^8 r^{-8}), \]
where
\[ T = -r^{-4} (dr \otimes dr + r^2 \alpha_1 \otimes \alpha_1 - r^2 \alpha_2 \otimes \alpha_2 - r^2 \alpha_3 \otimes \alpha_3) \]
\[ = -r^{-6} \left\{ (x_1^2 + x_2^2 - x_3^2 - x_4^2) (dx_1 \otimes dx_1 + dx_2 \otimes dx_2 - dx_3 \otimes dx_3 - dx_4 \otimes dx_4) \right. \]
\[ + 2 (x_1 x_3 + x_2 x_4) (dx_1 \otimes dx_3 + dx_3 \otimes dx_1 + dx_2 \otimes dx_4 + dx_4 \otimes dx_2) \]
\[ + 2 (x_1 x_4 - x_2 x_3) (dx_1 \otimes dx_4 + dx_4 \otimes dx_1 - dx_2 \otimes dx_3 - dx_3 \otimes dx_2) \left\} \right. \]
Near infinity, we have the asymptotic expansions

\[ g_{eh,\varepsilon} = g_{eucl} + \frac{1}{2} \varepsilon^4 T + O(\varepsilon^8 r^{-8}) \]

and

\[ \hat{g}_{eh,\varepsilon} = g_{eucl} + \frac{1}{2} \varepsilon^4 \hat{T} + O(\varepsilon^8 r^{-8}), \]

where

\[ T = -r^{-4} (dr \otimes dr + r^2 \alpha_1 \otimes \alpha_1 - r^2 \alpha_2 \otimes \alpha_2 - r^2 \alpha_3 \otimes \alpha_3) \]

\[ = -r^{-6} \left\{ (x_1^2 + x_2^2 - x_3^2 - x_4^2) (dx_1 \otimes dx_1 + dx_2 \otimes dx_2 - dx_3 \otimes dx_3 - dx_4 \otimes dx_4) + 2 (x_1 x_3 + x_2 x_4) (dx_1 \otimes dx_3 + dx_3 \otimes dx_1 + dx_2 \otimes dx_4 + dx_4 \otimes dx_2) + 2 (x_1 x_4 - x_2 x_3) (dx_1 \otimes dx_4 + dx_4 \otimes dx_1 - dx_2 \otimes dx_3 - dx_3 \otimes dx_2) \right\} \]

and \( \hat{T} = \)

\[ = -r^{-6} \left\{ (x_1^2 + x_2^2 - x_3^2 - x_4^2) (dx_1 \otimes dx_1 + dx_2 \otimes dx_2 - dx_3 \otimes dx_3 - dx_4 \otimes dx_4) + 2 (x_1 x_3 - x_2 x_4) (dx_1 \otimes dx_3 + dx_3 \otimes dx_1 - dx_2 \otimes dx_4 - dx_4 \otimes dx_2) + 2 (x_1 x_4 + x_2 x_3) (dx_1 \otimes dx_4 + dx_4 \otimes dx_1 + dx_2 \otimes dx_3 + dx_3 \otimes dx_2) \right\}. \]
We note that the metrics $g_{\text{eh},\varepsilon}$ and $\hat{g}_{\text{eh},\varepsilon}$ and the tensors $T$ and $\hat{T}$ are all invariant under the maps

$$(x_1, x_2, x_3, x_4) \mapsto (x_2, -x_1, x_3, x_4),$$

$$(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, x_4, -x_3),$$

$$(x_1, x_2, x_3, x_4) \mapsto (x_3, x_4, x_1, x_2),$$

$$(x_1, x_2, x_3, x_4) \mapsto (-x_3, x_4, -x_1, x_2).$$
We note that the metrics $g_{\text{eh},\varepsilon}$ and $\hat{g}_{\text{eh},\varepsilon}$ and the tensors $T$ and $\hat{T}$ are all invariant under the maps

\[
(x_1, x_2, x_3, x_4) \mapsto (x_2, -x_1, x_3, x_4),
\]

\[
(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, x_4, -x_3),
\]

\[
(x_1, x_2, x_3, x_4) \mapsto (x_3, x_4, x_1, x_2),
\]

\[
(x_1, x_2, x_3, x_4) \mapsto (-x_3, x_4, -x_1, x_2).
\]

To analyze the kernel of $\Delta_{L,g_{\text{eh}}}$, we consider the frame $r \frac{\partial}{\partial r}, V_1, V_2, V_3$ dual to the co-frame $\frac{1}{r} dr, \alpha_1, \alpha_2, \alpha_3$. 
We note that the metrics $g_{eh,\varepsilon}$ and $\hat{g}_{eh,\varepsilon}$ and the tensors $T$ and $\hat{T}$ are all invariant under the maps

\[
\begin{align*}
(x_1, x_2, x_3, x_4) &\mapsto (x_2, -x_1, x_3, x_4), \\
(x_1, x_2, x_3, x_4) &\mapsto (x_1, x_2, x_4, -x_3), \\
(x_1, x_2, x_3, x_4) &\mapsto (x_3, x_4, x_1, x_2), \\
(x_1, x_2, x_3, x_4) &\mapsto (-x_3, x_4, -x_1, x_2).
\end{align*}
\]

To analyze the kernel of $\Delta_{L,g_{eh}}$, we consider the frame $r \frac{\partial}{\partial r}, V_1, V_2, V_3$ dual to the co-frame $\frac{1}{r} dr, \alpha_1, \alpha_2, \alpha_3$.

Clearly

\[
\begin{align*}
V_1 &= x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_4} - x_4 \frac{\partial}{\partial x_3}, \\
V_2 &= x_1 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_1} + x_4 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_4}, \\
V_3 &= x_1 \frac{\partial}{\partial x_4} - x_4 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_2}.
\end{align*}
\]
We next define

\[ o_1, \varepsilon = g_{eh, \varepsilon} - \frac{1}{2} \mathcal{L}_r \frac{\partial}{\partial r} g_{eh, \varepsilon} = \frac{1}{2} \varepsilon \frac{\partial}{\partial \varepsilon} g_{eh, \varepsilon} = \varepsilon^4 T + O(\varepsilon^8 r^{-8}), \]

\[ o_2, \varepsilon = \frac{1}{2} \mathcal{L} \frac{r^2 v_2}{\sqrt{\varepsilon^4 + r^4}} g_{eh, \varepsilon}, \]

\[ o_3, \varepsilon = \frac{1}{2} \mathcal{L} \frac{r^2 v_3}{\sqrt{\varepsilon^4 + r^4}} g_{eh, \varepsilon}. \]
We next define

\[ o_1, \varepsilon = g_{eh, \varepsilon} - \frac{1}{2} \mathcal{L}_r \frac{\partial}{\partial r} g_{eh, \varepsilon} \]

\[ = \frac{1}{2} \varepsilon \frac{\partial}{\partial \varepsilon} g_{eh, \varepsilon} = \varepsilon^4 T + O(\varepsilon^8 r^{-8}), \]

\[ o_2, \varepsilon = \frac{1}{2} \mathcal{L} \frac{r^2 v_2}{\sqrt{\varepsilon^4 + r^4}} g_{eh, \varepsilon}, \]

\[ o_3, \varepsilon = \frac{1}{2} \mathcal{L} \frac{r^2 v_3}{\sqrt{\varepsilon^4 + r^4}} g_{eh, \varepsilon}. \]

Equivalently,

\[ o_1, \varepsilon = -\frac{\varepsilon^4 r^2}{(\varepsilon^4 + r^4)^{3/2}} (dr \otimes dr + r^2 \alpha_1 \otimes \alpha_1) + \frac{\varepsilon^4}{(\varepsilon^4 + r^4)^{1/2}} (\alpha_2 \otimes \alpha_2 + \alpha_3 \otimes \alpha_3) \]

\[ o_2, \varepsilon = \frac{\varepsilon^4}{\varepsilon^4 + r^4} (r dr \otimes \alpha_2 + r \alpha_2 \otimes dr - r^2 \alpha_1 \otimes \alpha_3 - r^2 \alpha_3 \otimes \alpha_1), \]

\[ o_3, \varepsilon = \frac{\varepsilon^4}{\varepsilon^4 + r^4} (r dr \otimes \alpha_3 + r \alpha_3 \otimes dr + r^2 \alpha_1 \otimes \alpha_2 + r^2 \alpha_2 \otimes \alpha_1). \]
Proposition (O. Biquard 2013; D. Page 1978) For each \( i \in \{1, 2, 3\} \), the tensor \( o_{i, \varepsilon} \) has the following properties:

(i) \( \text{tr}_{g_{\text{eh}, \varepsilon}} o_{i, \varepsilon} = 0 \).
(ii) \( \text{div}_{g_{\text{eh}, \varepsilon}} o_{i, \varepsilon} = 0 \).
(iii) \( \Delta_{L, g_{\text{eh}, \varepsilon}} o_{i, \varepsilon} = 0 \).
(iv) \( \int_{\mathbb{R}^4 \setminus \{0\}} |o_{1, \varepsilon}|^2_{g_{\text{eh}, \varepsilon}} \, d\text{vol}_{g_{\text{eh}, \varepsilon}} = 2 \pi^2 \varepsilon^4 \).
Proposition (O. Biquard 2013; D. Page 1978) For each $i \in \{1, 2, 3\}$, the tensor $o_{i,\varepsilon}$ has the following properties:

(i) $\text{tr}_{g_{\text{eh}},\varepsilon} o_{i,\varepsilon} = 0$.

(ii) $\text{div}_{g_{\text{eh}},\varepsilon} o_{i,\varepsilon} = 0$.

(iii) $\Delta_{L,g_{\text{eh}},\varepsilon} o_{i,\varepsilon} = 0$.

(iv) $\int_{\mathbb{R}^4 \setminus \{0\}} |o_{1,\varepsilon}|^2_{g_{\text{eh}},\varepsilon} \, d\text{vol}_{g_{\text{eh}},\varepsilon} = 2\pi^2 \varepsilon^4$.

Proposition (O. Biquard 2013; O. Biquard, Y. Rollin 2009) Let $(M_{\text{eh}}, g_{\text{eh}})$ denote the Eguchi-Hanson manifold with parameter $\varepsilon = 1$, and let $o_i := o_{i,1}$. Then the following statements hold:

(i) Let $h$ be a symmetric $(0, 2)$-tensor on the Eguchi-Hanson manifold satisfying $|h| \leq (1 + r)^{-\sigma}$ for some $\sigma > 0$ and $\Delta_{L,g_{\text{eh}}} h = 0$. Then $h \in \text{span}\{o_1, o_2, o_3\}$.

(ii) Let $h$ be a symmetric $(0, 2)$-tensor on the Eguchi-Hanson manifold satisfying $|h| \leq (1 + r)^{-\sigma - 1}$ and $|\nabla h| \leq (1 + r)^{-\sigma - 2}$ for some $\sigma > 0$. Then

$$\int_{M_{\text{eh}}} \langle \Delta_{L,g_{\text{eh}}} h, h \rangle \, d\text{vol}_{g_{\text{eh}}} \leq 0.$$ 

Moreover, equality holds if and only if $h \in \text{span}\{o_1, o_2, o_3\}$.
We choose the torus $\mathbb{T}^4 := \mathbb{R}^4/(2\mathbb{Z})^4$ and the orbifold $\mathbb{T}^4/\mathbb{Z}_2$. 
We choose the torus $\mathbb{T}^4 := \mathbb{R}^4/(2\mathbb{Z})^4$ and the orbifold $\mathbb{T}^4/\mathbb{Z}_2$.

The orbifold is covered by $\mathbb{R}^4$ and the preimage of the singular points is $\mathbb{Z}^4$. 
We choose the torus $\mathbb{T}^4 := \mathbb{R}^4/(2\mathbb{Z})^4$ and the orbifold $\mathbb{T}^4/\mathbb{Z}_2$.

The orbifold is covered by $\mathbb{R}^4$ and the preimage of the singular points is $\mathbb{Z}^4$.

We want to construct a solution of the Jacobi equation with prescribed singular behavior at the singular points of the orbifold.
We choose the torus $\mathbb{T}^4 := \mathbb{R}^4/(2\mathbb{Z})^4$ and the orbifold $\mathbb{T}^4/\mathbb{Z}_2$.

The orbifold is covered by $\mathbb{R}^4$ and the preimage of the singular points is $\mathbb{Z}^4$.

We want to construct a solution of the Jacobi equation with prescribed singular behavior at the singular points of the orbifold.

Equivalently its lift to $\mathbb{R}^4$ is a periodic with periods $(2\mathbb{Z})^4$ harmonic tensor field with prescribed singular behavior at the points of $\mathbb{Z}^4$. 

We choose the torus $\mathbb{T}^4 := \mathbb{R}^4/(2\mathbb{Z})^4$ and the orbifold $\mathbb{T}^4/\mathbb{Z}_2$.

The orbifold is covered by $\mathbb{R}^4$ and the preimage of the singular points is $\mathbb{Z}^4$.

We want to construct a solution of the Jacobi equation with prescribed singular behavior at the singular points of the orbifold.

Equivalently its lift to $\mathbb{R}^4$ is a periodic with periods $(2\mathbb{Z})^4$ harmonic tensor field with prescribed singular behavior at the points of $\mathbb{Z}^4$.

The group $\mathcal{G}$ of symmetries we impose (lifted to $\mathbb{R}^4$) consists of the symmetries which either fix both $\mathbb{Z}_4^{\text{even}}$ and $\mathbb{Z}_4^{\text{odd}}$ or exchange them, where

\[
\mathbb{Z}_4^{\text{even}} = \{(a_1, a_2, a_3, a_4) \in \mathbb{Z}^4 : a_1 + a_2 + a_3 + a_4 \text{ is even}\} \quad \text{and} \quad \mathbb{Z}_4^{\text{odd}} = \{(a_1, a_2, a_3, a_4) \in \mathbb{Z}^4 : a_1 + a_2 + a_3 + a_4 \text{ is odd}\}.
\]
\( \mathcal{G} \) is generated by the following.

\[
\begin{align*}
(x_1, x_2, x_3, x_4) &\mapsto (x_2, -x_1, x_3, x_4), \\
(x_1, x_2, x_3, x_4) &\mapsto (x_1, x_2, x_4, -x_3), \\
(x_1, x_2, x_3, x_4) &\mapsto (x_3, x_4, x_1, x_2), \\
(x_1, x_2, x_3, x_4) &\mapsto (-x_3, x_4, -x_1, x_2), \\
(x_1, x_2, x_3, x_4) &\mapsto (1 - x_1, x_2, x_3, x_4), \\
(x_1, x_2, x_3, x_4) &\mapsto (x_1, 1 - x_2, x_3, x_4), \\
(x_1, x_2, x_3, x_4) &\mapsto (x_1, x_2, 1 - x_3, x_4), \\
(x_1, x_2, x_3, x_4) &\mapsto (x_1, x_2, x_3, 1 - x_4), \\
(x_1, x_2, x_3, x_4) &\mapsto (x_1 + 2, x_2, x_3, x_4), \\
(x_1, x_2, x_3, x_4) &\mapsto (x_1, x_2 + 2, x_3, x_4), \\
(x_1, x_2, x_3, x_4) &\mapsto (x_1, x_2, x_3 + 2, x_4), \\
(x_1, x_2, x_3, x_4) &\mapsto (x_1, x_2, x_3, x_4 + 2).
\end{align*}
\]

The first four generate the stabilizer of the origin and are symmetries of the EH metric as mentioned earlier.
$G$ is generated by the following.

\[
\begin{align*}
(x_1, x_2, x_3, x_4) &\mapsto (x_2, -x_1, x_3, x_4), \\
(x_1, x_2, x_3, x_4) &\mapsto (x_1, x_2, x_4, -x_3), \\
(x_1, x_2, x_3, x_4) &\mapsto (x_3, x_4, x_1, x_2), \\
(x_1, x_2, x_3, x_4) &\mapsto (-x_3, x_4, -x_1, x_2), \\
(x_1, x_2, x_3, x_4) &\mapsto (1 - x_1, x_2, x_3, x_4), \\
(x_1, x_2, x_3, x_4) &\mapsto (x_1, 1 - x_2, x_3, x_4), \\
(x_1, x_2, x_3, x_4) &\mapsto (x_1, x_2, 1 - x_3, x_4), \\
(x_1, x_2, x_3, x_4) &\mapsto (x_1, x_2, x_3, 1 - x_4), \\
(x_1, x_2, x_3, x_4) &\mapsto (x_1 + 2, x_2, x_3, x_4), \\
(x_1, x_2, x_3, x_4) &\mapsto (x_1, x_2 + 2, x_3, x_4), \\
(x_1, x_2, x_3, x_4) &\mapsto (x_1, x_2, x_3 + 2, x_4), \\
(x_1, x_2, x_3, x_4) &\mapsto (x_1, x_2, x_3, x_4 + 2).
\end{align*}
\]

The first four generate the stabilizer of the origin and are symmetries of the EH metric as mentioned earlier.
Moreover they kill $o_{2, \varepsilon}, o_{3, \varepsilon}$ but respect $o_{1, \varepsilon}$. 
Moreover they kill $o_2,\varepsilon, o_3,\varepsilon$ but respect $o_1,\varepsilon$.

The symmetries which fix $\mathbb{Z}^4_{\text{even}}$ respect orientation and the other ones do not.
Moreover they kill $o_{2,\varepsilon}, o_{3,\varepsilon}$ but respect $o_{1,\varepsilon}$.

The symmetries which fix $\mathbb{Z}_{\text{even}}^4$ respect orientation and the other ones do not.

Of the 10-parameter family of deformations of the torus only scaling is allowed by $\mathcal{G}$.
Moreover they kill $o_2, \varepsilon, o_3, \varepsilon$ but respect $o_1, \varepsilon$.

The symmetries which fix $\mathbb{Z}_4$ respect orientation and the other ones do not.

Of the 10-parameter family of deformations of the torus only scaling is allowed by $\mathcal{G}$.

We define the singular solution by

$$H = \frac{1}{2} \sum_{a \in \mathbb{Z}_4^{\text{even}}} \tau_a^* T + \frac{1}{2} \sum_{a \in \mathbb{Z}_4^{\text{odd}}} \tau_a^* \hat{T},$$

where for each $a \in \mathbb{Z}^4$, we denote by $\tau_a$ the translation $x \mapsto x - a$. 
Moreover they kill $o_{2,\varepsilon}, o_{3,\varepsilon}$ but respect $o_{1,\varepsilon}$.

The symmetries which fix $\mathbb{Z}_4$ respect orientation and the other ones do not.

Of the 10-parameter family of deformations of the torus only scaling is allowed by $\mathcal{G}$.

We define the singular solution by

$$H = \frac{1}{2} \sum_{a \in \mathbb{Z}_4^{\text{even}}} \tau_a^* T + \frac{1}{2} \sum_{a \in \mathbb{Z}_4^{\text{odd}}} \tau_a^* \hat{T},$$

where for each $a \in \mathbb{Z}_4$, we denote by $\tau_a$ the translation $x \mapsto x - a$.

Although each series does not converge absolutely, $H$ is well defined because there are cancellations in the sense

$$\tau_{(a_1,a_2,a_3,a_4)}^* T(x) + \tau_{(a_3,-a_4,-a_1,a_2)}^* T(x) = O(|a|^{-5}),$$

$$\tau_{(a_1,a_2,a_3,a_4)}^* \hat{T}(x) + \tau_{(a_3,a_4,-a_1,-a_2)}^* \hat{T}(x) = O(|a|^{-5})$$

as $|a| \to \infty$. 
We define now the approximately Ricci-flat metric $\bar{g}_{\varepsilon,\delta}$ depending on two given positive numbers: $\varepsilon$ (the scale of the EH) and $\delta$ (the scale of the annulus where the gluing occurs), where we assume $\varepsilon \ll \delta^4 \ll 1$. 
We define now the approximately Ricci-flat metric $\bar{g}_{\varepsilon,\delta}$ depending on two given positive numbers: $\varepsilon$ (the scale of the EH) and $\delta$ (the scale of the annulus where the gluing occurs), where we assume $\varepsilon \ll \delta^4 \ll 1$.

On $B_{\delta/2}^{in} := \{x : |x| \leq \frac{1}{2} \delta \}$, we define $\bar{g}_{\varepsilon,\delta} = g_{eh,\varepsilon}$.
We define now the approximately Ricci-flat metric $\bar{g}_{\varepsilon, \delta}$ depending on two given positive numbers: $\varepsilon$ (the scale of the EH) and $\delta$ (the scale of the annulus where the gluing occurs), where we assume $\varepsilon \ll \delta^4 \ll 1$.

On $B_{\delta/2}^{in} := \{x : |x| \leq \frac{1}{2} \delta \}$, we define $\bar{g}_{\varepsilon, \delta} = g_{eh, \varepsilon}$.

On $B_{\delta}^{out} := [-\frac{1}{2}, \frac{1}{2}]^4 \setminus \{x : |x| \leq \delta \}$, we define

$$\bar{g}_{\varepsilon, \delta} = g_{eucl} + \varepsilon^4 H.$$
We define now the approximately Ricci-flat metric $\bar{g}_{\varepsilon, \delta}$ depending on two given positive numbers: $\varepsilon$ (the scale of the EH) and $\delta$ (the scale of the annulus where the gluing occurs), where we assume $\varepsilon \ll \delta^4 \ll 1$.

On $B_{\delta/2}^{in} := \{ x : |x| \leq \frac{1}{2} \delta \}$, we define $\bar{g}_{\varepsilon, \delta} = g_{\text{eh}, \varepsilon}$.

On $B_{\delta}^{out} := [-\frac{1}{2}, \frac{1}{2}]^4 \setminus \{ x : |x| \leq \delta \}$, we define

$$\bar{g}_{\varepsilon, \delta} = g_{\text{eucl}} + \varepsilon^4 H.$$ 

Finally, in the intermediate region $A_{\delta} := \{ x : \frac{1}{2} \delta \leq |x| \leq \delta \}$, we define

$$\bar{g}_{\varepsilon, \delta} = (1 - \chi(|x|/\delta)) g_{\text{eh}, \varepsilon} + \chi(|x|/\delta) \left( g_{\text{eucl}} + \varepsilon^4 H \right),$$

where $\chi$ is a cutoff function satisfying $\chi = 0$ on $(-\infty, \frac{2}{3}]$ and $\chi = 0$ on $[\frac{5}{6}, \infty)$.
Clearly $H$ is invariant under $\mathcal{G}$. 
Clearly $H$ is invariant under $\mathcal{G}$.

We can extend then $\bar{g}_{\varepsilon,\delta}$ smoothly to a metric on $\mathbb{R}^4 \setminus \mathbb{Z}^4$ which is invariant under $\mathcal{G}$.
Clearly $H$ is invariant under $G$.

We can extend then $\bar{g}_{\varepsilon,\delta}$ smoothly to a metric on $\mathbb{R}^4 \setminus \mathbb{Z}^4$ which is invariant under $G$.

The resulting metric $\bar{g}_{\varepsilon,\delta}$ on $\mathbb{R}^4 \setminus \mathbb{Z}^4$ is singular at each lattice point.
Clearly $H$ is invariant under $\mathcal{G}$.

We can extend then $\bar{g}_{\varepsilon,\delta}$ smoothly to a metric on $\mathbb{R}^4 \setminus \mathbb{Z}^4$ which is invariant under $\mathcal{G}$.

The resulting metric $\bar{g}_{\varepsilon,\delta}$ on $\mathbb{R}^4 \setminus \mathbb{Z}^4$ is singular at each lattice point.

In fact, if $a \in \mathbb{Z}^4_{\text{even}}$, then we have $\bar{g}_{\varepsilon,\delta} = \tau_a^* g_{\text{eh},\varepsilon}$ in a neighborhood of $a$. 
Clearly \( H \) is invariant under \( \mathcal{G} \).

We can extend then \( \bar{g}_{\varepsilon, \delta} \) smoothly to a metric on \( \mathbb{R}^4 \setminus \mathbb{Z}^4 \) which is invariant under \( \mathcal{G} \).

The resulting metric \( \bar{g}_{\varepsilon, \delta} \) on \( \mathbb{R}^4 \setminus \mathbb{Z}^4 \) is singular at each lattice point.

In fact, if \( a \in \mathbb{Z}_{\text{even}}^4 \), then we have \( \bar{g}_{\varepsilon, \delta} = \tau_a^* g_{eh, \varepsilon} \) in a neighborhood of \( a \).

Similarly, if \( a \in \mathbb{Z}_{\text{odd}}^4 \), then we have \( \bar{g}_{\varepsilon, \delta} = \tau_a^* \hat{g}_{eh, \varepsilon} \) in a neighborhood of \( a \).
Clearly $H$ is invariant under $\mathcal{G}$.

We can extend then $\bar{g}_{\varepsilon,\delta}$ smoothly to a metric on $\mathbb{R}^4 \setminus \mathbb{Z}^4$ which is invariant under $\mathcal{G}$.

The resulting metric $\bar{g}_{\varepsilon,\delta}$ on $\mathbb{R}^4 \setminus \mathbb{Z}^4$ is singular at each lattice point.

In fact, if $a \in \mathbb{Z}^4_{\text{even}}$, then we have $\bar{g}_{\varepsilon,\delta} = \tau^*_a g_{eh,\varepsilon}$ in a neighborhood of $a$.

Similarly, if $a \in \mathbb{Z}^4_{\text{odd}}$, then we have $\bar{g}_{\varepsilon,\delta} = \tau^*_a \hat{g}_{eh,\varepsilon}$ in a neighborhood of $a$.

Thus, if we take the quotient by translations and antipodal reflection, then the metric $\bar{g}_{\varepsilon,\delta}$ descends to a smooth metric on the quotient manifold $\mathcal{M}$. 
On $B^{in}_{\delta/2}$ we have $\text{Ric}_{\bar{g}_{\varepsilon, \delta}} = 0$. 
On $B^\text{in}_{\delta/2}$ we have $\text{Ric}_{\bar{g}_{\epsilon,\delta}} = 0$.

In the intermediate region $A_\delta$ we have $|\text{Ric}_{\bar{g}_{\epsilon,\delta}}| \leq C \epsilon^4 \delta^{-2}$.
On $B_{\delta/2}^{in}$ we have $\text{Ric}_{\bar{g}_{\varepsilon,\delta}} = 0$.

In the intermediate region $A_\delta$ we have $|\text{Ric}_{\bar{g}_{\varepsilon,\delta}}| \leq C \varepsilon^4 \delta^{-2}$.

For $x \in B_\delta^{out}$ we have $|\text{Ric}_{\bar{g}_{\varepsilon,\delta}}| \leq C \varepsilon^8 |x|^{-10}$.
On $B_{\delta/2}^{in}$ we have $\text{Ric}_{\bar{g}_{\varepsilon,\delta}} = 0$.

In the intermediate region $A_{\delta}$ we have $|\text{Ric}_{\bar{g}_{\varepsilon,\delta}}| \leq C \varepsilon^4 \delta^{-2}$.

For $x \in B_{\delta}^{out}$ we have $|\text{Ric}_{\bar{g}_{\varepsilon,\delta}}| \leq C \varepsilon^8 |x|^{-10}$.

Analogous estimates hold for the derivatives of $\text{Ric}_{\bar{g}_{\varepsilon,\delta}}$. 
On $B^{in}_{\delta/2}$ we have $\text{Ric}_{\bar{g}_{\varepsilon,\delta}} = 0$.

In the intermediate region $A_{\delta}$ we have $|\text{Ric}_{\bar{g}_{\varepsilon,\delta}}| \leq C \varepsilon^4 \delta^{-2}$.

For $x \in B^{out}_{\delta}$ we have $|\text{Ric}_{\bar{g}_{\varepsilon,\delta}}| \leq C \varepsilon^8 |x|^{-10}$.

Analogous estimates hold for the derivatives of $\text{Ric}_{\bar{g}_{\varepsilon,\delta}}$.

We find it useful to denote by $\bar{o}_{1,\varepsilon,\delta}$ the trace-free part of the tensor $\frac{1}{2} \varepsilon \frac{\partial}{\partial \varepsilon} \bar{g}_{\varepsilon,\delta}$ with respect to the metric $\bar{g}_{\varepsilon,\delta}$.
On $B_{\delta/2}^{in}$ we have $\text{Ric}_{\bar{g}_{\varepsilon,\delta}} = 0$.

In the intermediate region $A_{\delta}$ we have $|\text{Ric}_{\bar{g}_{\varepsilon,\delta}}| \leq C \varepsilon^4 \delta^{-2}$.

For $x \in B_{\delta}^{out}$ we have $|\text{Ric}_{\bar{g}_{\varepsilon,\delta}}| \leq C \varepsilon^8 |x|^{-10}$.

Analogous estimates hold for the derivatives of $\text{Ric}_{\bar{g}_{\varepsilon,\delta}}$.

We find it useful to denote by $\bar{o}_{1,\varepsilon,\delta}$ the trace-free part of the tensor $\frac{1}{2} \varepsilon \frac{\partial}{\partial \varepsilon} \bar{g}_{\varepsilon,\delta}$ with respect to the metric $\bar{g}_{\varepsilon,\delta}$.

Clearly, $\bar{o}_{1,\varepsilon,\delta} = o_{1,\varepsilon}$ on $B_{\delta/2}^{in}$, so we can think of $\bar{o}_{1,\varepsilon,\delta}$ as an extension of $o_{1,\varepsilon}$ to the manifold $M$. 
On $B_{\delta/2}^{in}$ we have $\text{Ric}_{\bar{g}_{\varepsilon,\delta}} = 0$.

In the intermediate region $A_{\delta}$ we have $|\text{Ric}_{\bar{g}_{\varepsilon,\delta}}| \leq C \varepsilon^4 \delta^{-2}$.

For $x \in B_{\delta}^{out}$ we have $|\text{Ric}_{\bar{g}_{\varepsilon,\delta}}| \leq C \varepsilon^8 |x|^{-10}$.

Analogous estimates hold for the derivatives of $\text{Ric}_{\bar{g}_{\varepsilon,\delta}}$.

We find it useful to denote by $\bar{o}_{1,\varepsilon,\delta}$ the trace-free part of the tensor $\frac{1}{2} \varepsilon \frac{\partial}{\partial \varepsilon} \bar{g}_{\varepsilon,\delta}$ with respect to the metric $\bar{g}_{\varepsilon,\delta}$.

Clearly, $\bar{o}_{1,\varepsilon,\delta} = o_{1,\varepsilon}$ on $B_{\delta/2}^{in}$, so we can think of $\bar{o}_{1,\varepsilon,\delta}$ as an extension of $o_{1,\varepsilon}$ to the manifold $M$.

$\Delta_{L,\bar{g}_{\varepsilon,\delta}} \bar{o}_{1,\varepsilon,\delta}$ satisfies the same estimates with $\text{Ric}_{\bar{g}_{\varepsilon,\delta}}$ above and $\bar{o}_{1,\varepsilon,\delta}$ can be used as approximate kernel.
We have the main estimate

$$-2 \int_{[-\frac{1}{2}, \frac{1}{2}]^4 \setminus \{0\}} \langle \tilde{\sigma}_1, \varepsilon, \delta \rangle, \text{Ric} \tilde{g}_\varepsilon, \delta \rangle \tilde{g}_\varepsilon, \delta \; d\text{vol} \tilde{g}_\varepsilon, \delta = 32 \pi^2 \omega \varepsilon^8 + O(\varepsilon^8 \delta^2).$$

where

$$\omega := \sum_{a \in \mathbb{Z}_\text{odd}^4} |a|^{-10} (|a|^4 - 6 (a_1^2 + a_2^2) (a_3^2 + a_4^2)) \approx 7.70.$$
We have the main estimate
\[-2 \int_{[-\frac{1}{2}, \frac{1}{2}]^4 \setminus \{0\}} \langle \tilde{\sigma}_1, \varepsilon, \delta, \text{Ric}_{\tilde{g}_{\varepsilon, \delta}} \rangle_{\tilde{g}_{\varepsilon, \delta}} \, d\text{vol}_{\tilde{g}_{\varepsilon, \delta}} = 32\pi^2 \omega \varepsilon^8 + O(\varepsilon^8 \delta^2),\]
where
\[\omega := \sum_{a \in \mathbb{Z}^4_{\text{odd}}} |a|^{-10} (|a|^4 - 6 (a_1^2 + a_2^2) (a_3^2 + a_4^2)) \approx 7.70.\]

The nonvanishing of the right hand side (equivalently of \(\omega\)) shows that we cannot correct to Ricci-flatness.
We have the main estimate

\[-2 \int_{[-\frac{1}{2}, \frac{1}{2}]^4 \setminus \{0\}} \langle \bar{o}_1, \varepsilon, \delta, \operatorname{Ric} \bar{g}_\varepsilon, \delta \rangle \bar{g}_\varepsilon, \delta \ d\operatorname{vol} \bar{g}_\varepsilon, \delta = 32\pi^2 \omega \varepsilon^8 + O(\varepsilon^8 \delta^2).\]

where

\[\omega := \sum_{a \in \mathbb{Z}^4_{\text{odd}}} |a|^{-10} (|a|^4 - 6 (a_1^2 + a_2^2) (a_3^2 + a_4^2)) \approx 7.70.\]

The nonvanishing of the right hand side (equivalently of \(\omega\)) shows that we cannot correct to Ricci-flatness.

**Proof of the estimate.** Working modulo error terms of appropriate side and using Green's second identity we find that the left hand side equals \(\frac{1}{2} \varepsilon^8 \times\)

\[
\left( \sum_{a \in \mathbb{Z}^4_{\text{even}} \setminus \{0\}} \int_{\{|x|=\delta\}} \left( \langle T, D_\nu(\tau_a^* T) \rangle_{\operatorname{eucl}} - \langle \tau_a^* T, D_\nu T \rangle_{\operatorname{eucl}} \right) d\mu_{\operatorname{eucl}} \right)
+ \sum_{a \in \mathbb{Z}^4_{\text{odd}}} \int_{\{|x|=\delta\}} \left( \langle T, D_\nu(\tau_a^* \hat{T}) \rangle_{\operatorname{eucl}} - \langle \tau_a^* \hat{T}, D_\nu T \rangle_{\operatorname{eucl}} \right) d\mu_{\operatorname{eucl}}.\]
Using Green’s second identity again we evaluate each boundary integral as a linear combination the values of second derivatives of the coefficients of the coefficients in the expansions of $\tau^*_a T$ or $\tau^*_a \hat{T}$ respectively.

It turns out that the integrals with $a \in \mathbb{Z}_4$ even all vanish confirming that the interaction of EH of the same orientation does not create an obstruction.

The integral with $a \in \mathbb{Z}_4$ odd turns out to equal $64\pi^2 |a| - 10 (|a|^4 - 6 (a^2_1 + a^2_2)(a^2_3 + a^2_4))$ completing the proof and confirming that an EH neck "sees" necks of the opposite orientation.

By the definition of $\bar{o}_1$, $\varepsilon$, $\delta$, the $L^2$ norm of $\bar{g}_{\varepsilon,\delta}$ that on $B$ in $\delta/2$ we have $\bar{g}_{\varepsilon,\delta} = g_{eh,\varepsilon}$, $\partial / \partial \varepsilon \bar{g}_{\varepsilon,\delta} = 2 \varepsilon \bar{o}_1$, $\partial / \partial \varepsilon \bar{g}_{\varepsilon,\delta} = \bar{g}_{\varepsilon,\delta}$, and $\bar{o}_1$, $\varepsilon$, $\delta$ estimates on the error terms we have

$$\int_{-\frac{1}{2}, \frac{1}{2}} 4 \{0\} \langle \bar{o}_1, \varepsilon, \delta \rangle \bar{g}_{\varepsilon,\delta} d vol \bar{g}_{\varepsilon,\delta} = 4\pi^2 \varepsilon^3 + O(\varepsilon^7 \delta^{-4}).$$
Using Green’s second identity again we evaluate each boundary integral as a linear combination the values of second derivatives of the coefficients of the coefficients in the expansions of $\tau_a^* T$ or $\tau_a^* \hat{T}$ respectively.

It turns out that the integrals with $a \in \mathbb{Z}_4^{\text{even}}$ all vanish confirming that the interaction of EH of the same orientation does not create an obstruction.
Using Green’s second identity again we evaluate each boundary integral as a linear combination the values of second derivatives of the coefficients of the coefficients in the expansions of $\tau_* T$ or $\tau_* \hat{T}$ respectively.

It turns out that the integrals with $a \in \mathbb{Z}_4^{\text{even}}$ all vanish confirming that the interaction of EH of the same orientation does not create an obstruction.

The integral with $a \in \mathbb{Z}_4^{\text{odd}}$ turns out to equal $64\pi^2 |a|^{-10} (|a|^4 - 6 (a_1^2 + a_2^2)(a_3^2 + a_4^2))$ completing the proof and confirming that an EH neck “sees” necks of the opposite orientation.
Using Green’s second identity again we evaluate each boundary integral as a linear combination the values of second derivatives of the coefficients of the coefficient in the expansions of $\tau_a^* T$ or $\tau_a^* \hat{T}$ respectively.

It turns out that the integrals with $a \in \mathbb{Z}_4^{\text{even}}$ all vanish confirming that the interaction of EH of the same orientation does not create an obstruction.

The integral with $a \in \mathbb{Z}_4^{\text{odd}}$ turns out to equal $64\pi^2 |a|^{-10} (|a|^4 - 6 (a_1^2 + a_2^2) (a_3^2 + a_4^2))$ completing the proof and confirming that an EH neck “sees” necks of the opposite orientation.

By the definition of $\bar{o}_{1,\epsilon,\delta}$, the $L^2$ norm of $\bar{o}_{1,\epsilon}$, that on $B_{\delta/2}^{\text{in}}$ we have $\bar{g}_{\epsilon,\delta} = g_{\text{eh},\epsilon}$, $\frac{\partial}{\partial \epsilon} \bar{g}_{\epsilon,\delta} = \frac{2}{\epsilon} o_{1,\epsilon}$, and $\bar{o}_{1,\epsilon,\delta} = o_{1,\epsilon}$, and estimates on the error terms we have

$$\int_{\left[-\frac{1}{2}, \frac{1}{2}\right]^4 \setminus \{0\}} \left\langle \bar{o}_{1,\epsilon,\delta}, \frac{\partial}{\partial \epsilon} \bar{g}_{\epsilon,\delta} \right\rangle \bar{g}_{\epsilon,\delta} \, d\text{vol} \bar{g}_{\epsilon,\delta} = 4\pi^2 \epsilon^3 + O(\epsilon^7 \delta^{-4}).$$
We want to study now how the size of the EH necks would evolve by the Ricci flow.
We want to study now how the size of the EH necks would evolve by the Ricci flow.

Projecting the Ricci flow equation to $\bar{\omega}_1, \varepsilon, \delta$ and using the main estimate and the last estimate we find

$$\frac{d}{dt} \varepsilon = 8\omega \varepsilon^5 + o(\varepsilon^5).$$

The positivity of $\omega$ then drives the necks to grow.

Solving this heuristically derived ODE we get solutions $\varepsilon(t)$ for $t \in (-\infty, T]$ with $\varepsilon(t) \to 0$ as $t \to -\infty$.

This motivates the following Theorem.
We want to study now how the size of the EH necks would evolve by the Ricci flow.

Projecting the Ricci flow equation to $\bar{\mathcal{O}}_{1,\varepsilon,\delta}$ and using the main estimate and the last estimate we find

$$\frac{d}{dt} \varepsilon = 8 \omega \varepsilon^5 + o(\varepsilon^5).$$

The positivity of $\omega$ then drives the necks to grow.
We want to study now how the size of the EH necks would evolve by the Ricci flow.

Projecting the Ricci flow equation to $\bar{\sigma}_1, \varepsilon, \delta$ and using the main estimate and the last estimate we find

$$\frac{d}{dt} \varepsilon = 8\omega \varepsilon^5 + o(\varepsilon^5).$$

The positivity of $\omega$ then drives the necks to grow.

Solving this heuristically derived ODE we get solutions $\varepsilon(t)$ for $t \in (-\infty, T]$ with $\varepsilon(t) \to 0$ as $t \to -\infty$. 
We want to study now how the size of the EH necks would evolve by the Ricci flow.

Projecting the Ricci flow equation to $\bar{\sigma}_1, \varepsilon, \delta$ and using the main estimate and the last estimate we find

$$\frac{d}{dt} \varepsilon = 8 \omega \varepsilon^5 + o(\varepsilon^5).$$

The positivity of $\omega$ then drives the necks to grow.

Solving this heuristically derived ODE we get solutions $\varepsilon(t)$ for $t \in (-\infty, T]$ with $\varepsilon(t) \to 0$ as $t \to -\infty$.

This motivates the following Theorem.
Theorem

There exists a compact ancient solution to the Ricci flow in dimension 4 symmetric under the group $\mathcal{G}$ defined above and with the following property.

For $-t$ sufficiently large, the manifold can be viewed as a desingularization of a flat torus with 16 orbifold points.

More precisely, we divide the 16 orbifold points into two classes according to a checkerboard pattern.

Near 8 orbifold points, the metric is a small perturbation of a positively-oriented near the remaining 8 orbifold points, the metric is a small perturbation of a negatively-oriented Eguchi-Hanson metric.

As $t \to -\infty$, the size of the Eguchi-Hanson instantons shrinks to zero, and we have $\sup |R_{g}(t)| g(t) = (c + o(1)) \frac{1}{2}$, where $c$ is a positive constant.

Finally, the Ricci curvature of $g(t)$ satisfies $\sup |\text{Ric}_{g}(t)| g(t) = O \left( \left( -t^{-1/2} + \kappa \right) \right)$ as $t \to -\infty$, where $\kappa > 0$ can be chosen arbitrarily small.
Theorem
There exists a compact ancient solution to the Ricci flow in dimension 4 symmetric under the group \( G \) defined above and with the following property. For \(-t\) sufficiently large, the manifold can be viewed as a desingularization of a flat torus with 16 orbifold points.

More precisely, we divide the 16 orbifold points into two classes according to a checkerboard pattern. Near 8 orbifold points, the metric is a small perturbation of a positively-oriented metric near the remaining 8 orbifold points, the metric is a small perturbation of a negatively-oriented Eguchi-Hanson metric. As \( t \to -\infty \), the size of the Eguchi-Hanson instantons shrinks to zero, and we have

\[
\sup |R_m g(t)| g(t) = (c + o(1)) (-t)^{1/2},
\]

where \( c \) is a positive constant.

Finally, the Ricci curvature of \( g(t) \) satisfies

\[
\sup |Ric g(t)| g(t) = O\left(\left(\frac{-1}{2} + \kappa\right)^{1/2}\right)
\]
as \( t \to -\infty \), where \( \kappa > 0 \) can be chosen arbitrarily small.
Theorem

There exists a compact ancient solution to the Ricci flow in dimension 4 symmetric under the group $G$ defined above and with the following property. For $-t$ sufficiently large, the manifold can be viewed as a desingularization of a flat torus with 16 orbifold points. More precisely, we divide the 16 orbifold points into two classes according to a checkerboard pattern.
Theorem
There exists a compact ancient solution to the Ricci flow in dimension 4 symmetric under the group $\mathcal{G}$ defined above and with the following property. For $-t$ sufficiently large, the manifold can be viewed as a desingularization of a flat torus with 16 orbifold points. More precisely, we divide the 16 orbifold points into two classes according to a checkerboard pattern. Near 8 orbifold points, the metric is a small perturbation of a positively-oriented metric near the remaining 8 orbifold points, the metric is a small perturbation of a negatively-oriented Eguchi-Hanson metric.
Theorem
There exists a compact ancient solution to the Ricci flow in dimension 4 symmetric under the group $G$ defined above and with the following property. For $-t$ sufficiently large, the manifold can be viewed as a desingularization of a flat torus with 16 orbifold points. More precisely, we divide the 16 orbifold points into two classes according to a checkerboard pattern. Near 8 orbifold points, the metric is a small perturbation of a positively-oriented metric near the remaining 8 orbifold points, the metric is a small perturbation of a negatively-oriented Eguchi-Hanson metric. As $t \to -\infty$, the size of the Eguchi-Hanson instantons shrinks to zero, and we have $\sup |Rm_{g(t)}|_{g(t)} = (c + o(1))(-t)^{1/2}$, where $c$ is a positive constant.
Theorem
There exists a compact ancient solution to the Ricci flow in dimension 4 symmetric under the group $G$ defined above and with the following property. For $-t$ sufficiently large, the manifold can be viewed as a desingularization of a flat torus with 16 orbifold points. More precisely, we divide the 16 orbifold points into two classes according to a checkerboard pattern. Near 8 orbifold points, the metric is a small perturbation of a positively-oriented near the remaining 8 orbifold points, the metric is a small perturbation of a negatively-oriented Eguchi-Hanson metric. As $t \to -\infty$, the size of the Eguchi-Hanson instantons shrinks to zero, and we have $\sup |Rm_{g(t)}|_{g(t)} = (c + o(1)) (-t)^{\frac{1}{2}}$, where $c$ is a positive constant. Finally, the Ricci curvature of $g(t)$ satisfies $\sup |Ric_{g(t)}|_{g(t)} = O((-t)^{-\frac{1}{2} + \kappa})$ as $t \to -\infty$, where $\kappa > 0$ can be chosen arbitrarily small.
The strategy of the proof is to correct the approximate ancient solution suggested earlier to an exact solution by using the Schauder fixed point theorem in a way similar to gluing constructions.
The strategy of the proof is to correct the approximate ancient solution suggested earlier to an exact solution by using the Schauder fixed point theorem in a way similar to gluing constructions.

The approximate solution we perturb is \( \bar{g}(-32\omega t)^{-\frac{1}{4}}, (-t)^{-\frac{1}{400}} \) defined for \( t \in (-\infty, -\Lambda] \).
The strategy of the proof is to correct the approximate ancient solution suggested earlier to an exact solution by using the Schauder fixed point theorem in a way similar to gluing constructions.

The approximate solution we perturb is $\bar{g}(-32\omega t)^{-\frac{1}{4}}(\omega t)^{-\frac{1}{400}}$, defined for $t \in (-\infty, -\Lambda]$.

The perturbation is controlled by a tensor field $k$ on $M \times (-\infty, -\Lambda]$ invariant under $\mathcal{G}$ and two real valued functions $\eta$ and $\beta$ on $(-\infty, -\Lambda]$. 

$\eta$ modifies the scale $\epsilon(t)$ by the equation $\epsilon(t) = (-32\omega t + \int_{-\Lambda}^{t} \eta(s) ds)^{-\frac{1}{4}}$ and its purpose is to compensate for the error term in the direction of the approximate kernel. It plays the role of "unbalancing" parameters in gluing problems corresponding to the obstruction of EH scaling.
The strategy of the proof is to correct the approximate ancient solution suggested earlier to an exact solution by using the Schauder fixed point theorem in a way similar to gluing constructions.

The approximate solution we perturb is \( \bar{g} \left( -32 \omega t \right)^{-\frac{1}{4}}, (-t)^{-\frac{1}{400}} \) defined for \( t \in (-\infty, -\Lambda] \).

The perturbation is controlled by a tensor field \( k \) on \( M \times (-\infty, -\Lambda] \) invariant under \( G \) and two real valued functions \( \eta \) and \( \beta \) on \( (-\infty, -\Lambda] \).

\( \eta \) modifies the scale \( \varepsilon(t) \) by the equation

\[
\varepsilon(t) = \left( -32 \omega t + \int_t^{-\Lambda} \eta(s) \, ds \right)^{-\frac{1}{4}}
\]

and its purpose is to compensate for the error term in the direction of the approximate kernel. It plays the role of “unbalancing” parameters in gluing problems corresponding to the obstruction of EH scaling.
\(\beta\) effectively modifies the scale of the underlying orbifold which is allowed by \(\mathcal{G}\). It plays the role of “unbalancing” parameters in gluing problems corresponding to that obstruction. Its effect is minor however unlike the previous one.
- β effectively modifies the scale of the underlying orbifold which is allowed by $G$. It plays the role of “unbalancing” parameters in gluing problems corresponding to that obstruction. Its effect is minor however unlike the previous one.

- $k$ modifies the metric $\bar{g}_{\epsilon}(t), \delta(t)$ where $\delta(t) = (-t)^{-\frac{1}{400}}$. 
\(\beta\) effectively modifies the scale of the underlying orbifold which is allowed by \(\mathcal{G}\). It plays the role of “unbalancing” parameters in gluing problems corresponding to that obstruction. Its effect is minor however unlike the previous one.

\(k\) modifies the metric \(\bar{g}_{\varepsilon(t),\delta(t)}\) where \(\delta(t) = (-t)^{-\frac{1}{400}}\).

To apply the Schauder fixed point theorem we define a map \(\mathcal{J}\) which sends a triplet \((k, \eta(\cdot), \beta(\cdot))\) \(\in \mathcal{A}^\alpha_{\gamma,\sigma,\Lambda}\) to the triplet \((h, \xi(\cdot), \nu(\cdot))\).
\( \beta \) effectively modifies the scale of the underlying orbifold which is allowed by \( \mathcal{G} \). It plays the role of “unbalancing” parameters in gluing problems corresponding to that obstruction. Its effect is minor however unlike the previous one.

\( k \) modifies the metric \( \bar{g}_{\varepsilon(t), \delta(t)} \) where \( \delta(t) = (-t)^{-\frac{1}{400}} \).

To apply the Schauder fixed point theorem we define a map \( \mathcal{J} \) which sends a triplet \( (k, \eta(\cdot), \beta(\cdot)) \in \mathcal{A}_{\gamma, \sigma, \Lambda}^\alpha \) to the triplet \( (h, \xi(\cdot), \nu(\cdot)) \).

\( \mathcal{J} \) is constructed in such a way that if \( (k, \eta(\cdot), \beta(\cdot)) \) is a fixed point, then the metrics \( g(t) \) satisfy

\[
g(t) := e^{\int_t^{-\Lambda} \beta(s) \, ds} \left( \bar{g}_{\varepsilon(t), \delta(t)} + k(t) \right),
\]

satisfy

\[
\frac{\partial}{\partial t} g(t) = -2 \text{Ric}_g(t) + \mathcal{L}_Y g(t),
\]

where \( Y = \text{div}_{\bar{g}_{\varepsilon(t), \delta(t)}} k - \frac{1}{2} \nabla \text{tr}_{\bar{g}_{\varepsilon(t), \delta(t)}} k \).
By pulling back the metrics $g(t)$ under the flow of
diffeomorphisms generated by $Y(t)$, we obtain a solution to
the Ricci flow which is defined for $t \in (-\infty, -\Lambda]$. 

As in the elliptic case the definition and estimation of $J$
involves solving a linear inhomogeneous equation modulo (or
orthogonally to) the kernel with appropriate estimates and
estimating the quadratic terms.

The quadratic terms are defined for a given metric $g$
and a symmetric $(0, 2)$-tensor $k$ satisfying $|k|_g \leq 1/2$
by $Q_g(k) = 2\text{Ric}_g + k - 2\text{Ric}_g + \Delta L$
where $Y = \text{div} g k - 1/2 \nabla \text{tr} g k$.

Note that $Q_g(k)$ can be expanded as
$Q_g(k) = k^* \nabla^2 k + \nabla k^* \nabla k + Rm^*(k^* k) + \text{higher order terms}$.

The linear results are based on a blow-up approach as in
results of Mazzeo-Pacard, together with the Liouville-type
theorems based on understanding the kernel.
By pulling back the metrics $g(t)$ under the flow of
diffeomorphisms generated by $Y(t)$, we obtain a solution to
the Ricci flow which is defined for $t \in (-\infty, -\Lambda]$.

As in the elliptic case the definition and estimation of $J$
involves solving a linear inhomogeneous equation modulo (or
orthogonally to) the kernel with appropriate estimates and
estimating the quadratic terms.
By pulling back the metrics $g(t)$ under the flow of diffeomorphisms generated by $Y(t)$, we obtain a solution to the Ricci flow which is defined for $t \in (-\infty, -\Lambda]$.

As in the elliptic case the definition and estimation of $\mathcal{J}$ involves solving a linear inhomogeneous equation modulo (or orthogonally to) the kernel with appropriate estimates and estimating the quadratic terms.

The quadratic terms are defined for a given metric $g$ and a symmetric $(0, 2)$-tensor $k$ satisfying $|k|_g \leq \frac{1}{2}$, by

$$Q_g(k) = 2 \text{Ric}_{g+k} - 2 \text{Ric}_g + \Delta_{L,g} k - \mathcal{L}_Y(g + k),$$

where $Y = \text{div}_g k - \frac{1}{2} \nabla \text{tr}_g k$. 

Note that $Q_g(k)$ can be expanded as

$$Q_g(k) = k^\ast \nabla_2 k + \nabla k^\ast \nabla k + Rm_{g+k} k^\ast k + \text{higher order terms}.$$
By pulling back the metrics $g(t)$ under the flow of diffeomorphisms generated by $Y(t)$, we obtain a solution to the Ricci flow which is defined for $t \in (-\infty, -\Lambda]$.

As in the elliptic case the definition and estimation of $\mathcal{J}$ involves solving a linear inhomogeneous equation modulo (or orthogonally to) the kernel with appropriate estimates and estimating the quadratic terms.

The quadratic terms are defined for a given metric $g$ and a symmetric $(0, 2)$-tensor $k$ satisfying $|k|_g \leq \frac{1}{2}$, by

$$Q_g(k) = 2 \text{Ric}_g + k - 2 \text{Ric}_g + \Delta_{L,g} k - \mathcal{L}_Y (g + k),$$

where $Y = \text{div}_g k - \frac{1}{2} \nabla \text{tr}_g k$.

Note that $Q_g(k)$ can be expanded as

$$Q_g(k) = k \ast \nabla^2 k + \nabla k \ast \nabla k + \text{Rm}_g \ast k \ast k + \text{higher order terms}.$$
By pulling back the metrics $g(t)$ under the flow of diffeomorphisms generated by $Y(t)$, we obtain a solution to the Ricci flow which is defined for $t \in (-\infty, -\Lambda]$.

As in the elliptic case the definition and estimation of $\mathcal{J}$ involves solving a linear inhomogeneous equation modulo (or orthogonally to) the kernel with appropriate estimates and estimating the quadratic terms.

The quadratic terms are defined for a given metric $g$ and a symmetric $(0, 2)$-tensor $k$ satisfying $|k|_g \leq \frac{1}{2}$, by

$$Q_g(k) = 2 \text{Ric}_g + k - 2 \text{Ric}_g + \Delta_{L,g} k - \mathcal{L}_Y(g + k),$$

where $Y = \text{div}_g k - \frac{1}{2} \nabla \text{tr}_g k$.

Note that $Q_g(k)$ can be expanded as

$$Q_g(k) = k \ast \nabla^2 k + \nabla k \ast \nabla k + \text{Rm}_g \ast k \ast k + \text{higher order terms}.$$
Given real numbers $\gamma > 0$ and $\sigma \in (0, 2)$, we can find real numbers $\Lambda > 0$ and $C > 0$ with the following property (under reasonable assumptions). Let $h$ be a solution of the inhomogeneous heat equation $\frac{\partial}{\partial t} h(t) = \Delta_{L,\bar{\gamma}(t),\delta(t)} h(t) + \psi(t)$ which is defined on $M \times (-\infty, -\Lambda]$ and satisfies
\[
\sup_{M \times (-\infty, -\Lambda]} (-t)^\gamma (\varepsilon(t) + r)^\sigma |h(t)|_{\bar{\gamma}(t),\delta(t)} < \infty.
\]
Moreover, we assume that $h$ is invariant under the group $\mathcal{G}$ and satisfies the orthogonality conditions
\[
\int_{[-\frac{1}{2}, \frac{1}{2}]^4 \setminus \{0\}} \langle h(t), \bar{\mathcal{O}}_{1,\varepsilon(t),\delta(t)} \rangle_{\bar{\gamma}(t),\delta(t)} d\text{vol}_{\bar{\gamma}(t),\delta(t)} = 0
\]
and
\[
\int_{[-\frac{1}{2}, \frac{1}{2}]^4 \setminus \{0\}} \langle h(t), \bar{\mathcal{G}}_{\varepsilon(t),\delta(t)} \rangle_{\bar{\gamma}(t),\delta(t)} d\text{vol}_{\bar{\gamma}(t),\delta(t)} = 0
\]
for all $t \in (-\infty, -\Lambda]$. Then
\[
\sup_{M \times (-\infty, -\Lambda]} (-t)^\gamma ((-t)^{-\frac{1}{4}} + r)^\sigma |h(t)|_{\bar{\gamma}(t),\delta(t)} \leq C \sup_{M \times (-\infty, -\Lambda]} (-t)^\gamma ((-t)^{-\frac{1}{4}} + r)^{\sigma+2} |\psi(t)|_{\bar{\gamma}(t),\delta(t)}.
\]
Let \((M_{eh}, g_{eh})\) denote the Eguchi-Hanson manifold with parameter \(\varepsilon = 1\), and let \(o_i := o_{i,1}\). Let \(h\) be a solution of the heat equation
\[
\frac{\partial}{\partial t} h = \Delta_{L,g_{eh}} h
\]
on \(M_{eh} \times (-\infty, 0]\) with the property that
\[
|h| \leq (1 + r)^{-\sigma}
\]
for some \(\sigma > 0\). If \(\int_{M_{eh}} \langle h(t), o_i \rangle = 0\) for all \(i \in \{1, 2, 3\}\) and all \(t \in (-\infty, 0]\), then \(h\) vanishes identically.
FINAL COMMENTS

- It would be interesting to find out if there is a large neck size where we have balancing.
FINAL COMMENTS

- It would be interesting to find out if there is a large neck size where we have balancing.
- A related interesting open question is what happens to our ancient solution beyond the range of sufficiently large $-t$.
FINAL COMMENTS

- It would be interesting to find out if there is a large neck size where we have balancing.
- A related interesting open question is what happens to our ancient solution beyond the range of sufficiently large $-t$.
- Can numerics help?
FINAL COMMENTS

► It would be interesting to find out if there is a large neck size where we have balancing.

► A related interesting open question is what happens to our ancient solution beyond the range of sufficiently large $-t$.

► Can numerics help?

► An interesting observation is that if we consider less symmetric configurations we should be able to construct metrics by gluing 16 small Eguchi-Hanson metrics on the orbifold as before, which under the Ricci flow evolve so that one of the Eguchi-Hanson metrics becomes extinct by shrinking to zero size, while the rest stay close to fixed nonzero sizes.