Gluing Eguchi-Hanson Metrics and a Question of Page
(by Simon Brendle and NK)

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- In Brendle-Kapouleas: Comm. Pure Appl. Math., 70(7):1366-1401, 2017, we study a question Page asked in 1981:
- In 1978 Gibbons-Pope and Page in an effort to understand the geometry of the Ricci-flat metrics on the K3 surface proposed the following gluing construction for those metrics:
- Starting with a flat torus $T^4 = \mathbb{R}^4 \,/\, \mathbb{Z}^4$, quotient out further with a reflection with respect to a point. This produces a flat orbifold with 16 singular points modeled after $\mathbb{R}^4/\mathbb{Z}_2$.

- Resolve the singular points by attaching in their place Eguchi-Hansom (EH) manifolds scaled down by small factors.

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- EH manifolds are Ricci-flat hyper-Kähler four-manifolds which are topologically $T\mathbb{S}^2$ and they carry the metric

$$g_{\mathrm{eh},\varepsilon} = \frac{r^2}{(\varepsilon^4 + r^4)^{\frac{1}{2}}} \left(dr \otimes dr + r^2 \, \alpha_1 \otimes \alpha_1 \right) + (\varepsilon^4 + r^4)^{\frac{1}{2}} \left(\alpha_2 \otimes \alpha_2 + \alpha_3 \otimes \alpha_3 \right) \sim \varepsilon^2 g_{\mathrm{eh},1},$$

where

$$\alpha_1 = \frac{1}{r^2} \left(x_1 \, dx_2 - x_2 \, dx_1 + x_3 \, dx_4 - x_4 \, dx_3 \right),$$
$$\alpha_2 = \frac{1}{r^2} \left(x_1 \, dx_3 - x_3 \, dx_1 + x_4 \, dx_2 - x_2 \, dx_4 \right),$$
$$\alpha_3 = \frac{1}{r^2} \left(x_1 \, dx_4 - x_4 \, dx_1 + x_2 \, dx_3 - x_3 \, dx_2 \right).$$

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\end{aligned}
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- They are Asymptotically Locally Euclidean (ALE) with infinity asymptotically $\mathbb{R}^4/\mathbb{Z}^2$, hyper-Kähler, and self-dual.

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- This construction was carried out rigorously by Topiwala, LeBrun-Singer, and Donaldson.
- Since there is a unique Ricci-flat metric for given Kähler class there are no obstructions of the type we will discuss later.

- Page in 1981 asked the question whether modifying the construction by reversing the orientation of some but not all the EH manifolds being attached could produce new Ricci-flat non-Kähler metrics (known such metrics are extremely few).

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- The signature of the new manifolds is different and they cannot be Kähler.
- It turns out that for such a construction there are obstructions which have to be understood.
- In this direction the understanding of the kernel of the Lichnerowicz Laplacian by Biquard and Page is important. Recall that the Lichnerowicz Laplacian is defined by

$$\Delta_L h_{ik} = \Delta h_{ik} + 2R_{ijkl}h^{jl} - \mathrm{Ric}_i^l h_{kl} - \mathrm{Ric}_k^l h_{il}.$$

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- The approach (called *linearized doubling* in the minimal doubling case) constructs as an intermediate step singular solutions of the Jacobi equation on the given object (in our case the orbifold and in the minimal doubling case the surface being doubled).

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- The approach (called *linearized doubling* in the minimal doubling case) constructs as an intermediate step singular solutions of the Jacobi equation on the given object (in our case the orbifold and in the minimal doubling case the surface being doubled).
- The obstructions (that is balancing/unbalancing questions) can be (approximately) calculated in terms of these singular solutions, which are easier to understand, because they satisfy a *linear* instead of a *nonlinear* equation.

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- To discuss the construction in detail we let $\hat{g}_{\mathrm{eh},\varepsilon}$ denote the pull-back of $g_{\mathrm{eh},\varepsilon}$ under the map $(x_1, x_2, x_3, x_4) \mapsto (-x_1, x_2, x_3, x_4)$.

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- Clearly, $\hat{g}_{\mathrm{eh},\varepsilon}$ is a Ricci flat metric.

Near infinity, we have the asymptotic expansions

$$g_{\mathrm{eh},\varepsilon} = g_{\mathrm{eucl}} + \frac{1}{2}\,\varepsilon^4\, T + O(\varepsilon^8\, r^{-8})$$

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$$\begin{aligned} T &= -r^{-4}\,(dr\otimes dr + r^2\,\alpha_1\otimes\alpha_1 - r^2\,\alpha_2\otimes\alpha_2 - r^2\,\alpha_3\otimes\alpha_3) \\ &= -r^{-6}\,\big\{(x_1^2+x_2^2-x_3^2-x_4^2)\,(dx_1\otimes dx_1+dx_2\otimes dx_2-dx_3\otimes dx_3-dx_4\otimes dx_4) \\ &+ 2\,(x_1x_3+x_2x_4)\,(dx_1\otimes dx_3 + dx_3\otimes dx_1 + dx_2\otimes dx_4 + dx_4\otimes dx_2) \\ &+ 2\,(x_1x_4-x_2x_3)\,(dx_1\otimes dx_4 + dx_4\otimes dx_1 - dx_2\otimes dx_3 - dx_3\otimes dx_2)\big\} \end{aligned}$$

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- We note that the metrics $g_{\mathrm{eh},\varepsilon}$ and $\hat{g}_{\mathrm{eh},\varepsilon}$ and the tensors T and $\hat{T}$ are all invariant under the maps

$$\begin{aligned}
(x_1, x_2, x_3, x_4) &\mapsto (x_2, -x_1, x_3, x_4),\\
(x_1, x_2, x_3, x_4) &\mapsto (x_1, x_2, x_4, -x_3),\\
(x_1, x_2, x_3, x_4) &\mapsto (x_3, x_4, x_1, x_2),\\
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- To analyze the kernel of $\Delta_{L,g_{\mathrm{eh}}}$, we consider the frame $r\,\frac{\partial}{\partial r}, V_1, V_2, V_3$ dual to the co-frame $\frac{1}{r}\,dr, \alpha_1, \alpha_2, \alpha_3$.

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- Clearly

$$
\begin{aligned}
V_1 &= x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_4} - x_4 \frac{\partial}{\partial x_3},\\
V_2 &= x_1 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_1} + x_4 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_4},\\
V_3 &= x_1 \frac{\partial}{\partial x_4} - x_4 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_2}.
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- We next define

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\begin{aligned}
o_{1,\varepsilon} &= g_{\mathrm{eh},\varepsilon} - \frac{1}{2}\,\mathcal{L}_{r\frac{\partial}{\partial r}} g_{\mathrm{eh},\varepsilon} \\
&\qquad\qquad = \frac{1}{2}\,\varepsilon\,\frac{\partial}{\partial \varepsilon} g_{\mathrm{eh},\varepsilon} = \varepsilon^4\, T + O(\varepsilon^8\, r^{-8}), \\
o_{2,\varepsilon} &= \frac{1}{2}\,\mathcal{L}_{\frac{r^2 V_2}{\sqrt{\varepsilon^4 + r^4}}} g_{\mathrm{eh},\varepsilon}, \\
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- Equivalently,

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\begin{aligned}
o_{1,\varepsilon} &= -\frac{\varepsilon^4\, r^2}{(\varepsilon^4 + r^4)^{\frac{3}{2}}}\left(dr \otimes dr + r^2\,\alpha_1 \otimes \alpha_1\right) + \frac{\varepsilon^4}{(\varepsilon^4 + r^4)^{\frac{1}{2}}}\left(\alpha_2 \otimes \alpha_2 + \right. \\
o_{2,\varepsilon} &= \frac{\varepsilon^4}{\varepsilon^4 + r^4}\left(r\,dr \otimes \alpha_2 + r\,\alpha_2 \otimes dr - r^2\,\alpha_1 \otimes \alpha_3 - r^2\,\alpha_3 \otimes \alpha_1\right), \\
o_{3,\varepsilon} &= \frac{\varepsilon^4}{\varepsilon^4 + r^4}\left(r\,dr \otimes \alpha_3 + r\,\alpha_3 \otimes dr + r^2\,\alpha_1 \otimes \alpha_2 + r^2\,\alpha_2 \otimes \alpha_1\right).
\end{aligned}
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Proposition(O. Biquard 2013; D. Page 1978) For each $i \in \{1,2,3\}$, the tensor $o_{i,\varepsilon}$ has the following properties:

(i) $\mathrm{tr}_{g_{\mathrm{eh},\varepsilon}} o_{i,\varepsilon} = 0$.

(ii) $\mathrm{div}_{g_{\mathrm{eh},\varepsilon}} o_{i,\varepsilon} = 0$.

(iii) $\Delta_{L,g_{\mathrm{eh},\varepsilon}} o_{i,\varepsilon} = 0$.

(iv) $\int_{\mathbb{R}^4\setminus\{0\}} |o_{1,\varepsilon}|^2_{g_{\mathrm{eh},\varepsilon}} \, d\mathrm{vol}_{g_{\mathrm{eh},\varepsilon}} = 2\pi^2\, \varepsilon^4$.

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Proposition(O. Biquard 2013; O. Biquard, Y. Rollin 2009) Let $(M_{\mathrm{eh}}, g_{\mathrm{eh}})$ denote the Eguchi-Hanson manifold with parameter $\varepsilon = 1$, and let $o_i := o_{i,1}$. Then the following statements hold:

(i) Let h be a symmetric $(0,2)$-tensor on the Eguchi-Hanson manifold satisfying $|h| \le (1+r)^{-\sigma}$ for some $\sigma > 0$ and $\Delta_{L,g_{\mathrm{eh}}} h = 0$. Then $h \in \mathrm{span}\{o_1, o_2, o_3\}$.

(ii) Let h be a symmetric $(0,2)$-tensor on the Eguchi-Hanson manifold satisfying $|h| \le (1+r)^{-\sigma-1}$ and $|\nabla h| \le (1+r)^{-\sigma-2}$ for some $\sigma > 0$. Then

$$\int_{M_{\mathrm{eh}}} \langle \Delta_{L,g_{\mathrm{eh}}} h, h\rangle \, d\mathrm{vol}_{g_{\mathrm{eh}}} \le 0.$$

Moreover, equality holds if and only if $h \in \mathrm{span}\{o_1, o_2, o_3\}$.

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- Equivalently its lift to $\mathbb{R}^4$ is a periodic with periods $(2\mathbb{Z})^4$ harmonic tensor field with prescribed singular behavior at the points of $\mathbb{Z}^4$.
- The group $\mathcal{G}$ of symmentries we impose (lifted to $\mathbb{R}^4$) consists of the symmetries which either fix both $\mathbb{Z}^4_{\text{even}}$ and $\mathbb{Z}^4_{\text{odd}}$ or exchange them, where

$$\mathbb{Z}^4_{\text{even}} = \{(a_1, a_2, a_3, a_4) \in \mathbb{Z}^4 : a_1 + a_2 + a_3 + a_4 \text{ is even}\} \text{ and}$$
$$\mathbb{Z}^4_{\text{odd}} = \{(a_1, a_2, a_3, a_4) \in \mathbb{Z}^4 : a_1 + a_2 + a_3 + a_4 \text{ is odd}\}.$$

- $\mathcal{G}$ is generated by the following.

$$
\begin{aligned}
(x_1, x_2, x_3, x_4) &\mapsto (x_2, -x_1, x_3, x_4),\\
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(x_1, x_2, x_3, x_4) &\mapsto (x_3, x_4, x_1, x_2),\\
(x_1, x_2, x_3, x_4) &\mapsto (-x_3, x_4, -x_1, x_2),\\
(x_1, x_2, x_3, x_4) &\mapsto (1 - x_1, x_2, x_3, x_4),\\
(x_1, x_2, x_3, x_4) &\mapsto (x_1, 1 - x_2, x_3, x_4),\\
(x_1, x_2, x_3, x_4) &\mapsto (x_1, x_2, 1 - x_3, x_4),\\
(x_1, x_2, x_3, x_4) &\mapsto (x_1, x_2, x_3, 1 - x_4),\\
(x_1, x_2, x_3, x_4) &\mapsto (x_1 + 2, x_2, x_3, x_4),\\
(x_1, x_2, x_3, x_4) &\mapsto (x_1, x_2 + 2, x_3, x_4),\\
(x_1, x_2, x_3, x_4) &\mapsto (x_1, x_2, x_3 + 2, x_4),\\
(x_1, x_2, x_3, x_4) &\mapsto (x_1, x_2, x_3, x_4 + 2).
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(x_1, x_2, x_3, x_4) &\mapsto (1 - x_1, x_2, x_3, x_4),\\
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(x_1, x_2, x_3, x_4) &\mapsto (x_1, x_2 + 2, x_3, x_4),\\
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(x_1, x_2, x_3, x_4) &\mapsto (x_1, x_2, x_3, x_4 + 2).
\end{aligned}
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- The first four generate the stabilizer of the origin and are symmetries of the EH metric as mentioned earlier.

- Moreover they kill $o_{2,\varepsilon}, o_{3,\varepsilon}$ but respect $o_{1,\varepsilon}$.

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- The symmetries which fix $\mathbb{Z}^4_{\mathrm{even}}$ respect orientation and the other ones do not.
- Of the 10-parameter family of deformations of the torus only scaling is allowed by $\mathcal{G}$.
- We define the singular solution by

$$H = \frac{1}{2} \sum_{a \in \mathbb{Z}^4_{\mathrm{even}}} \tau_a^* T + \frac{1}{2} \sum_{a \in \mathbb{Z}^4_{\mathrm{odd}}} \tau_a^* \hat{T},$$

  where for each $a \in \mathbb{Z}^4$, we denote by τ_a the translation $x \mapsto x - a$.

- Moreover they kill $o_{2,\varepsilon}, o_{3,\varepsilon}$ but respect $o_{1,\varepsilon}$.
- The symmetries which fix $\mathbb{Z}^4_{\mathrm{even}}$ respect orientation and the other ones do not.
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  where for each $a \in \mathbb{Z}^4$, we denote by τ_a the translation $x \mapsto x - a$.
- Although each series does not converge absolutely, H is well defined because there are cancellations in the sense

$$\tau^*_{(a_1,a_2,a_3,a_4)} T(x) + \tau^*_{(a_3,-a_4,-a_1,a_2)} T(x) = O(|a|^{-5}),$$
$$\tau^*_{(a_1,a_2,a_3,a_4)} \hat{T}(x) + \tau^*_{(a_3,a_4,-a_1,-a_2)} \hat{T}(x) = O(|a|^{-5})$$

  as $|a| \to \infty$.

- We define now the approximately Ricci-flat metric $\bar{g}_{\varepsilon,\delta}$ depending on two given positive numbers: ε (the scale of the EH) and δ (the scale of the annulus where the gluing occurs), where we assume $\varepsilon \ll \delta^4 \ll 1$.

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- Finally, in the intermediate region $A_\delta := \{x : \frac{1}{2}\,\delta \leq |x| \leq \delta\,\}$, we define

$$\bar{g}_{\varepsilon,\delta} \;\; = \;\; (1 \; - \; \chi(|x|/\delta))\, g_{\mathrm{eh},\varepsilon} \; + \; \chi(|x|/\delta)\left(g_{\mathrm{eucl}} \; + \; \varepsilon^4 H\right),$$

where χ is a cutoff function satisfying $\chi = 0$ on $(-\infty, \frac{2}{3}]$ and $\chi = 0$ on $[\frac{5}{6}, \infty)$.

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- Similarly, if $a \in \mathbb{Z}^4_{\mathrm{odd}}$, then we have $\bar{g}_{\varepsilon,\delta} = \tau_a^* \hat{g}_{\mathrm{eh},\varepsilon}$ in a neighborhood of a.
- Thus, if we take the quotient by translations and antipodal reflection, then the metric $\bar{g}_{\varepsilon,\delta}$ descends to a smooth metric on the quotient manifold M.

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- We find it useful to denote by $\bar{o}_{1,\varepsilon,\delta}$ the trace-free part of the tensor $\frac{1}{2}\,\varepsilon\,\frac{\partial}{\partial\varepsilon}\bar{g}_{\varepsilon,\delta}$ with respect to the metric $\bar{g}_{\varepsilon,\delta}$.

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- $\Delta_{L,\bar{g}_{\varepsilon,\delta}}\bar{o}_{1,\varepsilon,\delta}$ satisfies the same estimates with $\mathrm{Ric}_{\bar{g}_{\varepsilon,\delta}}$ above and $\bar{o}_{1,\varepsilon,\delta}$ can be used as approximate kernel.

- We have the main estimate

$$-2\int_{[-\frac{1}{2},\frac{1}{2}]^4\setminus\{0\}} \langle \bar{o}_{1,\varepsilon,\delta}, \mathrm{Ric}_{\bar{g}_{\varepsilon,\delta}} \rangle_{\bar{g}_{\varepsilon,\delta}} \, d\mathrm{vol}_{\bar{g}_{\varepsilon,\delta}} = 32\pi^2 \omega\, \varepsilon^8 + O(\varepsilon^8\, \delta^2).$$

where

$$\omega := \sum_{a\in\mathbb{Z}^4_{\mathrm{odd}}} |a|^{-10} \left(|a|^4 - 6\,(a_1^2 + a_2^2)\,(a_3^2 + a_4^2)\right) \approx 7.70.$$

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- The nonvanishing of the right hand side (equivalently of ω) shows that we cannot correct to Ricci-flatness.
- *Proof of the estimate.* Working modulo error terms of appropriate side and using Green's second identity we find that the left hand side equals $\frac{1}{2}\,\varepsilon^8\times$

$$\left(\sum_{a\in\mathbb{Z}^4_{\mathrm{even}}\setminus\{0\}} \int_{\{|x|=\delta\}} \left(\langle T, D_\nu(\tau_a^* T)\rangle_{g_{\mathrm{eucl}}} - \langle \tau_a^* T, D_\nu T\rangle_{g_{\mathrm{eucl}}}\right) d\mu_{g_{\mathrm{eucl}}}\right.$$

$$\left. + \sum_{a\in\mathbb{Z}^4_{\mathrm{odd}}} \int_{\{|x|=\delta\}} \left(\langle T, D_\nu(\tau_a^* \hat{T})\rangle_{g_{\mathrm{eucl}}} - \langle \tau_a^* \hat{T}, D_\nu T\rangle_{g_{\mathrm{eucl}}}\right) d\mu_{g_{\mathrm{eucl}}}\right).$$

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- The integral with $a \in \mathbb{Z}^4_{\mathrm{odd}}$ turns out to equal $64\pi^2 \left|a\right|^{-10} \left(\left|a\right|^4 - 6\left(a_1^2 + a_2^2\right)\left(a_3^2 + a_4^2\right)\right)$ completing the proof and confirming that an EH neck "sees" necks of the opposite orientation.

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- By the definition of $\bar{o}_{1,\varepsilon,\delta}$, the L^2 norm of $\bar{o}_{1,\varepsilon}$, that on $B^{in}_{\delta/2}$ we have $\bar{g}_{\varepsilon,\delta} = g_{\mathrm{eh},\varepsilon}$, $\frac{\partial}{\partial \varepsilon} \bar{g}_{\varepsilon,\delta} = \frac{2}{\varepsilon} \, o_{1,\varepsilon}$, and $\bar{o}_{1,\varepsilon,\delta} = o_{1,\varepsilon}$, and estimates on the error terms we have

$$\int_{[-\frac{1}{2},\frac{1}{2}]^4 \setminus \{0\}} \left\langle \bar{o}_{1,\varepsilon,\delta}, \frac{\partial}{\partial \varepsilon} \bar{g}_{\varepsilon,\delta} \right\rangle_{\bar{g}_{\varepsilon,\delta}} d\mathrm{vol}_{\bar{g}_{\varepsilon,\delta}} = 4\pi^2 \varepsilon^3 + O(\varepsilon^7 \delta^{-4}).$$

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- Projecting the Ricci flow equation to $\bar{o}_{1,\varepsilon,\delta}$ and using the main estimate and the last estimate we find

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- This motivates the following Theorem.

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Theorem

There exists a compact ancient solution to the Ricci flow in dimension 4 symmetric under the group $\mathcal{G}$ defined above and with the following property. For $-t$ sufficiently large, the manifold can be viewed as a desingularization of a flat torus with 16 orbifold points. More precisely, we divide the 16 orbifold points into two classes according to a checkerboard pattern. Near 8 orbifold points, the metric is a small perturbation of a positively-oriented near the remaining 8 orbifold points, the metric is a small perturbation of a negatively-oriented Eguchi-Hanson metric. As $t \to -\infty$, the size of the Eguchi-Hanson instantons shrinks to zero, and we have $\sup |\mathrm{Rm}_{g(t)}|_{g(t)} = (c + o(1))\,(-t)^{\frac{1}{2}}$, where c is a positive constant. Finally, the Ricci curvature of $g(t)$ satisfies $\sup |\mathrm{Ric}_{g(t)}|_{g(t)} = O((-t)^{-\frac{1}{2}+\kappa})$ as $t \to -\infty$, where $\kappa > 0$ can be chosen arbitrarily small.

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- The perturbation is controlled by a tensor field k on $M \times (-\infty, -\Lambda]$ invariant under $\mathcal{G}$ and two real valued functions η and β on $(-\infty, -\Lambda]$.
- η modifies the scale $\varepsilon(t)$ by the equation

$$\varepsilon(t) = \Big(-32\omega t + \int_t^{-\Lambda} \eta(s)\, ds \Big)^{-\frac{1}{4}}$$

  and its purpose is to compensate for the error term in the direction of the approximate kernel. It plays the role of "unbalancing" parameters in gluing problems corresponding to the obstruction of EH scaling.

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- To apply the Schauder fixed point theorem we define a map $\mathcal{J}$ which sends a triplet $(k, \eta(\cdot), \beta(\cdot)) \in \mathcal{A}^{\alpha}_{\gamma,\sigma,\Lambda}$ to the triplet $(h, \xi(\cdot), \nu(\cdot))$.

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- $\mathcal{J}$ is constructed in such a way that if $(k, \eta(\cdot), \beta(\cdot))$ is a fixed point, then the metrics $g(t)$ satisfy

$$g(t) := e^{\int_t^{-\Lambda} \beta(s)\, ds} (\bar{g}_{\varepsilon(t),\delta(t)} + k(t)),$$

satisfy

$$\frac{\partial}{\partial t} g(t) = -2 \operatorname{Ric}_{g(t)} + \mathcal{L}_Y g(t),$$

where $Y = \operatorname{div}_{\bar{g}_{\varepsilon(t),\delta(t)}} k - \frac{1}{2} \nabla \operatorname{tr}_{\bar{g}_{\varepsilon(t),\delta(t)}} k$.

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- The quadratic terms are defined for a given metric g and a symmetric $(0,2)$-tensor k satisfying $|k|_g \leq \frac{1}{2}$, by

$$Q_g(k) = 2\,\mathrm{Ric}_{g+k} - 2\,\mathrm{Ric}_g + \Delta_{L,g} k - \mathcal{L}_Y(g+k),$$

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  $$Q_g(k) = k * \nabla^2 k + \nabla k * \nabla k + \mathrm{Rm}_g * k * k + \text{higher order terms}.$$

- The linear results are based on a blow-up approach as in results of Mazzeo-Pacard, together with the Liouville-type theorems based on understanding the kernel.

Given real numbers $\gamma > 0$ and $\sigma \in (0, 2)$, we can find real numbers $\Lambda > 0$ and $C > 0$ with the following property (under reasonable assumptions). Let h be a solution of the inhomogeneous heat equation $\frac{\partial}{\partial t} h(t) = \Delta_{L, \bar{g}_{\varepsilon(t),\delta(t)}} h(t) + \psi(t)$ which is defined on $M \times (-\infty, -\Lambda]$ and satisfies $\sup_{M \times (-\infty,-\Lambda]} (-t)^\gamma \, (\varepsilon(t) + r)^\sigma \, |h(t)|_{\bar{g}_{\varepsilon(t),\delta(t)}} < \infty$. Moreover, we assume that h is invariant under the group $\mathcal{G}$ and satisfies the orthogonality conditions

$$\int_{[-\frac{1}{2},\frac{1}{2}]^4 \setminus \{0\}} \langle h(t), \bar{o}_{1,\varepsilon(t),\delta(t)} \rangle_{\bar{g}_{\varepsilon(t),\delta(t)}} \, d\mathrm{vol}_{\bar{g}_{\varepsilon(t),\delta(t)}} = 0$$

and

$$\int_{[-\frac{1}{2},\frac{1}{2}]^4 \setminus \{0\}} \langle h(t), \bar{g}_{\varepsilon(t),\delta(t)} \rangle_{\bar{g}_{\varepsilon(t),\delta(t)}} \, d\mathrm{vol}_{\bar{g}_{\varepsilon(t),\delta(t)}} = 0$$

for all $t \in (-\infty, -\Lambda]$. Then

$$\begin{aligned} \sup_{M \times (-\infty,-\Lambda]} (-t)^\gamma \, ((-t)^{-\frac{1}{4}} + r)^\sigma \, |h(t)|_{\bar{g}_{\varepsilon(t),\delta(t)}} &\leq \\ \leq C \sup_{M \times (-\infty,-\Lambda]} (-t)^\gamma \, &((-t)^{-\frac{1}{4}} + r)^{\sigma+2} \, |\psi(t)|_{\bar{g}_{\varepsilon(t),\delta(t)}}. \end{aligned}$$

Let $(M_{\mathrm{eh}}, g_{\mathrm{eh}})$ denote the Eguchi-Hanson manifold with parameter $\varepsilon = 1$, and let $o_i := o_{i,1}$. Let h be a solution of the heat equation $\frac{\partial}{\partial t} h = \Delta_{L,g_{\mathrm{eh}}} h$ on $M_{\mathrm{eh}} \times (-\infty, 0]$ with the property that $|h| \leq (1+r)^{-\sigma}$ for some $\sigma > 0$. If $\int_{M_{\mathrm{eh}}} \langle h(t), o_i \rangle = 0$ for all $i \in \{1, 2, 3\}$ and all $t \in (-\infty, 0]$, then h vanishes identically.

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- Can numerics help?
- An interesting observation is that if we consider less symmetric configurations we should be able to construct metrics by gluing 16 small Eguchi-Hanson metrics on the orbifold as before, which under the Ricci flow evolve so that one of the Eguchi-Hanson metrics becomes extinct by shrinking to zero size, while the rest stay close to fixed nonzero sizes.