U(2)-invariant Ricci flow in 4D and partial regularity of Ricci flows

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(on work by Alexander Appleton, partly joint with Jon Wilkening)

UC Berkeley

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Structure of Talk

1. Introduction (Ricci flow, $U(2)$-invariant metrics + examples)
2. Numerical simulations
3. Rigorous results
4. Partial Regularity Theory for general Ricci flows

Bibliography

- A. Appleton, *A family of non-collapsed steady Ricci solitons in even dimensions greater or equal to four,* (2017), arXiv:1708.00161v4
- A. Appleton, *Eguchi-Hanson singularities in U(2)-invariant Ricci flow,* (2019), arXiv:1903.09936

Introduction

Ricci flow

$(M, g(t)), \quad t \in [0, T)$

$$\partial_t g(t) = -2 \operatorname{Ric}_{g(t)}$$

Special metrics:

- $\operatorname{Ric} = 0$
  $\rightsquigarrow \quad g(t) = g$, constant flow
- $\operatorname{Ric} = \frac{1}{2} g$
  $\rightsquigarrow \quad g(t) = -t\, g$, $t < 0$, has a singularity at time 0
- gradient steady soliton $\quad \operatorname{Ric} + \nabla^2 f = 0$
  $\rightsquigarrow \quad g(t) = \phi_t^* g$, constant modulo diffeomorphisms
- gradient shrinking soliton $\quad \operatorname{Ric} + \nabla^2 f = \frac{1}{2} g$
  $\rightsquigarrow \quad g(t) = -t\, \phi_t^* g$, $t < 0$, scales down modulo diffeomorphisms

Goal: Understand singularity formation for "general" initial conditions.

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Setup: Cohomogeneity-1 symmetry $U(2) \curvearrowright (M^4, g)$

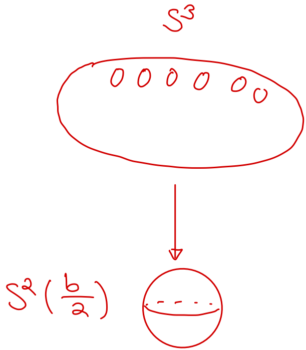
Principal orbits: $S^3/\mathbb{Z}_k$ $(k \geq 1 \text{ fixed})$

$S^3 \subset \mathbb{C}^2$ $S^1 \subset U(2)$

$\mathbb{Z}_k \subset S^1 \curvearrowright S^3$ scalar mult.

Hopf fibration

$\ell(\text{Hopf fiber}) = \frac{2\pi a}{k}$



$\Rightarrow$ Hopf fibers + U(2)-action descend to $S^3/\mathbb{Z}_k$

U(2)-invariant metric:

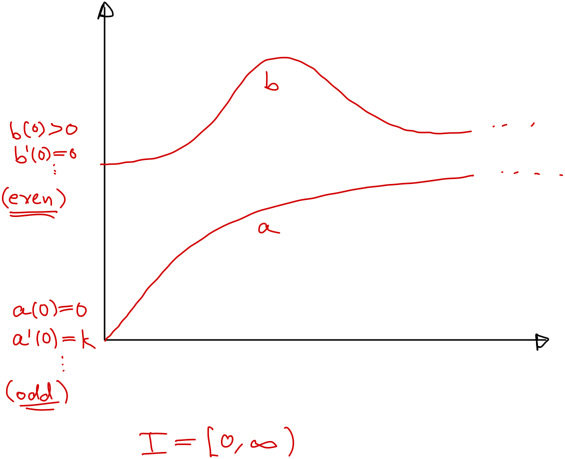
$\alpha, \beta, \gamma \in \Omega^1(S^3/\mathbb{Z}_k)$ $SU(2)$-invariant local ONF

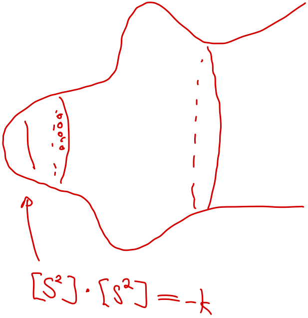
$g = a^2\, \alpha \otimes \alpha + b^2\, (\beta \otimes \beta + \gamma \otimes \gamma)$ "Berger sphere"

Exceptional orbits: S^2 $(U(2) \to SU(2) \to SO(3) \curvearrowright S^2)$

$\Longrightarrow \quad (M^4, g)$ is determined by $k \geq 1$ and $a, b : I \to \mathbb{R}_{\geq 0}$

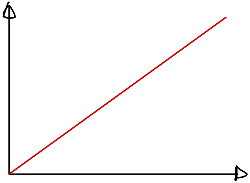
$$g = ds^2 + a^2(s)\,\alpha \otimes \alpha + b^2(s)\,(\beta \otimes \beta + \gamma \otimes \gamma)$$

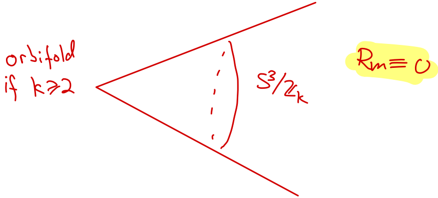




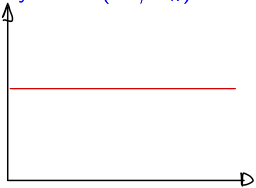
Examples

1. Euclidean orbifold $\mathbb{R}^4/\mathbb{Z}_k$ $\quad a = b = s$

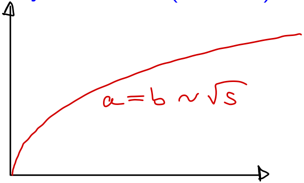


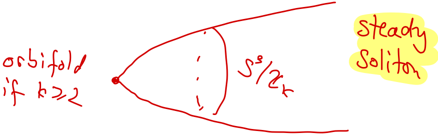


2. Cylinder $(S^3/\mathbb{Z}_k) \times \mathbb{R}$ $\quad a = b = const$

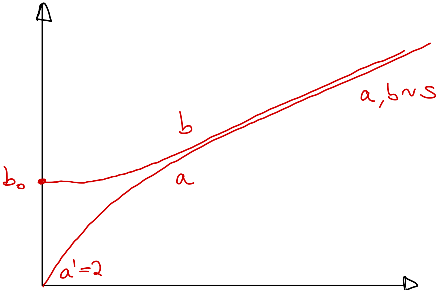


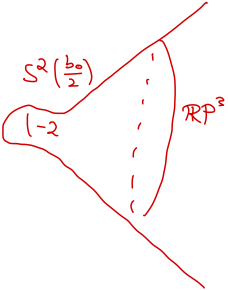
3. Bryant soliton (orbifold)





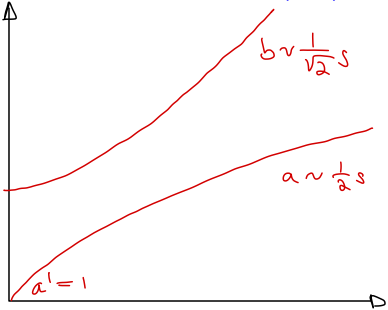
❹ Eguchi-Hanson $\quad k = 2$





Ric = 0
Hyperkähler
ALE
≈ $\mathcal{O}(-2)$

❺ Feldman-Ilmanen-Knopf (FIK) shrinker $\quad k = 1$



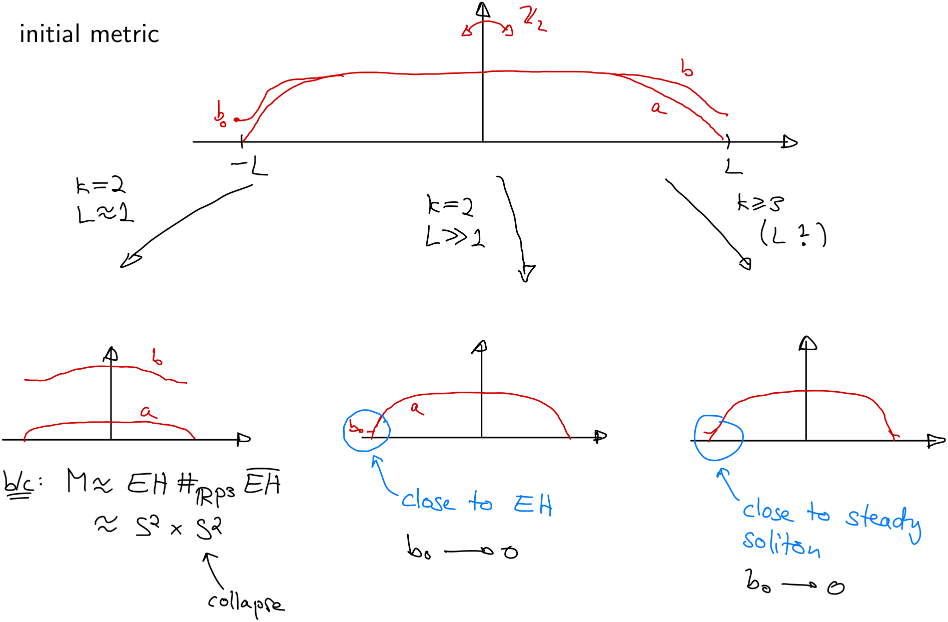
Kähler
shrinking
soliton

$\mathcal{O}(-1)$

Numerical simulation of $U(2)$-invariant RFs

(A. Appleton & J. Wilkening)

initial metric



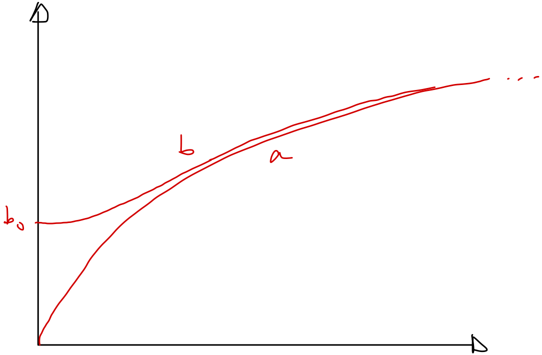
Rigorous results

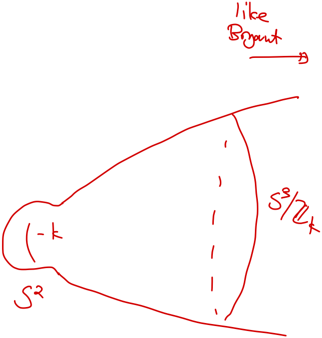
Results regarding singularity models

Appleton 2017

If $k \geq 3$, then $\exists$ $U(2)$-invariant, gradient steady solitons on $\mathcal{O}(-k)$ s.t. $a, b \sim \sqrt{r}$ at ∞.

Question: Uniqueness?



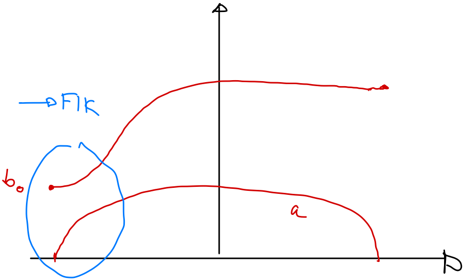


Also: $\exists$ Taub-NUT-like steady solitons (Appleton, Stolarski, Wink)

Results regarding singularity formation

Maximo 2014

Suppose $k = 1$, $M \approx \overline{\mathbb{C}P}^2 \# \mathbb{C}P^2$.
Then $\exists$ large class of $U(2)$-invariant initial conditions $g(0)$ such that the flow develops a Type-I singularity ($|\mathrm{Rm}| \leq C/(T-t)$) that is modeled on the FIK-shrinker.



$\partial_t b_0^2 = -4$

Note: These flows are Kähler.

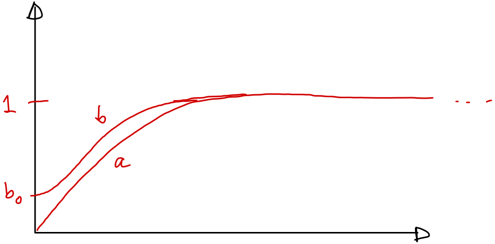
If $g(t)$ is Kähler, then $\partial_t b_0^2 = 4(k-2)$, so no singularity can occur if $k \geq 2$.

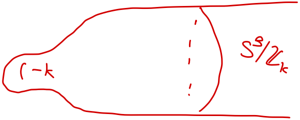
So if $k \geq 2$, then we need to consider non-Kähler initial data.

Initial metric: Suppose $M = \mathcal{O}(-k)$ and $g(0)$ given by $a, b : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$

Asymptotically cylindrical: $a, b \to 1$ as $s \to \infty$

+ technical assumptions





Appleton 2019

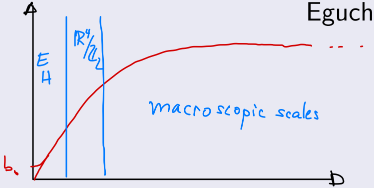
If $k \geq 2$, then the flow $g(t)$, $t \in [0, T)$, develops a Type-II singularity that is modeled on an eternal flow $(M', g'(t))$, $t \in \mathbb{R}$.

Type I $\quad b_0^2 \sim |Rm|^{-1}(0,t) \sim c(T-t)$

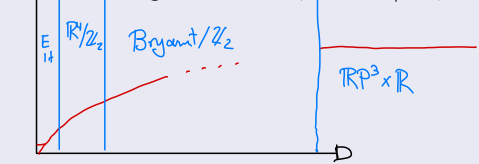
Type II $\quad b_0^2 \sim \ldots \sim o(T-t)$

If $k = 2$, then the singularity is modeled on the Eguchi-Hanson metric.
Moreover, one of 2 possible behaviors occurs:

1. Bubbling off: The metric converges to an orbifold at scale 1.
   Possible blow-up models:
   Eguchi-Hanson, $\mathbb{R}^4/\mathbb{Z}_2$



2. Collapse at infinity: The cylindrical end becomes singular at the same time.
   Possible blow-up models:
   Eguchi-Hanson, $\mathbb{R}^4/\mathbb{Z}_2$, Bryant$/\mathbb{Z}_2$, $\mathbb{R}P^3 \times \mathbb{R}$



Questions: Which behavior occurs?
Do we see the steady solitons from before if $k \geq 3$?

Proof strategy

1+1 PDE in 3 functions p, a, b, where

$g = p^2 ds^2 + a^2\alpha^2 + b^2(\beta^2+\gamma^2)$, $\qquad a' := \frac{1}{p}\frac{da}{ds}, \ldots$

$$\partial_t p = \frac{a''}{a} + 2\frac{b''}{b}$$
$$\partial_t a = -\frac{a^3}{2b^4} + 2\frac{a'b'}{b} + a''$$
$$\partial_t b = -\frac{4}{b} + \frac{a^2}{b^3} + \frac{a'b'}{a} + \frac{(b')^2}{b} + b''$$

No hope to control solutions without geometric intuition!

Recall: $\mathcal{O}(-2)$ has a Hyperkähler metric, the Eguchi-Hanson metric
$\rightsquigarrow$ 2 interesting (almost) complex structures I, J

Kähler conditions for I and J:

$$b' - \frac{a}{b} = 0, \qquad a' + \frac{a^2}{b^2} - 2 = 0$$

These are preserved by the RF.

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$I(ds) = a\alpha \qquad I\beta = \gamma$

$J(ds) = b\beta \qquad J(a\alpha) = b\beta$

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(I) $$b' - \frac{a}{b} = 0,$$ (J) $$a' + \frac{a^2}{b^2} - 2 = 0$$

These are preserved by the RF.

Set

$$y := b' - \frac{a}{b}, \qquad\qquad x := a' + \frac{a^2}{b^2} - 2$$

Compute evolution (using Mathematica)

$$\partial_t y = y'' + (\dots)y' + (\dots)y$$

$$\partial_t x = x'' + (\dots)x' + (\dots)x - \left(\frac{4a^2}{b^4} + \frac{2}{b^2}\right) y^2$$

Preserved conditions:

1. $y \leq 0$, $y \geq 0$ ("sub/super-Kähler")
2. $x \leq 0$

Note:

- If $y \leq 0$, then $\partial_t b_0^2 \leq 0$, so assume $y \leq 0$ from now on.
- If $x \equiv 0$, then $y \equiv 0$, so $(M, g) =$ Eguchi-Hanson.

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Further preserved conditions:

1. $a' \geq 0,\ b' \geq 0$
2. $Q := a/b \leq 1$
3. $T_1 = a' + 2Q^2 - 2 \geq -Z$
4. $T_2 = Qy - x \geq -Z$
5. $T_3 = a' - Qb' - Q^2 + 1 \geq -Z$

Lemma

Consider a $U(2)$-invariant ancient flow $(M = \mathcal{O}(-k), g(t))$, $t \leq 0$, that is κ-noncollapsed at all scales (+ additional conditions). Then $T_1, T_2, T_3, \geq 0$.

Proof T_1, T_2, T_3 are scaling invariant. Suppose, by contradiction, that

$$T_1(p_i, t_i) \longrightarrow \inf_{M \times (-\infty, 0]} T_1 = -Z < 0$$

Rescale by $|\mathrm{Rm}|(p_i, t_i)$, take limit based at (p_i, t_i)

$\rightsquigarrow$ T_1 attains a minimum $-Z$ at the limiting basepoint $(p_\infty, 0)$.

This contradicts the strong maximum principle if $Z > 0$.

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Lemma

If $k = 2$, then the ancient flow from the previous lemma must be a stationary Eguchi-Hanson metric.

Proof via "successive constraining"

Find a continuous family of functions $\{f_\theta(s)\}_{\theta \in [0,1]}$ such that

1. $f_0 = 0$
2. $f_1 = 1$
3. $x \geq -f_\theta(Q) Q^2$ is preserved
4. nice behavior at infinity

Consider

$$I := \{\theta \in [0,1] \mid x \geq -f_\theta(Q) Q^2\} \qquad \text{(closed)}$$

Strong Maximum Principle $\implies$ I is open.

$T_1 \geq 0 \implies 1 \in I$

So $I = [0,1] \implies x \geq 0$

$T_2 \geq 0 \implies x \leq Qy \leq 0 \implies x \equiv 0$

Lemma

If $k = 2$, then the ancient flow from the previous lemma must be a stationary Eguchi-Hanson metric.

Proof via "successive constraining"

Find a continuous family of functions $\{f_\theta(s)\}_{\theta \in [0,1]}$ such that

1. $f_0 = 0$
2. $f_1 = 1$
3. $x \geq -f_\theta(Q)Q^2$ is preserved
4. nice behavior at infinity

$Q = \frac{a}{b}$

Consider

$$I := \{\theta \in [0,1] \mid x \geq -f_\theta(Q)Q^2\} \qquad \text{(closed)}$$

Strong Maximum Principle $\Longrightarrow$ I is open.

$T_1 \geq 0 \Longrightarrow 1 \in I$

So $I = [0,1] \Longrightarrow x \geq 0$

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Partial regularity theory for general RFs

(without the $U(2)$-symmetry assumption)

Gradient shrinking solitons (M, g, f)

$$\mathrm{Ric} + \nabla^2 f - \frac{1}{2} g = 0$$

$\rightsquigarrow$ Associated (selfsimilar) Ricci flow: $g(t) := |t| \phi_t^* g, \quad t < 0$

The singularity model of $(M, (g(t))_{t<0})$ is the flow itself.

Example: Round shrinking cylinder $M = S^{k \geq 2} \times \mathbb{R}^{n-k}$

$$g = 2(k-1)\, g_{S^k} + g_{\mathbb{R}^{n-k}}, \qquad f = \frac{1}{4} \sum_{i=k+1}^{n} x_i^2$$

Folklore Conjecture

For any Ricci flow "most" singularity models are gradient shrinking solitons.

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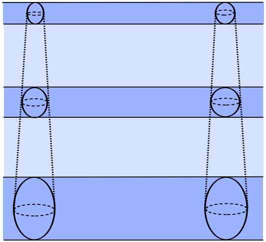

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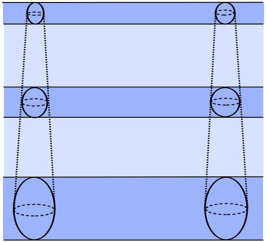

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For any Ricci flow "most" singularity models are gradient shrinking solitons.

Recall: Possible blow-up models in Appleton's example:

Eguchi-Hanson, $\mathbb{R}^4/\mathbb{Z}_2$, $\big($Bryant$/\mathbb{Z}_2$, $\mathbb{R}P^3 \times \mathbb{R}\big)$

Folklore Conjecture (revised)

For any Ricci flow, "most" singularity models are possibly singular gradient shrinking solitons, where the singular set has codimension ≥ 4.

Goal: Verify this conjecture.

Main Tool: New Compactness and Partial Regularity Theory for Ricci flows (We focus on the 4D case here)

Recall the Einstein case:

Cheeger, Colding, Naber, Tian

If (M_i^4, g_i, x_i) is a sequence of pointed Einstein metrics ($\mathrm{Ric} = \lambda_i g_i$, $|\lambda_i| \leq 1$) that is non-collapsed, i.e.,

$$|B(x_i, r)| \geq v > 0,$$

then a subsequence (M_i, g_i, x_i) converges to an Einstein orbifold with isolated singularities in the Gromov-Hausdorff sense. The convergence is smooth on the regular part.

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Compactness and partial regularity for RFs

Theorem (B. 2020)

Consider a sequence of 4-dimensional, pointed Ricci flows:

$$(M_i^4, (g_i(t))_{t\in(-T_i,0]}, (x_i, 0)), \qquad T_\infty := \lim_{i\to\infty} T_i > 0.$$

Then a subsequence $\mathbb{F}$-converges to a metric flow over $(-T_\infty, 0]$:

$$(M_i^4, (g_i(t))_{t\in(-T_i,0]}, (\nu_{x_i,0})) \xrightarrow[i\to\infty]{\mathbb{F}} (\mathcal{X}, d, (\nu_{x_\infty})).$$

Suppose that the following non-collapsing condition holds:

$$\mathcal{N}_{x_i,0}(\tau_0) \geq -Y_0 > -\infty.$$

Then $\mathcal{X}$ is smooth away from a singular set of codimension ≥ 4. The tangent flows at the singular points are either $\mathbb{R}^4/\Gamma$ (i.e., orbifold singularities) or smooth gradient shrinking soliton orbifolds.

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Consequences

Finite-time singularity formation:

Theorem (B. 2020)

The revised Folklore Conjecture is true:

- Any RF $(M^4, (g(t)_{t\in[0,T)})$ has a "singular time-slice" M_T at time T.
- The tangent flows (= blow-up models) based at $(x, T) \in M_T$ are singular gradient shrinking soliton orbifolds

Long-time asymptotics:

Theorem (B. 2020)

If $(M^4, (g(t))_{t\geq 0})$ is immortal, then for $t \gg 1$

$$M = M_{\text{thick}}(t) \quad \cup \quad M_{\text{thin}}(t)$$

such that the flow on $M_{\text{thick}}(t)$ converges to an Einstein orbifold ($\text{Ric}_{g_\infty} = -g_\infty$) and the flow on $M_{\text{thin}}(t)$ is collapsed.

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